MUNI SCI
Department
of Theoretical Physics
and Astrophysics

Integrable deformations of strings

Habilitation Thesis

Linus Wulff

Brno 2021

Abstract

This thesis deals with deformations of string sigma models which have the property that they preserve integrability. This means that starting from an integrable string sigma model and deforming it one obtains a new integrable model, which reduces to the original one when the deformation parameter is taken to zero. There are different types of such deformations but a large class, which we will focus on here, are the so-called Yang-Baxter deformations. They are defined by a constant matrix R which solves the classical Yang-Baxter equation. After introducing these deformations in the simplest setting of the Principal Chiral Model we will describe their close relation to the transformation known as non-abelian T-duality. In the case of string theory there are additional conditions on the sigma model. In particular, it must be Weyl invariant. We show that Yang-Baxter deformations preserve the Weyl invariance to at least two loop order in the sigma model perturbation theory, provided R satisfies a so-called unimodularity condition. The proof of this important fact is greatly simplified by working in a formalism with an enlarged symmetry group, known as Double Field Theory. This also allows us to find the first quantum (α') correction to these deformed models.

List of included papers

This thesis is based on the following papers:

[I] S. Hronek and L. Wulff, "Relaxing unimodularity for Yang-Baxter deformed strings," JHEP **10** (2020), 065 [arXiv:2007.15663 [hep-th]].

[II] R. Borsato and L. Wulff, "Quantum Correction to Generalized T Dualities," Phys. Rev. Lett. **125** (2020) no.20, 201603 [arXiv:2007.07902 [hep-th]].

[III] R. Borsato, A. Vilar López and L. Wulff, "The first α'-correction to homogeneous Yang-Baxter deformations using $O(d,d)$," JHEP **07** (2020) no.07, 103 [arXiv:2003.05867 [hep-th]].

[IV] R. Borsato and L. Wulff, "Two-loop conformal invariance for Yang-Baxter deformed strings," JHEP **03** (2020), 126 [arXiv:1910.02011 [hep-th]].

[V] R. Borsato and L. Wulff, "Marginal deformations of WZW models and the classical Yang–Baxter equation," J. Phys. A **52** (2019) no.22, 225401 [arXiv:1812.07287 [hep-th]].

[VI] R. Borsato and L. Wulff, "Non-abelian T-duality and Yang-Baxter deformations of Green-Schwarz strings," JHEP **08** (2018), 027 [arXiv:1806.04083 [hep-th]].

[VII] R. Borsato and L. Wulff, "On non-abelian T-duality and deformations of supercoset string sigma-models," JHEP **10** (2017), 024 [arXiv:1706.10169 [hep-th]].

[VIII] R. Borsato and L. Wulff, "Integrable Deformations of T-Dual σ Models," Phys. Rev. Lett. **117** (2016) no.25, 251602 [arXiv:1609.09834 [hep-th]].

[IX] R. Borsato and L. Wulff, "Target space supergeometry of η and λ-deformed strings," JHEP **10** (2016), 045 [arXiv:1608.03570 [hep-th]].

Reprints can be found at the end of this thesis. All authors made an equal contribution to the papers.

Acknowledgments

I wish to thank my co-authors, in particular Riccardo Borsato, for enjoyable collaborations on the fascinating topic of integrable deformations. I thank Stanislav Hronek for helpful comments on a draft of this thesis.

Contents

Chapter 1

Introduction

This thesis summarizes work done over the past five years on the topic of integrable deformations of string sigma models. It consists of nine papers together with an introductory part explaining some of the important results in those papers and giving important background. To keep things as simple as possible we focus on the Principal Chiral Model for introducing the Yang-Baxter deformation and non-abelian T-duality (Chapter 2), and on the bosonic string when we discuss deformations of string sigma models and Weyl invariance (Chapters 3 & 4). Many extensions of the results, as well as a more complete list of references, can be found in the papers.

Two-dimensional non-linear sigma models have many applications in physics. Here we will mainly be interested in the application to string theory where certain two-dimensional sigma models describe the dynamics of the string itself. The action of the non-linear sigma model takes the form

$$S = \int d^2\xi \, \partial_+ x^m \partial_- x^n \left(G_{mn}(x) + B_{mn}(x) \right) \,, \tag{1.1}$$

where the 2d coordinates are ξ^0, ξ^1 and we have defined the light-cone derivatives $\partial_\pm = \partial/\partial\xi^0 \pm \partial/\partial\xi^1$. This is a theory of D scalar fields, $x^m = x^m(\xi)$, $m = 0, \ldots, D-1$, from the 2d point of view. The reason we call them x is that these fields can be interpreted as coordinates of a D-dimensional space, which we will often refer to as the background (or target space). The matrices $G_{mn}(x)$ and $B_{mn}(x)$ are symmetric and anti-symmetric respectively and the first is required to be non-degenerate and therefore can be interpreted as a metric on the D-dimensional background. For the applications we have in mind both the 2d metric and the background metric G_{mn} will be taken to be Lorentzian. The simplest example being $G_{mn} = \eta_{mn} = \mathrm{diag}(-1, 1, \ldots, 1)$, the D-dimensional Minkowski metric and $B_{mn} = 0$. In that case we have a theory of D free scalars which we can easily solve. However, for general functions $G_{mn}(x)$ and $B_{mn}(x)$ we don't know how to solve the theory, particularly at the quantum level. But it turns out that there exist special choices of G_{mn} and B_{mn} for which the theory can be essentially solved, in particular the energy spectrum can be fully determined. This happens when the theory has hidden symmetries which lead to infinitely many conserved charges. Such theories are called integrable.[1]

[1] In general an integrable theory should have as many conservation laws as degrees of freedom. In our case we are dealing with a field theory which has degrees of freedom living at each point in spacetime so we need infinitely many conservation laws.

An example where this happens is when G_{mn} is the metric on a group manifold while $B_{mn} = 0$, as we will see in the next chapter. In this case the model is known as the Principal Chiral Model (PCM). Such integrable models are very useful since they are solvable, at least in a certain sense, and therefore one can learn a lot about them and they can serve as toy models for more complicated systems. However, finding models that are integrable is very hard. A strategy which has proven very useful is to ask instead the following question: suppose we have an integrable sigma model such as the PCM, is there a way to modify it a little bit while still preserving the integrability of the model. This turns out to be possible in many cases and the resulting model is then called an *integrable deformation* of the original model. A large class of such integrable deformations are the so-called Yang-Baxter (YB) deformations, which we will introduce in the next chapter and which will be the main topic of this thesis.

The rest of the introductory part of this thesis is organized as follows. In chapter 2 we will introduce the YB deformation of the PCM. Then we will introduce an operation known as non-abelian T-duality (NATD), which is a certain non-local field redefinition in the sigma model. Finally we will see how the so-called homogeneous YB deformations can be constructed using NATD. The PCM is not a string sigma model but it is useful as a simple toy model to introduce the basic ideas.

In chapter 3 we will introduce the bosonic string and the requirement of Weyl invariance which forces the background fields to solve a generalization of Einstein's equations. Then we will generalize the YB deformation to the case of the bosonic string. To verify that the deformed model solves the Weyl invariance conditions directly turns out to be very hard. For this reason a reformulation of these conditions, featuring a larger $O(D, D)$ symmetry, is introduced. This formulation, known as Double Field Theory (DFT), makes it easy to see that the deformed model solves the correct equations at the classical level. We then turn to the problem of including quantum corrections in the 2d theory, which are parametrized by α', the inverse string tension. At one-loop order one finds that the deformed background solves the correct equations if a certain algebraic condition, known as the unimodularity condition, is fulfilled. Finally, in chapter 4 we analyze what happens at the two-loop order, which corresponds to including the first α'-correction in the equations for the background. This means that the background should solve Einstein's equations with a correction involving the Riemann tensor squared. It turns out to be possible to incorporate this correction in the DFT description, which makes it possible to determine what happens to the YB deformation at this order. We will see that a certain correction to the YB deformation is required to solve the equations at first order in α'.

Finally we end with some conclusions and a list of important further developments.

Chapter 2

Yang-Baxter deformations and non-abelian T-duality

In this chapter we will introduce the integrable Yang-Baxter deformation [1, 2] in the setting of the Principal Chiral Model. We will then show that in the homogeneous case, which we will define, this deformation can be constructed using a transformation known as non-abelian T-duality [3].

2.1 Yang-Baxter deformations of the Principal Chiral Model

Consider a 2d sigma model with target space a group manifold. This is known as the Principal Chiral Model (PCM) and the action is

$$S_{\mathrm{PCM}} = \int d^2\xi \ \mathrm{tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) = \int d^2\xi \ \mathrm{tr}(\partial_+ g g^{-1}\partial_- g g^{-1}) \,, \tag{2.1}$$

for the field $g(\xi) \in G$ valued in (some representation of) the group G. This model has a global $G \times G$ symmetry corresponding to the fact that multiplying g by a constant element of G from the left or from the right leaves the PCM action invariant. Because of this the equations of motion of the model take the form of a conservation equation for a symmetry current

$$\partial_+ j_- + \partial_- j_+ = 0 \,, \tag{2.2}$$

where we have introduced the right-invariant Maurer-Cartan form valued in the Lie algebra of G

$$j_\pm = \partial_\pm g g^{-1} \,. \tag{2.3}$$

By construction j satisfies a flatness condition

$$\partial_+ j_- - \partial_- j_+ - [j_+, j_-] = 0 \,. \tag{2.4}$$

A very important property of this model is that it is integrable, in particular it possesses infinitely many conserved quantities. This follows from the fact that the equations of motion can be expressed as the flatness of a one-parameter family of connections known as a Lax connection,

$$\partial_+ L_- - \partial_- L_+ + [L_+, L_-] = 0\,, \tag{2.5}$$

where

$$L_\pm(z) = -\frac{1}{1 \pm z} j_\pm\,, \tag{2.6}$$

as is easily verified using the flatness of j. Here $z \in \mathbb{C}$ is an auxiliary parameter known as the spectral parameter. We can now argue that this leads to infinitely many conserved quantities as follows. For simplicity we take the theory to be defined on a compact spatial circle. Computing the holonomy of the connection around this circle

$$M(z) = \mathcal{P} e^{\int_{S^1} d\xi^1 L_1(z)}\,, \tag{2.7}$$

one finds that it is conserved. To see this consider deforming the integration contour by translating it forward in time by a little bit. The change in the holonomy is given by an integral of the curvature of L, but this vanishes by the flatness condition (2.5). Therefore the eigenvalues of $M(z)$ are conserved and by Taylor expanding in z we obtain an infinite set of conserved quantities. This demonstrates the integrability of the model.[1]

Now consider deforming the PCM action by introducing a constant operator acting on the second current factor in the PCM Lagrangian in (2.1). Without loss of generality we can write this operator as $(1 + \eta R)^{-1}$ where η is the deformation parameter. The deformed action is

$$S_{\mathrm{YB}} = \int d^2\xi \; \mathrm{tr}\left(j_+ \frac{1}{1 + \eta R} j_- \right)\,, \tag{2.8}$$

where we take $R : \mathfrak{g} \to \mathfrak{g}$ to be an arbitrary constant linear operator on the Lie algebra of G. The original PCM is clearly recovered when the deformation is removed, i.e. when we take $\eta \to 0$. For a general R this deformation will break the integrability of the model, leading to a model over which one has much less control. However, it is interesting to ask whether special choices of R exist which *preserve* the integrability of the model. To analyze this question we look at the equations of motion of the deformed model. Let us first note that the deformation we have introduced generically breaks the left G symmetry but always preserves the right-acting copy of G since the action is written in terms of the right-invariant current. Defining the deformed currents[2]

$$J_+ = \frac{1}{1 + \eta R^{\mathrm{T}}} \partial_+ g g^{-1}\,, \qquad J_- = \frac{1}{1 + \eta R} \partial_- g g^{-1}\,, \tag{2.9}$$

the equations of motion again take the form of a conservation equation

$$\partial_+ \tilde{J}_- + \partial_- \tilde{J}_+ = 0\,, \tag{2.10}$$

where $\tilde{J}_\pm = \mathrm{Ad}_g\, J_\pm = g^{-1} J_\pm g$ is the Noether current for the right-acting G symmetry. We now want to ask when this model can be integrable. Let us assume that the Lax connection can again be expressed in terms of the components of the conserved currents, so that

$$L_\pm = a_\pm^{-1} \tilde{J}_\pm\,, \tag{2.11}$$

[1] Strictly speaking one should also show that they are independent and mutually Poisson commuting. This requires a bit more work.

[2] The transpose of R is defined in the standard way as $\mathrm{tr}(XR(Y)) = \mathrm{tr}(R^{\mathrm{T}}(X)Y)$.

for some coefficients $a_\pm$ depending on η and the spectral parameter z. The flatness condition for L then reads

$$a_+\partial_+\tilde{J}_- - a_-\partial_-\tilde{J}_+ + [\tilde{J}_+, \tilde{J}_-] = 0\,. \tag{2.12}$$

From the flatness condition for j in (2.4) we can find a deformed flatness condition for $\tilde{J}$ by writing $j_+ = \mathrm{Ad}_g^{-1}\tilde{J}_+ + \eta R^{\mathrm{T}}J_+$ and similarly for j_- but with R instead of R^{T}. One finds, after multiplying with Ad_g,

$$\partial_+\tilde{J}_- - \partial_-\tilde{J}_+ + [\tilde{J}_+, \tilde{J}_-] + \eta\,\mathrm{Ad}_g\, R\partial_+J_- - \eta\,\mathrm{Ad}_g\, R^{\mathrm{T}}\partial_-J_+ - \eta^2\,\mathrm{Ad}_g[R^{\mathrm{T}}J_+, RJ_-] = 0\,. \tag{2.13}$$

Using this equation to eliminate the commutator term in (2.12) we find, keeping terms to first order in η,

$$(a_+ - 1)\partial_+\tilde{J}_- - (a_- - 1)\partial_-\tilde{J}_+ - \eta\,\mathrm{Ad}_g\, R\partial_+j_- + \eta\,\mathrm{Ad}_g\, R^{\mathrm{T}}\partial_-j_+ + \mathcal{O}(\eta^2) = 0\,. \tag{2.14}$$

Noting that $a_\pm = 1 \pm z + \mathcal{O}(\eta)$ [3] and using the equations of motion (2.10) the vanishing of the terms linear in η implies

$$\big(R + R^{\mathrm{T}} + (a_+ + a_-)_1\big)\,[j_+, j_-] = 0\,, \tag{2.15}$$

where $(\cdots)_1$ denotes the term of first order in η. This condition says that the symmetric part of R should be proportional to the identity operator. But it is evident from (2.8) that a contribution to R proportional to the identity can be absorbed into a redefinition of η and a rescaling of the action. Doing this we find that R should be anti-symmetric, $R^{\mathrm{T}} = -R$, and $a_+ + a_-$ should not have a term linear in η. At the second order in η a similar calculation gives the condition

$$[Rj_+, Rj_-] - R[Rj_+, j_-] - R[j_+, Rj_-] = -\frac{1}{2}(a_+ + a_-)_2[j_+, j_-]\,. \tag{2.16}$$

This requires that R satisfy the (modified) classical Yang-Baxter equation

$$[RX, RY] - R([RX, Y] + [X, RY]) = c[X, Y] \qquad \forall X, Y \in \mathfrak{g}\,, \tag{2.17}$$

for some constant c, and that $a_+ + a_- = 2 - 2c\eta^2 + \mathcal{O}(\eta^3)$. The homogeneous case $c = 0$ is the classical Yang-Baxter equation and the inhomogeneous case $c \neq 0$ is the modified classical Yang-Baxter equation. When $c \neq 0$ we can rescale R to set $c = \pm 1$. We have found that, at least assuming the Lax connection is expressed in terms of the components of the conserved Noether current, R must be an anti-symmetric R-matrix solving the (modified) classical Yang-Baxter equation. In fact these conditions on R are also sufficient for the deformed model to be integrable, namely it is not hard to see that flatness of the connection

$$L_\pm = \frac{1 + c\eta^2}{1 \pm z}\tilde{J}_\pm\,, \tag{2.18}$$

is equivalent to the equations of motion of the deformed model (2.8). The deformed flatness condition (2.13) is useful when checking this. Using the YB equation it takes the simpler form

$$\partial_+\tilde{J}_- - \partial_-\tilde{J}_+ + (1 + c\eta^2)[\tilde{J}_+, \tilde{J}_-] = -\eta R_g(\partial_+\tilde{J}_- + \partial_-\tilde{J}_+)\,, \tag{2.19}$$

where we have defined $R_g = \mathrm{Ad}_g\, R\,\mathrm{Ad}_g^{-1}$. Note that the RHS is proportional to the equations of motion, (2.10).

[3] The sign difference compared to (2.6) comes from the fact that for $\eta = 0$ the two Lax connections differ by a gauge transformation.

2.2 Non-abelian T-duality for PCM

There exists a very interesting change of variables that one can perform for sigma models like the PCM. This change of variables is actually non-local and leads to a model whose Lagrangian looks completely different, but which is nevertheless (classically) equivalent to the original model. This change of variables goes under the name of non-abelian T-duality (NATD). The simplest example of NATD for the PCM goes as follows. We start with the action for the PCM

$$S = \int d^2\xi \;\mathrm{tr}(g^{-1}\partial_+ g g^{-1}\partial_- g)\,. \tag{2.20}$$

But now we rewrite this action in first order form as

$$S_{\mathrm{F}} = \int d^2\xi \;\mathrm{tr}(A_+A_- + \nu F_{+-}(A))\,, \tag{2.21}$$

where A is a gauge field and $F(A)$ is its field strength, i.e. $F_{+-}(A) = \partial_+A_- - \partial_-A_+ + [A_+, A_-]$. The field $\nu \in \mathfrak{g}$ is a Lagrange multiplier enforcing the constraint that the field strength of A vanish. The general solution to this constraint is that A is pure gauge, i.e. (locally) $A_\pm = g^{-1}\partial_\pm g$, and plugging this into the action gives back the original action (2.20). This shows that the two models are equivalent. The remarkable thing is now that there is another way to get a second order action from (2.21). Noting that the action is actually quadratic in A we may instead integrate out A. The equations of motion for A give

$$A_\pm \mp \partial_\pm\nu \pm [\nu, A_\pm] = 0 \qquad \Rightarrow \qquad A^r_\pm = \pm\left[(1 \pm \nu\cdot f)^{-1}\right]^r{}_s\partial_\pm\nu^s\,, \tag{2.22}$$

where $(\nu\cdot f)^r{}_s = \nu^t f_{ts}{}^r$ and we have written $A = A^rT_r$, $\nu = \nu^rT_r$ with T_r generators of $\mathfrak{g}$ satisfying $[T_r, T_s] = f_{rs}{}^tT_t$ and $\mathrm{tr}(T_rT_s) = \kappa_{rs}$ with κ_{rs} non-degenerate but not necessarily positive(negative) definite. Plugging this solution for $A_\pm$ into the action (2.21) we obtain the NATD action

$$S_{\mathrm{NATD}} = \int d^2\xi\,\partial_+\nu^r\left(\frac{\kappa}{1-\nu\cdot f}\right)_{rs}\partial_-\nu^s\,. \tag{2.23}$$

Example. The simplest example is when G is abelian (in this case we are really dealing with abelian T-duality). Let us take it to be two-dimensional with $\mathfrak{g}$ generated by T_1, T_2. Since the structure constants vanish the duality becomes trivial in this case, taking $g = e^{x^rT_r}$ we have

$$S = \int d^2\xi\,\partial_+x^r\partial_-x^s\kappa_{rs}\,, \quad \rightarrow \quad S_{\mathrm{NATD}} = \int d^2\xi\,\partial_+\nu^r\partial_-\nu^s\kappa_{rs}\,. \tag{2.24}$$

The two models are clearly the same, at least locally. Note that global properties of the original model are lost under NATD. For example if we started from a torus where $x^{1,2}$ have finite range the duality process does not tell us what the range of $\nu^{1,2}$ is, it could be taken finite or infinite. The models could only be globally equivalent in the former case. We can make this example more interesting by noting that we can add a total derivative term to the original action before dualizing. Following the same steps are before one then finds

$$S = \int d^2\xi\,\partial_+x^r\partial_-x^s(\kappa_{rs} + \zeta\varepsilon_{rs})\,, \quad \rightarrow \quad S_{\mathrm{NATD}} = \int d^2\xi\,\partial_+\nu^r\left(\frac{\kappa}{1+\zeta\kappa^{-1}\varepsilon}\right)_{rs}\partial_-\nu^s\,. \tag{2.25}$$

We have introduced the parameter ζ multiplying the total derivative. Taking $\zeta \to 0$ gives back the previous case. Note that the term we have added is locally a total derivative, but is only a total derivative globally if the x^1 or x^2-direction is not compact. The NATD model simplifies to

$$S_{\mathrm{NATD}} = \frac{\det \kappa}{\det \kappa + \zeta^2} \int d^2\xi \, \partial_+ \nu^r \partial_- \nu^s (\kappa_{rs} - \zeta \varepsilon_{rs}) \,, \tag{2.26}$$

which, upon rescaling ν^r, is again locally equivalent to the original model. While this example is still too trivial to be really interesting it does suggest that including the possibility of adding total derivatives to the action before dualizing could lead to interesting effects. We will come back to this after considering more general NATD on a subgroup.

2.3 NATD with respect to a subgroup

We started with a model with a global symmetry group $G \times G$ and ended up with a dual model whose symmetry group has only one factor of G.[4] In fact NATD typically breaks some of the global symmetry. What we considered so far is the simplest example of NATD, but a NATD model can be constructed for any subgroup of the isometry group $G \times G$. The example we considered above corresponds to dualizing with respect to the left copy of $G \subset G \times G$. Let us now describe how to dualize with respect to a subgroup K of the left copy of G, $K \subset G \subset G \times G$.[5]

We start by writing the group element as $g = kf$ with $k \in K$ and $f \in G$ so that

$$g^{-1}dg = \mathrm{Ad}_f(k^{-1}dk) + f^{-1}df \,. \tag{2.27}$$

To avoid a redundant description we take k to depend on the coordinates we want to dualize x and f to depend on the remaining 'spectator' coordinates y. Setting $j = dff^{-1}$ the first order action (2.21) becomes

$$S_{\mathrm{F}} = \int d^2\xi \; \mathrm{tr}((A_+ + j_+)(A_- + j_-) + \nu F_{+-}(A)) \,, \qquad A_\pm \in \mathfrak{k} \,. \tag{2.28}$$

As before, integrating out ν implies that A is pure gauge, i.e. $A = k^{-1}dk$ for some $k \in K$, recovering the original PCM action. Since $A \in \mathfrak{k}$ it follows that ν should belong to its dual (with respect to the metric on $\mathfrak{g}$), denoted $\mathfrak{k}^*$. If $\mathfrak{k}$ is generated by $\{T_{\underline{r}}\}$ then $\mathfrak{k}^*$ is generated by $\{T^{\underline{r}}\}$ with $\mathrm{tr}(T^{\underline{r}} T_{\underline{s}}) = \delta^{\underline{r}}_{\underline{s}}$. The equations of motion for $A_\pm$ imply that

$$A_+ = (\mathcal{O}^{-1})^{\mathrm{T}}(\partial_+ \nu - P^{\mathrm{T}} j_+) \,, \qquad A_- = \mathcal{O}^{-1}(-\partial_- \nu - P^{\mathrm{T}} j_-) \,, \tag{2.29}$$

where

$$\mathcal{O} = P^{\mathrm{T}}(1 - \mathrm{ad}_\nu)P \,. \tag{2.30}$$

Here P is a projection operator from $\mathfrak{g}$ down to $\mathfrak{k}$. From the definition of the transpose, $\mathrm{tr}(XP(Y)) = \mathrm{tr}(P^{\mathrm{T}}(X)Y)$, it follows that P^{T} projects on the dual Lie algebra $\mathfrak{k}^*$. Note

[4] It acts as $\nu \to \mathrm{Ad}_g(\nu) = g^{-1}\nu g$ with g a constant element of the group G.

[5] More generally one could consider a subgroup of the full $G \times G$, but this is more conveniently described using a $(G \times G)/G$ coset sigma model.

also that the definition of the adjoint action, $\mathrm{ad}_\nu X = [\nu, X]$, implies that $\mathrm{ad}_\nu^{\mathrm{T}} = -\mathrm{ad}_\nu$. The inverse of $\mathcal{O}$ refers to the subspace where it is defined, in particular we have $\mathcal{O}^{-1}\mathcal{O} = P$ and $\mathcal{O}\mathcal{O}^{-1} = P^T$. This (partial) inverse must exist for the dual model to be well-defined. Substituting the solution for $A_\pm$ into the Lagrangian gives the NATD model

$$S_{\mathrm{NATD}} = \int d^2\xi \ \mathrm{tr}\left(j_+ j_- + (\partial_+\nu - j_+)\mathcal{O}^{-1}(\partial_-\nu + j_-)\right) . \tag{2.31}$$

For $K = G$ the j's are absent and this reduces to our previous result (2.23).

Now we are ready to generalize this by including total derivative terms before dualization. It might seem that this cannot lead to anything interesting but as we will see, due to the fact that NATD is a *non-local* field redefinition, this can lead to dual models which are different even locally.

2.4 YB deformations from NATD

When we considered the example of abelian T-duality in the previous section we noted that we could include a total derivative term in the action before dualizing. We will now analyze this possibility more carefully and see that it leads to a surprising connection to Yang-Baxter models. We therefore generalize our starting point to

$$S = \int d^2\xi \ \left(\mathrm{tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) - \zeta\omega(g^{-1}\partial_+ g, g^{-1}\partial_- g)\right) , \tag{2.32}$$

where ζ is a free parameter and $\omega : \mathfrak{g} \oplus \mathfrak{g} \to \mathbb{R}$ is anti-symmetric and linear in the two arguments. In order not to change the local physics of the original model we require the extra term to be a total derivative (locally). Thinking of this term as a two-form the condition is that it should be closed, i.e.

$$0 = d\omega(g^{-1}dg,\, g^{-1}dg) = -2\omega(g^{-1}dg \wedge g^{-1}dg,\, g^{-1}dg) , \tag{2.33}$$

which requires that

$$\omega([X,Y],Z) + \omega([Y,Z],X) + \omega([Z,X],Y) = 0 \qquad \forall X,Y,Z \in \mathfrak{g} . \tag{2.34}$$

This is in fact the definition of a Lie algebra 2-cocycle. We therefore learn that it is possible to modify our starting point in this way whenever such a 2-cocycle exists. We may write $\omega(T_r, T_s) = \omega_{rs}$ and in terms of these components the 2-cocycle condition reads

$$\omega_{r[s} f_{tu]}{}^r = 0 . \tag{2.35}$$

Suppose we have a subgroup $K \subset G$ which admits a 2-cocycle ω. We can then add the term

$$-\zeta \int \omega(k^{-1}dk, k^{-1}dk) , \tag{2.36}$$

to the PCM action, or, equivalently, the term

$$-\zeta \int \omega(A, A) = -\zeta \int \omega_{rs} A^r \wedge A^s , \tag{2.37}$$

to the first order action (2.28). We can now perform the NATD on K following the same steps as in section 2.3. The result is the "Deformed T-Dual" (DTD) action

$$S_{\mathrm{DTD}} = \int d^2\xi \ \mathrm{tr}\left(j_+ j_- + (\partial_+ \nu - j_+)\mathcal{O}^{-1}(\partial_- \nu + j_-)\right) , \tag{2.38}$$

where now[6]

$$\mathcal{O} = P^{\mathrm{T}}(1 - \mathrm{ad}_\nu + \zeta\omega)P \,. \tag{2.39}$$

Setting the parameter ζ to zero we recover the NATD action (2.31). We have therefore constructed a deformation of the NATD action controlled by the deformation parameter ζ. However, it may happen that this is not an actual deformation but just a rewriting of the NATD action. This happens if it is possible to absorb the $\zeta\omega$ piece into ad_ν by a shift in ν. This in turn is possible precisely if ω takes the form $\omega_{rs} = X_t f_{rs}{}^t$ for some X_t, i.e. if the 2-cocycle ω is *exact*. Therefore the non-trivial deformations are characterized by 2-cocycles modulo exact ones, i.e. by elements of the second Lie algebra cohomology group $H^2(\mathfrak{k})$. A standard result in Lie algebra theory is that elements of $H^2(\mathfrak{k})$ classify non-trivial central extensions of $\mathfrak{k}$. Indeed, an equivalent way of arriving at (2.38) is to perform NATD on a central extension of $\mathfrak{k}$, characterized by ω, as originally suggested in [4]. Let us also note that since we started from the PCM, which is integrable, the model in (2.38) will also be integrable. This follows from the fact that adding the cocycle (closed B-field) term does not affect the local physics of the model, while the NATD transformation is a canonical transformation [5] and so must preserve the property of integrability.

So far this DTD model looks nothing like the YB deformation in (2.8). However, if ω is *non-degenerate* they are essentially the same, as we will now show. A Lie algebra admitting a non-degenerate 2-cocycle is called *quasi-Frobenius* (or symplectic). Therefore we will now consider the case where $\mathfrak{g}$ has a quasi-Frobenius subalgebra $\mathfrak{k} \subset \mathfrak{g}$ generated by $\{T_{\underline{r}}\}$ with 2-cocycle ω. In this case $\omega_{\underline{rs}}$ is invertible as a matrix and we set $R = \omega^{-1}$. By multiplying the free indices of the 2-cocycle condition (2.35) by R's we find that R satisfies the equation

$$R^{[\underline{r}|\underline{u}|} R^{\underline{s}|\underline{v}|} f_{\underline{uv}}{}^{\underline{t}]} = 0 \,. \tag{2.40}$$

This is nothing but the component form of the classical Yang-Baxter equation (2.17) with zero RHS ($c = 0$). Furthermore, R trivially extends from the subalgebra $\mathfrak{k}$ to all of $\mathfrak{g}$ by taking its remaining components to vanish. Therefore we have learned that non-degenerate 2-cocycles on a Lie algebra are in one-to-one correspondence with R-matrices solving the (non-modified) classical Yang-Baxter equation. This observation suggests a possible connection between the DTD model (2.38) and the YB deformation of the PCM (2.8). However, it is not straightforward to identify them as the former involves fields $f \in G$ (the spectators) and $\nu \in \mathfrak{k}^*$ while the latter involves only $g \in G$. To relate them we need to somehow re-express the Lie algebra valued field ν in terms of a group valued field h and then identify g with hf. It is not too hard to find a map that does the job working to the first few orders in ν. With a bit of work one finds that the correct all order map

[6] Note that we are thinking here of ω as a map from the Lie algebra $\mathfrak{k}$ to its dual $\mathfrak{k}^*$ by setting $\mathrm{tr}(Y\omega(X)) = \omega(X,Y)\ \forall X, Y \in \mathfrak{k}$.

is essentially given by the derivative of the exponential map, namely

$$\begin{aligned}\nu =& -\zeta P^{\mathrm{T}}\frac{1-\mathrm{Ad}_h^{-1}}{\log \mathrm{Ad}_h}\omega(\log h) = -\zeta P^{\mathrm{T}}\frac{1-e^{-\operatorname{ad}_X}}{\operatorname{ad}_X}\omega(X) \\ =& -\zeta P^{\mathrm{T}}\left(\omega(X)-\tfrac{1}{2}[X,\omega(X)]+\tfrac{1}{6}[X,[X,\omega(X)]]+\ldots\right) ,\end{aligned} \tag{2.41}$$

where we have written $h=e^X$ and expanded in powers of $X\in\mathfrak{k}$.

We will now show that performing this field redefinition in the DTD action (2.38) indeed leads to the YB action (2.8). The first step is to understand how $\partial_\pm\nu$ and ad_ν transform. To do this we note that the 2-cocycle condition on $\omega:\mathfrak{k}\to\mathfrak{k}^*$ implies that[7]

$$\omega([X,Y]) = P^{\mathrm{T}}[\omega(X),Y]+P^{\mathrm{T}}[X,\omega(Y)]\,, \qquad X,Y\in\mathfrak{k}\,. \tag{2.42}$$

Up to the projection by P^{T} this says that ω acts as a derivation with respect to the Lie bracket. If we formally extend the action of ω to act as a derivation also on the universal enveloping algebra, not just the Lie algebra, we may write (2.41) more simply as[8]

$$\nu = -\zeta P^{\mathrm{T}}(e^{-X}\omega(e^X)) = -\zeta P^{\mathrm{T}}(h^{-1}\omega(h))\,, \tag{2.43}$$

which is not hard to verify for the first few terms in the expansion of the exponential using the cocycle condition on ω. Using this expression we find

$$\begin{aligned}d\nu =& \zeta P^{\mathrm{T}}(h^{-1}dhh^{-1}\omega(h)) - \zeta P^{\mathrm{T}}(h^{-1}\omega(dh)) = -\zeta\omega(h^{-1}dh)+\zeta P^{\mathrm{T}}[h^{-1}dh,h^{-1}\omega(h)] \\ =& (P^{\mathrm{T}}-\mathcal{O})(h^{-1}dh)\,.\end{aligned} \tag{2.44}$$

Furthermore we have for $Y\in\mathfrak{k}$

$$P^{\mathrm{T}}\operatorname{ad}_\nu Y = -\zeta P^{\mathrm{T}}[P^{\mathrm{T}}(h^{-1}\omega(h)),Y] = -\zeta P^{\mathrm{T}}[h^{-1}\omega(h),Y] = -\zeta P^{\mathrm{T}}(\omega_h-\omega)Y\,, \tag{2.45}$$

where $\omega_h=\mathrm{Ad}_h\,\omega\,\mathrm{Ad}_h^{-1}$. Using this we can write

$$\mathcal{O}^{-1} = [P^{\mathrm{T}}(1+\eta R_h)\zeta\omega_h]^{-1} = \eta R_h[P^{\mathrm{T}}(1+\eta R_h)]^{-1} = \eta R_h(1+\eta R_h)^{-1} = 1-(1+\eta R_h)^{-1}\,, \tag{2.46}$$

where $\eta=\zeta^{-1}$, $R_h:\mathfrak{k}^*\to\mathfrak{k}$ is the inverse of ω_h and we used the fact that $R_h(1-P^{\mathrm{T}})=0$. Using these facts we find that the DTD Lagrangian in (2.38) becomes

$$\begin{aligned}&\operatorname{tr}\big((h^{-1}\partial_+h+j_+)(1+\eta R_h)^{-1}(h^{-1}\partial_-h+j_-)-\zeta h^{-1}\partial_+h\omega_h(h^{-1}\partial_-h)\big) \\ &= \operatorname{tr}\left(\partial_+gg^{-1}\frac{1}{1+\eta R}\partial_-gg^{-1}\right)+\eta^{-1}\omega(\partial_+hh^{-1},\partial_-hh^{-1})\,,\end{aligned} \tag{2.47}$$

[7]This follows from the fact that for any $Z\in\mathfrak{k}$ we have

$$\begin{aligned}\operatorname{tr}(\omega([X,Y])Z) =& \omega([X,Y],Z) = -\omega([Y,Z],X)-\omega([Z,X],Y) = -\operatorname{tr}(\omega([Y,Z])X)-\operatorname{tr}(\omega([Z,X])Y) \\ =& \operatorname{tr}([Y,Z]\omega(X))+\operatorname{tr}([Z,X]\omega(Y)) = -\operatorname{tr}(Z[Y,\omega(X)])+\operatorname{tr}(Z[X,\omega(Y)])\,.\end{aligned}$$

[8]This works in spite of the extra P^{T} on the RHS of (2.42) because of the fact that $P^{\mathrm{T}}[PX,(1-P^{\mathrm{T}})Y]=0$ for all $X,Y\in\mathfrak{g}$. This in turn follows by noting that for any $Z\in\mathfrak{g}$ we have

$$\operatorname{tr}(ZP^{\mathrm{T}}[PX,(1-P^{\mathrm{T}})Y]) = \operatorname{tr}([PZ,PX](1-P^{\mathrm{T}})Y) = \operatorname{tr}((1-P)[PZ,PX]Y)=0\,,$$

since $P(\mathfrak{g})=\mathfrak{k}$ is a subalgebra.

where $g = hf$. The first term on the RHS is precisely the Lagrangian of the YB model (2.8). The second term represents a closed B-field. Since this term is locally a total derivative it does not affect the local physics and can be dropped for most purposes. Therefore we have shown that

$$S_{\rm PCM} - \zeta \int \omega(k^{-1}dk, k^{-1}dk) \qquad \xrightarrow{\text{NATD on } K} \qquad S_{\rm YB} + \eta^{-1} \int \omega(dhh^{-1}, dhh^{-1})\,, \tag{2.48}$$

with the parameters related by $\zeta = \eta^{-1}$. Since the local physics is independent of the extra ω terms we have shown that *homogeneous* YB deformations, i.e. those with R matrix solving the classical YB equation (2.17) with $c = 0$, can be constructed as a NATD of the PCM. This observation has some important consequences, in particular

- Integrability of the YB deformation is now obvious since NATD preserves integrability.
- Since NATD can be performed for general 2d sigma models with isometries we can use it to give a more general definition of homogeneous YB deformations.
- We can classify the possible deformations by classifying quasi-Frobenius subalgebras of the isometry algebra.

Regarding the last point of classification there are several useful mathematical results. In particular a standard result is that a quasi-Frobenius subalgebra of a compact Lie algebra must be abelian. In this case, when the subalgebra that is dualized is abelian, one can show that the YB deformation coincides with another deformation that has been studied a lot in the string theory literature and that goes by the name of T-duality–shift–T-duality (TsT) [6]. Therefore another way of thinking about (homogeneous) YB deformations is as a non-abelian generalization of TsT transformations.

Chapter 3

String sigma model and Weyl invariance conditions

So far we have dealt with the PCM because it offers a simple setting to introduce the ideas of NATD and YB deformations. But as we have emphasized this is only a toy model. The PCM does not represent a consistent string sigma model since it fails to be conformal invariant at the quantum level. In string theory we start from a general model of 2d scalars coupled to 2d gravity[1] of the form (for simplicity we will consider only the bosonic string here)

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{-g}\,\partial_i x^m\partial_j x^n(g^{ij}G_{mn} + \varepsilon^{ij}B_{mn})\,, \tag{3.1}$$

where $G_{mn} = G_{nm}$ and $B_{mn} = -B_{nm}$ are functions of x^m while $g_{ij}(\xi)$ is the 2d *worldsheet* metric. The factor in front of the action $T = \frac{1}{4\pi\alpha'}$ is the string tension. This action is manifestly invariant under 2d diffeomorphisms. This gauge invariance can be used to eliminate two of the three components of the worldsheet metric g_{ij}. A very important fact about the above action is that it has another gauge symmetry which can eliminate also the last component of g_{ij}, so that the worldsheet metric is not dynamical. This is the invariance under Weyl transformations

$$g_{ij} \to e^{\lambda}g_{ij}\,, \tag{3.2}$$

with λ an arbitrary function of the worldsheet coordinates ξ^i. A cosmological constant term in the action would break this symmetry, which is why we did not add such a term in (3.1). The crucial point is that for the action (3.1) to make sense also at the quantum level we must make sure that the quantum theory remains invariant under the Weyl transformation (3.2). Using the diffeomorphisms to gauge fix $g_{ij} = e^{\sigma}\eta_{ij}$, with η_{ij} the 2d Minkowski metric, this turns into conformal symmetry of the 2d theory. In particular this implies that the theory looks the same on all length (or energy) scales. This requirement is very restrictive. Taylor expanding the functions G_{mn} and B_{mn} we have a Lagrangian with infinitely many couplings, corresponding to the coefficients in the Taylor expansion. But in a quantum field theory all couplings will typically depend on the energy scale at which we probe the theory. This running of the couplings is captured by their so-called

[1]The usual Einstein-Hilbert term is a total derivative in two dimensions so we don't include it.

β-functions. But, since energy scale is inversely related to distance scale, having couplings that depend on the energy scale is not consistent with conformal symmetry. Therefore in a conformal theory all the β-functions must vanish, which in our case leads to infinitely many constraints — one for each coefficient in the Taylor expansion of G_{mn} and B_{mn}. These conditions can be summarized as differential equations that G_{mn} and B_{mn} must satisfy. In fact, this turns out to not quite be enough. One must also modify the original action by a quantum *counter term* involving a new (scalar) function of x — the dilaton Φ. The quantum corrected action takes the form

$$S = -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{-g} \left(\partial_i x^m \partial_j x^n (g^{ij} G_{mn} + \varepsilon^{ij} B_{mn}) + \alpha' R^{(2)} \Phi \right) , \tag{3.3}$$

where $R^{(2)}$ is the Ricci scalar of the worldsheet metric g_{ij}. Note that the term we added is not Weyl invariant (unless Φ is a constant). It is precisely its non-zero Weyl transformation that compensates the non-invariance of the first term at the quantum level. A reflection of this is that the dilaton term comes with an extra factor of α', the quantum mechanical expansion parameter in this model. A quantum mechanical calculation, e.g. [7], shows that this model preserves Weyl invariance at the quantum level provided that the following β-function equations are satisfied

$$0 = R_{mn} + 2\nabla_m \nabla_n \Phi - \frac{1}{4} H_{mop} H_n{}^{op} + \mathcal{O}(\alpha') , \tag{3.4}$$

$$0 = \nabla^m H_{mno} - 2\nabla^m \Phi H_{mno} + \mathcal{O}(\alpha') , \tag{3.5}$$

$$0 = \nabla^2 \Phi - 2\nabla_m \Phi \nabla^m \Phi + \frac{1}{12} H_{mno} H^{mno} + \mathcal{O}(\alpha') , \tag{3.6}$$

where H is the field strength of B, i.e. $H_{mno} = 3\partial_{[m} B_{no]}$ and R_{mn} is the Ricci tensor constructed from the metric G_{mn}. In addition one finds that the dimension of spacetime must take the precise value $D = 26$, referred to as the critical dimension of the bosonic string. Remarkably, we recognize the first equation to be Einstein's equation (with particular matter sources given by Φ and B_{mn}). Therefore string theory contains Einstein's general relativity in a quantum mechanically consistent framework. The presence of higher quantum corrections, the $\mathcal{O}(\alpha')$ terms, shows that string theory gives Einstein's theory plus corrections. These α'-corrections will be important for us later. These β-function equations provide the dynamical equations of motion for the background fields G_{mn}, B_{mn} and Φ. They can be summarized in an action for the spacetime fields which takes the form

$$S = \frac{1}{2\kappa_0^2} \int d^{26}x \sqrt{-G} e^{-2\Phi} \left(R - \frac{1}{12} H_{mno} H^{mno} + 4\partial_m \Phi \partial^m \Phi + \mathcal{O}(\alpha') \right) . \tag{3.7}$$

This is the *low-energy effective action* for the bosonic string. In fact, the more realistic supersymmetric string theories have the same terms in their low-energy effective actions (although in that case the spacetime dimension is $D = 10$), plus additional terms involving other fields not present for the bosonic string. In addition to the α'-corrections there are also string loop corrections organized in powers of e^{Φ}. We will not have anything to say about those here.

3.1 Homogeneous Yang-Baxter deformations

We have seen in the previous chapter how, in the case of the PCM, homogeneous YB deformations can be constructed using non-abelian T-duality. But NATD can be carried out for a general 2d sigma model with isometries and therefore this construction is not limited to the PCM. At the classical level NATD can be shown to act as a canonical transformation, which means that it gives rise to a classically equivalent model. Adding the closed B-field term represented by the 2-cocycle on the subalgebra of the isometry algebra we are dualizing we obtain a YB deformed model. Applying this procedure to the classical bosonic string action (3.1) we obtain a new, classically equivalent model, which is a deformation of the one we started with. Let us call the new metric and B-field $\tilde{G}$ and $\tilde{B}$. With a little bit of work one finds that they are related to the original ones by the equation

$$\tilde{G} + \tilde{B} = (G + B)(1 + \eta\Theta(G + B))^{-1}\,, \tag{3.8}$$

where we are suppressing the indices m, n and we have introduced the object

$$\Theta^{mn} = k_r^m k_s^n R^{rs}\,, \tag{3.9}$$

where k_r^m are the components of the Killing vectors corresponding to the isometries dualized. As before R^{rs} is the R-matrix satisfying the CYBE, (2.17) with $c = 0$, and η is the deformation parameter. Let us show that this reproduces what we found in the case of the PCM. Writing the principal chiral model action in terms of G and B we read off

$$G_{mn} = \mathrm{tr}(g^{-1}\partial_m g g^{-1}\partial_n g)\,, \qquad B_{mn} = 0\,. \tag{3.10}$$

The isometries we are dualizing are of the form $g \to hg$ for a constant $h \in K \subset G$. The corresponding Killing vectors are read off from

$$g^{-1}\delta g = g^{-1}\epsilon g = \mathrm{Ad}_g\,\epsilon = \delta x^m g^{-1}\partial_m g\,, \qquad \delta x^m = \epsilon^r k_r^m\,, \tag{3.11}$$

and we find that

$$k_r{}^m(\partial_m g g^{-1})^s = \delta_r^s\,. \tag{3.12}$$

Using this fact (3.8) becomes

$$\begin{aligned}(\tilde{G} + \tilde{B})_{mn} &= \mathrm{tr}(g^{-1}\partial_m g g^{-1}\partial_n g) - \eta\,\mathrm{tr}(g^{-1}\partial_m g g^{-1}\partial_o g)\Theta^{op}\,\mathrm{tr}(g^{-1}\partial_p g g^{-1}\partial_n g) + \ldots \\ &= \mathrm{tr}(g^{-1}\partial_m g g^{-1}\partial_n g) - \eta\,\mathrm{tr}(\partial_m g g^{-1} R \partial_n g g^{-1}) + \ldots \\ &= \mathrm{tr}(\partial_m g g^{-1}\frac{1}{1 + \eta R}\partial_n g g^{-1})\,,\end{aligned} \tag{3.13}$$

in agreement with (2.8).

So far we considered only what happens at the level of the classical sigma model. In order to define the YB deformation in string theory we have to understand it at the quantum level. It is important to note first of all that, while at the classical level NATD is a symmetry of the theory, this can no longer be the case at the quantum level. At best NATD can be a map between in-equivalent string CFTs [8]. If this is the case NATD or YB deformations constructed from it would map solutions of the β-function equations (3.4)–(3.6) to new solutions. In particular we must be able to find also a formula for the dilaton $\tilde{\Phi}$ such that the fields of the deformed background solve the same equations. As a first step we will neglect the $O(\alpha')$ terms, which corresponds to working to one loop order in sigma model perturbation theory. The most straightforward approach would be to simply plug our expressions for the deformed metric and B-field in (3.8) into (3.4)–(3.6) and try to find a $\tilde{\Phi}$ such that they are satisfied. However, this is very difficult to do in practice due to the non-linearity of the map in (3.8). Just trying to compute the Riemann tensor from $\tilde{G}$ is quite a bit of work. Of course one can work out the first few orders in the η-expansion, but this is not very satisfactory. It turns out that a much more efficient way to address these questions is to use a reformulation of the string low-energy effective action which linearizes the map (3.8).

3.2 $O(D, D)$ invariant formulation of string effective action

The YB map (3.8) is complicated and non-linear which makes it hard to work with for the purposes we are interested in. Let us see if we can use different variables to make the map simpler. To simplify things we will start by assuming that $B = 0$. In the following we will also absorb the deformation parameter η into the definition of Θ. Separating (3.8) into the symmetric and anti-symmetric part we then have

$$\tilde{G} = G(1-\Theta^2 G)^{-1}\,, \qquad \tilde{B} = -G\Theta G(1-\Theta^2 G)^{-1}\,, \tag{3.14}$$

where $\Theta^2 = \Theta G \Theta$. From these expressions we see that

$$\tilde{G}^{-1} = G^{-1} - \Theta^2\,, \qquad \tilde{G} - \tilde{B}\tilde{G}^{-1}\tilde{B} = G\,, \qquad \tilde{B}\tilde{G}^{-1} = -G\Theta\,. \tag{3.15}$$

These expressions look much more promising. Allowing for non-zero B they become

$$\begin{aligned}&\tilde{G}^{-1} = G^{-1} + G^{-1}B\Theta + \Theta B G^{-1} - \Theta(G - BG^{-1}B)\Theta\,, \qquad \tilde{G} - \tilde{B}\tilde{G}^{-1}\tilde{B} = G - BG^{-1}B\,,\\ &\tilde{B}\tilde{G}^{-1} = BG^{-1} - (G - BG^{-1}B)\Theta\,.\end{aligned} \tag{3.16}$$

From this we conclude that if we could use instead of G and B two of the variables G^{-1}, $G - BG^{-1}B$ and BG^{-1} the map (3.8) would become linear. We can group these variables into a symmetric $2D \times 2D$-matrix

$$\mathcal{H}^{MN} = \begin{pmatrix} (G - BG^{-1}B)_{mn} & (BG^{-1})_m{}^n \\ -(G^{-1}B)^m{}_n & (G^{-1})^{mn} \end{pmatrix}, \tag{3.17}$$

in terms of which the YB deformation (3.8) becomes simply

$$\tilde{\mathcal{H}}^{MN} = O_P{}^M O_Q{}^N \mathcal{H}^{PQ}\,, \qquad O_M{}^N = \begin{pmatrix} \delta^m_n & \Theta^{mn} \\ 0 & \delta^n_m \end{pmatrix}. \tag{3.18}$$

In fact the matrix $O_M{}^N$ belongs to the group $O(D, D)$ since it preserves the split signature metric

$$\eta_{MN} = \begin{pmatrix} 0 & \delta^m_n \\ \delta^n_m & 0 \end{pmatrix}, \qquad O_M{}^P O_N{}^Q \eta_{PQ} = \eta_{MN}\,. \tag{3.19}$$

In fact it is easy to see that $\mathcal{H}^{MN}$ (or rather $\mathcal{H}^{M}{}_{N} = \mathcal{H}^{MP}\eta_{PN}$) itself belongs to $O(D,D)$! Therefore there is a natural action of $O(D,D)$ when we group the metric and B-field into the "generalized metric" $\mathcal{H}$ given by

$$\mathcal{H} \to O^T\mathcal{H}O \qquad O \in O(D,D)\,. \tag{3.20}$$

We see that if we could reformulate the low-energy effective action (3.7) for the string in terms of $\mathcal{H}$, rather that G and B, it would be much simpler to check that the equations of motion are preserved by the YB map (3.18). Remarkably it turns out that this is indeed possible.

To explain this we must go back to the notion of (abelian) T-duality. When string theory is compactified on an n-dimensional torus, T^n, T-dualities on the torus directions lead to an $O(n,n;\mathbb{Z})$ symmetry group (e.g. [7]). If we restrict our attention to the massless fields, as in the low-energy effective action, this becomes and $O(n,n;\mathbb{R})$ symmetry [9]. This symmetry is there in tree-level string theory (i.e. ignoring string loop corrections) but to all orders in the inverse string tension α' [10]. It has been suggested that if one could formulate a theory in D dimensions (recall that $D=26$ for the bosonic string) with $O(D,D)$ symmetry its compactification on T^n would automatically have the T-duality symmetry. In this way the T-duality symmetry of string theory would be made manifest. This idea goes by the name of Double Field Theory (DFT) [11, 12, 13]. The reason for the name is that to have manifest $O(D,D)$ symmetry one must double the number of spacetime dimensions, replacing x^m by an $O(D,D)$ vector $X^M = (\tilde{x}_m,\, x^m)$. However, the physical spacetime is still supposed to be D-dimensional and this is ensured by imposing the $O(D,D)$ invariant section condition (a.k.a. strong constraint)

$$\partial_M Y \partial^M Z = 0\,, \qquad \partial_M \partial^M Z = 0\,, \tag{3.21}$$

for all Y, Z built out of fields of the theory. Note that doubled indices $M, N, \ldots$ are raised and lowered with the $O(D,D)$ metric η (3.19). The standard solution to this constraint is to simply set $\tilde{x}_m = 0$ so that effectively $\partial_M = (0,\, \partial_m)$. In DFT the metric and B-field are combined precisely as we argued above into the generalized metric $\mathcal{H}^{MN}$ (3.17). This is quite natural since as we saw $O(D,D)$ acts in a natural way on this object. In the low energy description of string theory there is also the dilaton and in DFT it appears combined with the determinant of the metric as the generalized dilaton d (c.f. (3.7))

$$e^{-2d} = \sqrt{-G}e^{-2\Phi}\,. \tag{3.22}$$

The string low energy effective Lagrangian (3.7), at lowest order in α', can be written in terms of these fields as

$$L = e^{-2d}\mathcal{R}\,, \tag{3.23}$$

where the generalized Ricci scalar is defined as

$$\begin{aligned}\mathcal{R} =& 4\partial_M(\mathcal{H}^{MN}\partial_N d) - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d + \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}_{KL}\partial_N\mathcal{H}^{KL} \\ &- \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{LN}\,.\end{aligned} \tag{3.24}$$

Imposing the standard solution to the section condition $\partial_M = (0,\, \partial_m)$ this reduces to

$$L = \sqrt{-G}e^{-2\Phi}\left(R - \frac{1}{12}H_{mno}H^{mno} + 4\partial_m\Phi\partial^m\Phi\right) + 4\partial_m\left(\sqrt{-G}e^{-2\Phi}\partial^m\Phi\right)\,, \tag{3.25}$$

which, upon dropping the total derivative term, reproduces precisely the Lagrangian in (3.7). The fact that the low energy dynamics of the bosonic string can be cast in the form of (3.23) is remarkable – the global $O(D, D)$ symmetry is completely unexpected. This symmetry acts as

$$X^M \to X'^M = X^N O_N{}^M\,, \qquad \mathcal{H}^{MN}(X) \to O_P{}^M O_Q{}^N \mathcal{H}^{PQ}(X')\,, \qquad d(X) \to d(X')\,, \tag{3.26}$$

for $O_M{}^N$ a *constant* element of $O(D, D)$. Of course this is in the doubled space, before solving the section condition. But even after solving the section condition there is still an extra global symmetry that remains, namely the transformations above which preserve the choice $X^M = (0,\, x^m)$. These transformations are of the form

$$\begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix}\,, \tag{3.27}$$

i.e. of the same form as we found for YB in (3.18), but now with Θ an arbitrary *constant* anti-symmetric matrix. Note that this implies a very non-obvious symmetry of the low-energy effective action (3.7), namely that it is invariant under the YB map (3.8) (plus transformation of the dilaton read off from the fact that $d = \Phi - \frac{1}{2}\log\sqrt{-G}$ is invariant) but with Θ any constant anti-symmetric matrix.

This formulation has already taught us something interesting, but to understand how YB deformations preserve the β-function equations we must do a little more work. The reason is that for YB deformations Θ is not constant, so they are not simply a global $O(D, D)$ transformation. In fact Θ is constructed out of Killing vectors of the background in question and so depends on the background we are deforming. For this reason it cannot be a symmetry of the action. Instead it should map solutions of the equations of motion (β-function equations) into new solutions. To see how this happens, and also to address what happens when α'-corrections are included, it turns out to be convenient to work with a generalized vielbein rather than a generalized metric. We will now introduce this "frame-like" formulation of DFT.

3.3 Frame-like formulation of DFT

We are familiar with the fact that we can formulate ordinary Riemannian geometry either in terms of the metric or in terms of the vielbein. The fact that the vielbein contains more degrees of freedom than the metric is compensated by an extra gauge invariance, invariance under local Lorentz transformations with gauge group $O(D-1, 1)$. In analogy with this we introduce a generalized vielbein for the generalized metric

$$\mathcal{H}^{MN} = E_A{}^M E_B{}^N \mathcal{H}^{AB}\,. \tag{3.28}$$

The "flat" metric $\mathcal{H}^{AB}$ is the analog of the Minkowski metric in the ordinary Riemannian case and since we are just doubling the dimension we take it to have the form

$$\mathcal{H}^{AB} = \begin{pmatrix} \bar{\eta}_{ab} & 0 \\ 0 & \bar{\eta}^{ab} \end{pmatrix}\,, \tag{3.29}$$

where $\bar{\eta} = \mathrm{diag}(-1, 1, \ldots, 1)$ is the D-dimensional Minkowski metric. But we must remember that we also have another metric, namely the constant $O(D, D)$ metric η^{MN} (η_{MN}) in (3.19). We will demand that

$$\eta_{AB} = E_A{}^M E_B{}^N \eta_{MN} = \begin{pmatrix} \bar{\eta}^{ab} & 0 \\ 0 & -\bar{\eta}_{ab} \end{pmatrix} . \qquad (3.30)$$

The point of this constraint is that the analog of the local Lorentz symmetry in the standard case is $E_A{}^M \to \Lambda_A{}^B E_B{}^M$ where $\Lambda_A{}^B$ must preserve both $\mathcal{H}^{AB}$ in (3.29) and η^{AB} in (3.30) which means that it reduces to two copies of the Lorentz group, i.e.

$$\Lambda_A{}^B = \begin{pmatrix} (\Lambda^{(+)})^a{}_b & 0 \\ 0 & (\Lambda^{(-)})_a{}^b \end{pmatrix} , \qquad (3.31)$$

with $\Lambda^{(\pm)}$ two independent D-dimensional Lorentz transformations. It is now easy to see that we may take the generalized vielbein to have the form

$$E_A{}^M = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{(+)a}{}_m - e^{(+)an} B_{nm} & e^{(+)am} \\ -e^{(-)}_{am} - e^{(-)n}_a B_{nm} & e^{(-)m}_a \end{pmatrix} . \qquad (3.32)$$

Here the two sets of vielbeins $e^{(\pm)}$, with $e^{(\pm)m}_a e^{(\pm)n}_b \eta^{ab} = G^{mn}$, transform under the two copies of the Lorentz group as $e^{(\pm)m}_a \to \Lambda^{(\pm)b}_a e^{(\pm)m}_b$. We can use the double Lorentz symmetry to fix the gauge $e^{(+)} = e^{(-)} = e$. With this gauge fixing only the diagonal of the double Lorentz group, $\Lambda^{(+)} = \Lambda^{(-)} = \Lambda$, survives and becomes the usual Lorentz group. This gauge fixing is needed whenever we want to make contact with the usual gravity description which has only a single copy of the Lorentz group.

In ordinary Riemannian geometry the vielbein transforms under both diffeomorphisms and local Lorentz transformations. But we are familiar with the fact that we can work with objects which transform as scalars under diffeomorphisms, but transform non-trivially under local Lorentz transformations, namely the spin connection components $\omega_c{}^{ab} = e_c{}^m \omega_m{}^{ab}$ and derivatives with 'flat' indices $\partial_a = e_a{}^m \partial_m$. In fact, the spin connection is the only diffeomorphism scalar that can be constructed from the derivative of the vielbein.[2] In the present case we have a similar situation. The generalized vielbein transforms under both generalized diffeomorphisms and double Lorentz transformations. The generalized diffeomorphisms act on a doubled vector field by the generalized Lie derivative

$$V^M \to \mathcal{L}_\epsilon V^M = \epsilon^N \partial_N V^M + (\partial^M \epsilon_N - \partial_N \epsilon^M) V^N . \qquad (3.33)$$

Looking at the transformation of the generalized vielbein one sees that (with the standard solution of the section condition) ϵ^m is identified with the diffeomorphism and ϵ_m with the B-field gauge parameters, $\delta B_{mn} = 2\partial_{[m}\epsilon_{n]}$. The analog of the spin connection in this case would be the objects constructed from the derivative of the generalized vielbein which are generalized diffeomorphism scalars. It is not hard to see that the only combination that is a generalized diffeomorphism scalar is

$$F_{ABC} = 3\partial_{[A} E_B{}^M E_{C]M} , \qquad (3.34)$$

[2]Since $e_m{}^a(x) \to e_m{}^a(x+\epsilon) - \partial_m \epsilon^n e_n{}^a$ under an infinitesimal diffeomorphism the requirement that $c_c{}^{ab} e_a{}^m e_b{}^n \partial_n e_m{}^c$ transform as a scalar becomes $c_c{}^{ab} e_a{}^m e_b{}^n \partial_n \partial_m \epsilon^k e_k{}^c = 0$ which requires $c_c{}^{ab} = c_c{}^{[ab]}$ leading to the spin connection.

where we have defined $\partial_A = E_A{}^M \partial_M$. Note that F_{ABC} is also manifestly invariant under $O(D,D)$, since it just rotates the M-indices by a constant $O(D,D)$ matrix. Of course we also have the generalized dilaton, which is not a scalar due to the $\sqrt{-G}$ in (3.22), but its derivative can be combined with the derivative of the generalized vielbein to produce a second generalized diffeomorphism scalar

$$F_A = \partial^B E_B{}^M E_{AM} + 2\partial_A d\,. \tag{3.35}$$

The objects F_{ABC} and F_A are usually referred to as generalized fluxes, and this formulation of DFT is referred to as the flux formulation [14]. The advantage of working with the generalized fluxes is that both diffeomorphisms and B-field gauge transformations as well as global $O(D,D)$ symmetry is now manifest. The price we pay for this is that the double Lorentz transformations are not manifest. Instead the generalized fluxes transform similarly to connections as

$$\delta F_{ABC} = 3\partial_{[A}\lambda_{BC]} + 3\lambda_{[A}{}^D F_{BC]D}\,, \qquad \delta F_A = \partial^B \lambda_{BA} + \lambda_A{}^B F_B\,, \tag{3.36}$$

under an infinitesimal transformation $\delta E_A{}^M = \lambda_A{}^B E_B{}^M$. In the usual Riemannian case we would deal with this by constructing the gauge covariant curvature of the spin connection, i.e. the Riemann tensor, and writing actions in terms of that. In that way all the symmetries can be made manifest. This is not possible in the present case. The reason is that there is no analog of the Riemann tensor. For example, looking at the transformations it seems that one should consider the field strength $4\partial_{[A}F_{BCD]}$, which is invariant to leading order in fields. However, from the definition of F_{ABC} in terms of the generalized vielbein we have the Bianchi identity

$$4\partial_{[A}F_{BCD]} = 3F_{[AB|}{}^E F_{CD]E}\,, \tag{3.37}$$

so this would-be field strength is not an independent field. Similarly we have

$$2\partial_{[A}F_{B]} = -(\partial^C - F^C)F_{ABC}\,, \qquad 2\partial_{[A}\partial_{B]} = F_{ABC}\partial^C\,. \tag{3.38}$$

The absence of a Riemann tensor makes it more difficult to construct an action (and especially higher derivative corrections [15]). However, we will see that an action can nevertheless be constructed without too much effort. Before we do this we need one more fact. The presence of the constant $O(D,D)$ metric η^{AB} in (3.30) as well as the doubled metric $\mathcal{H}^{AB}$ in (3.29), which both square to one, means that we can construct constant projection operators

$$(P_\pm)^{AB} = \frac{1}{2}(\eta \pm \mathcal{H})^{AB}\,. \tag{3.39}$$

This means that we have a canonical splitting of the index $A = (\overline{a}, \underline{a})$, where the D-dimensional index $\overline{a}(\underline{a})$ corresponds to $P_+(P_-)$ projection. This means that the two generalized fluxes really contain six fields

$$F_{\overline{abc}}\,, \quad F_{\underline{a}\overline{bc}}\,, \quad F_{\overline{a}\underline{bc}}\,, \quad F_{\underline{abc}}\,, \qquad F_{\overline{a}}\,, \quad F_{\underline{a}}\,. \tag{3.40}$$

Their double Lorentz transformations are

$$\begin{aligned}
\delta F_{\overline{abc}} &= 3\partial_{[\overline{a}}\lambda_{\overline{bc}]} + 3\lambda_{[\overline{a}}{}^{\overline{d}}F_{\overline{bc}]\overline{d}}\,, & \delta F_{\underline{abc}} &= 3\partial_{[\underline{a}}\lambda_{\underline{bc}]} + 3\lambda_{[\underline{a}}{}^{\underline{d}}F_{\underline{bc}]\underline{d}}\,, \\
\delta F_{\underline{a}\overline{bc}} &= \partial_{\underline{a}}\lambda_{\overline{bc}} + \lambda_{\underline{a}}{}^{\underline{d}}F_{\underline{d}\overline{bc}} + 2\lambda_{[\overline{b}}{}^{\overline{d}}F_{|\underline{a}\overline{d}|\overline{c}]}\,, & \delta F_{\overline{a}\underline{bc}} &= \partial_{\overline{a}}\lambda_{\underline{bc}} + \lambda_{\overline{a}}{}^{\overline{d}}F_{\overline{d}\underline{bc}} + 2\lambda_{[\underline{b}}{}^{\underline{d}}F_{|\overline{a}\underline{d}|\underline{c}]}\,, \\
\delta F_{\overline{a}} &= \partial^{\overline{b}}\lambda_{\overline{ba}} + \lambda_{\overline{a}}{}^{\overline{b}}F_{\overline{b}}\,, & \delta F_{\underline{a}} &= \partial^{\underline{b}}\lambda_{\underline{ba}} + \lambda_{\underline{a}}{}^{\underline{b}}F_{\underline{b}}\,,
\end{aligned} \tag{3.41}$$

where $\lambda_{\overline{ab}}(\lambda_{\underline{ab}})$ are the non-zero components $\lambda^{(+)}(\lambda^{(-)})$ of λ_{AB}. The only two-derivative action we can construct using the fields in (3.40) consists of their squares plus the total derivative terms $\partial^{\overline{a}}F_{\overline{a}}$ and $\partial^{\underline{a}}F_{\underline{a}}$. It is not hard to check that the only combination invariant under the double Lorentz transformations is[3]

$$\mathcal{R} = 4\partial^{\overline{a}}F_{\overline{a}} - 2F^{\overline{a}}F_{\overline{a}} + F^{\underline{a}\overline{bc}}F_{\underline{a}\overline{bc}} + \tfrac{1}{3}F^{\overline{abc}}F_{\overline{abc}}\,. \tag{3.42}$$

With a bit of work one can rewrite this in terms of the generalized metric and dilaton and show that it coincides with our previous expression for the generalized Ricci scalar (3.24). Therefore the Lagrangian

$$L = e^{-2d}\mathcal{R}\,, \tag{3.43}$$

reproduces again (after solving the section condition in the standard way and fixing the gauge $e^{(+)} = e^{(-)}$) the low energy effective Lagrangian of the bosonic string in (3.7). The equations of motion following from this action are easily derived using the variation of the generalized fluxes

$$\delta F_{ABC} = 3\partial_{[A}\delta E_{BC]} + 3\delta E_{[A}{}^{D}F_{BC]D} \qquad \delta F_A = \partial^B\delta E_{BA} + \delta E_A{}^B F_B + 2\partial_A\delta d\,, \tag{3.44}$$

where we have defined $\delta E_{AB} = \delta E_A{}^M E_{BM} = \delta E_{[AB]}$ and one finds

$$\mathcal{R} = 0 \qquad \text{and} \qquad \mathcal{R}_{\overline{a}\underline{b}} = 0\,, \tag{3.45}$$

where the generalized Ricci tensor is given by[4]

$$\mathcal{R}_{\overline{a}\underline{b}} = \partial_{\overline{a}}F_{\underline{b}} + (\partial^{\underline{c}} - F^{\underline{c}})F_{\overline{a}\underline{bc}} - F_{\underline{c}\overline{da}}F^{\overline{d}\underline{c}}{}_{\underline{b}}\,. \tag{3.46}$$

These equations are equivalent to the string beta function equations (3.4)–(3.6) to lowest order in α' when we plug in the form of the generalized vielbein (3.32) and gauge fix the double Lorentz transformations by setting $e^{(+)} = e^{(-)}$. However, this rewriting is much more convenient for discussing generalizations of T-duality and in particular Yang-Baxter deformations, as we will now see.

3.4 One-loop Weyl invariance and unimodularity condition

Recall that our motivation for introducing the $O(D,D)$ covariant formalism in the previous section was to make it easier to understand YB deformations. Indeed we saw that the generalized metric transforms as in (3.18) which means that the generalized vielbein transforms as

$$\tilde{E}_A{}^M = E_A{}^N O_N{}^M\,, \qquad O_N{}^M = \delta_N^M + \Theta_N{}^M = \begin{pmatrix} \delta^n_m & \Theta^{nm} \\ 0 & \delta^m_n \end{pmatrix}. \tag{3.47}$$

This is a very simple linear transformation compared to the original non-linear transformation (3.8). While $O \in O(D,D)$ it is, in general, not constant. In fact Θ is constructed

[3]It can also be written as minus the same thing with the projections reversed.

[4]We can also define $\mathcal{R}_{\underline{a}\overline{b}}$ by reversing the projections. Then $\mathcal{R}_{\underline{a}\overline{b}} = \mathcal{R}_{\overline{b}\underline{a}}$, due to the Bianchi identities.

in terms of Killing vectors of the background (3.9) and these are in general not constant.[5] Since the generalized fluxes are only guaranteed to be invariant under constant $O(D,D)$ transformations we need to check how they transform under this transformation. From the definition in (3.34) we find

$$\tilde{F}_{ABC} = 3\tilde{\partial}_{[A}\tilde{E}_B{}^M\tilde{E}_{C]M} = F_{ABC} + 3E_{[A}{}^N\Theta_{|N}{}^K\partial_{K|}E_B{}^M E_{C]M} + 3\tilde{\partial}_{[A}\Theta_{|N|}{}^M E_B{}^N\tilde{E}_{C]M}\,. \tag{3.48}$$

Let us look at the terms involving Θ on the RHS. First we note that since the only non-zero component of $\Theta_M{}^N$ is Θ^{mn}, the contraction (with the $O(D,D)$ metric) of two Θ's will vanish. Therefore we may replace $\tilde{E}_{CM}$ in the last term with E_{CM}. Using $\tilde{\partial}_A = \partial_A + E_A{}^M\Theta_M{}^N\partial_N$ are left with terms linear and quadratic in Θ. Let us look first at the linear terms

$$3E_{[A}{}^N\Theta_{|N}{}^K\partial_{K|}E_B{}^M E_{C]M} + 3\partial_{[A}\Theta_{|N|}{}^M E_B{}^N E_{C]M}\,. \tag{3.49}$$

These terms actually cancel. To see this we recall that Θ is built from Killing vectors and the fact that G and B are invariant translates to the generalized Lie derivative (3.33) of $E_A{}^M$ along these being zero

$$0 = \mathcal{L}_{k_r}E_A{}^M = k_r^N\partial_N E_A{}^M + (\partial^M k_{rN} - \partial_N k_r^M)E_A{}^N\,, \qquad k_r^N = (0, k_r^n)\,. \tag{3.50}$$

Using this the first term becomes

$$\begin{aligned} &3E_{[A}{}^N\Theta_{|N}{}^K\partial_{K|}E_B{}^M E_{C]M} = -3R^{rs}k_{rN}(\partial^M k_{sK} - \partial_K k_s^M)E_{[A}{}^N E_B{}^K E_{C]M} \\ &= -3\partial_K\Theta_{MN}E_{[A}{}^N E_B{}^K E_{C]}{}^M = -3\partial_{[A}\Theta_{|N|}{}^M E_B{}^N E_{C]M}\,, \end{aligned} \tag{3.51}$$

completing the proof. This leaves only the term quadratic in Θ. However, this terms also vanishes due to the YB equation (2.40). Indeed we have

$$\begin{aligned} &3E_{[A}{}^K E_B{}^N E_{C]}{}^M\Theta_K{}^L\partial_L\Theta_{NM} = 6E_{[A}{}^K E_B{}^N E_{C]}{}^M R^{rs}R^{tu}k_{rK}k_s^L\partial_L k_{tN}k_{uM} \\ &= 6E_{[A}{}^K E_B{}^N E_{C]}{}^M R^{rs}R^{tu}k_{rK}k_{uM}k^l_{[s}\partial_l k_{t]N} = 3E_{[A}{}^K E_B{}^N E_{C]}{}^M R^{rs}R^{tu}k_{rK}k_{uM}f_{st}{}^v k_{vN} \\ &= 3E_{[A}{}^K E_B{}^N E_{C]}{}^M k_{rK}k_{uM}k_{vN}R^{rs}R^{tu}f_{st}{}^v = 0\,. \end{aligned} \tag{3.52}$$

In the third step we used the commutation relation of the Killing vectors $2k^l_{[r}\partial_l k^m_{s]} = f_{rs}{}^t k_t^m$. We have learned the remarkable fact that despite the $O(D,D)$ transformation involved being non-constant the generalized flux F_{ABC} is still invariant. What about the flux F_A? We have

$$\begin{aligned} \tilde{F}_A =& \tilde{\partial}^B\tilde{E}_B{}^M\tilde{E}_{AM} + 2\tilde{\partial}_A d = -\partial^M\tilde{E}_{AM} + 2\tilde{\partial}_A d \\ =& F_A - \partial_M E_A{}^N\Theta_N{}^M - E_A{}^N\partial_M\Theta_N{}^M + 2E_A{}^N\Theta_N{}^M\partial_M d\,. \end{aligned} \tag{3.53}$$

Note that we have assumed that the generalized dilaton does not transform, $\tilde{d} = d$, which is natural since this is what happens for ordinary T-duality. Using the invariance of $E_A{}^M$ under the isometries involved in Θ the first term involving Θ becomes

$$\tfrac{1}{2}R^{rs}f_{rs}{}^t k_{tM}E_A{}^M - R^{rs}k_{rN}\partial_A k_s^N = \tfrac{1}{2}R^{rs}f_{rs}{}^t k_{tM}E_A{}^M\,, \tag{3.54}$$

[5] If all the Killing vectors involved commute we can pick coordinates where they are constant. In this case the YB deformation becomes equivalent to a so-called TsT transformation.

since $k_r^M = (0, k_r^m)$. Using the fact that the generalized dilaton transforms as a scalar density, $k_r^N \partial_N d = \frac{1}{2}\partial_N k_r^N$, the last term cancels against a piece of the second term and the result is

$$\tilde{F}_A = F_A + E_A{}^M k_{tM} R^{rs} f_{rs}{}^t \,. \tag{3.55}$$

Unlike F_{ABC} we find that F_A is in general shifted by something proportional to the contraction of the R-matrix with the structure constants. Conversely, this extra term will vanish leaving also F_A invariant if the algebraic condition

$$R^{rs} f_{rs}{}^t = 0 \tag{3.56}$$

is satisfied. We call this the *unimodularity condition* because it is equivalent to the structure constants of the algebra of the Killing vectors involved in Θ having vanishing trace, $f_{rs}{}^s = 0$. To see this one contracts the YB equation (2.40) with $\omega_{rs} = (R^{-1})_{rs}$ leading to

$$R^{uv} f_{uv}{}^t = -R^{tu} f_{us}{}^s \,. \tag{3.57}$$

If the unimodularity condition is satisfied we have seen that both generalized fluxes are unchanged by the YB deformation

$$\tilde{F}_{ABC} = F_{ABC} \,, \qquad \tilde{F}_A = F_A \,. \tag{3.58}$$

This is quite remarkable when contrasted with the complicated transformation of the metric and B-field in (3.8). But not only are the fluxes invariant, their derivatives are as well. For example we have

$$\tilde{\partial}_A \tilde{F}_B = \tilde{\partial}_A F_B = \partial_A F_B + E_A{}^M \Theta_M{}^N \partial_N F_B = \partial_A F_B \,, \tag{3.59}$$

where we used the fact that $k_r^M \partial_M F_A = 0$ since k_r^M generate isometries of the original background. Clearly this observation extends to any number of derivatives of the fluxes. But this observation means that the equations of motion (3.45), (3.42) and (3.46) are invariant under the YB deformation. Therefore, if we start from a solution of these equations and apply a unimodular YB deformation we obtain another solution. Since we have seen that these equations coincide with the string β-function equations to lowest order in α' we find that the YB deformation maps consistent string backgrounds to other consistent backgrounds, at least to leading order in the α' expansion (corresponding to one loop order in sigma model perturbation theory).

What about non-unimodular deformations? In this case $\tilde{F}_A$ includes the extra shift in (3.55) so that the generalized fluxes are not invariant. In fact this extra shift proportional to the trace of the structure constants fits with what is known about non-abelian T-duality at one loop. In that case there is an anomaly which spoils the one-loop Weyl-invariance for non-unimodular groups [16, 17]. This fact would suggest that the non-unimodular YB models will not solve the one-loop β-function equations. However, a direct calculation, plugging in $\tilde{F}_{ABC} = F_{ABC}$ and $\tilde{F}_A$ into the equations (3.45), shows that it can nevertheless happen that the extra terms coming from the shift in F_A can decouple, giving again an admissible string background. This can happen in particular if $G + B$ is a degenerate matrix, and several examples are known. These examples are very special however and typically non-unimodular deformations fail to lead to admissible backgrounds at one-loop order.

Before we go on to consider what happens at two loops let us note an important fact. Starting from some admissible string background we construct the YB deformation in the doubled language by transforming the generalized vielbein (3.32) according to (3.47). But if we start with a generalized vielbein with $e^{(+)} = e^{(-)} = e$, which we need to do to connect to the standard gravity description, we obtain a deformed one with $\tilde{e}^{(+)} \neq \tilde{e}^{(-)}$. Therefore, to read off the deformed background we should perform one more step – a double Lorentz transformation which sets $\tilde{e}^{(+)} = \tilde{e}^{(-)} = \tilde{e}$. Assuming we start from $e^{(+)} = e^{(-)} = e$ one finds that after the transformation (3.47) the required double Lorentz transformation may be taken as

$$\Lambda^{(+)} = 1\,, \qquad (\Lambda^{(-)})_a{}^b = \tilde{\Lambda}_a{}^b = e_a{}^m e^b{}_n \left[(1 + (G-B)\Theta)\,(1-(G+B)\Theta)^{-1}\right]_m{}^n\,, \quad (3.60)$$

corresponding to picking the deformed vielbein to be

$$\tilde{e}^{am} = e^{an}\,(1 + (G-B)\Theta)_n{}^m\,. \qquad (3.61)$$

Of course, this extra double Lorentz transformation can be ignored at this stage since it is a symmetry of the theory. However, it will play an important role in the next section.

Chapter 4

Two-loop Weyl invariance and α'-correction

So far we have seen that the YB deformation maps solutions of the β-function equations for the bosonic string (3.4)–(3.6) to new solutions when α'-corrections are ignored. What happens when we take α'-corrections into account? We will now answer this question for the first α'-correction, ignoring $\mathcal{O}(\alpha'^2)$-terms. The first correction to the β-function equations for the bosonic string arises from a correction to the effective action involving the square of the Riemann tensor

$$S = S_0 + \alpha' S_1 + \mathcal{O}(\alpha'^2) \tag{4.1}$$

where S_0 was given in (3.7) and [18]

$$\begin{aligned} S_1 = \frac{1}{2\kappa_0^2} \int d^{26}x \sqrt{-G} e^{-2\Phi} \Big(&\tfrac{1}{4} R_{mnop} R^{mnop} - \tfrac{1}{8} R_{mnop} H^{mnq} H^{op}{}_q \\ &+ \tfrac{1}{96} H_{mno} H^m{}_{pq} H^{npr} H^{oq}{}_r - \tfrac{1}{32} (H^2)_{mn} (H^2)^{mn} \Big) . \end{aligned} \tag{4.2}$$

Checking directly that the YB map (3.8) maps solutions of the equations of motion corresponding to this corrected action to new solutions seems hopelessly complicated. But if we are again able to rewrite things in a manifestly $O(D,D)$ covariant form the proof becomes trivial. Remarkably, the α'-corrected action can again be written in an $O(D,D)$ invariant form. This is quite surprising given the fact that there is no $O(D,D)$ covariant analog of the Riemann tensor [19].[1] Instead this correction appears through a modification of the double Lorentz transformations [22]. To see why this happens it is instructive to consider first the case of the heterotic string.

4.1 α'-correction to double Lorentz transformation

In addition to the fields of the bosonic string the heterotic string also has gauge fields and fermions. We will set these to zero here for simplicity. Then the lowest order effective

[1] However, it can be understood from the fact that the correction can be generated [20] using a trick originally due to Bergshoeff and de Roo [21].

action is the same as for the bosonic string (3.7). However, the α'-corrections differ. In particular, a famous fact is that the Bianchi identity for the NSNS three-form $dH = 0$ is corrected in the heterotic string to (e.g. [23])

$$dH = \frac{\alpha'}{4}\,\mathrm{tr}(R\wedge R)\,. \tag{4.3}$$

This correction is required by the Green-Schwarz anomaly cancellation mechanism [24]. We may solve this condition by taking

$$H = dB + \frac{\alpha'}{4}\Omega_3\,, \qquad d\Omega_3 = \mathrm{tr}(R\wedge R)\,, \tag{4.4}$$

where $\Omega_3 = \mathrm{tr}(\omega\wedge d\omega) + \frac{2}{3}\,\mathrm{tr}(\omega\wedge\omega\wedge\omega)$ is a gravitational Chern-Simons three-form. But H is by definition invariant under Lorentz transformations while Ω_3 is not, and this requires a non-standard transformation for B. From the transformation of the spin connection, $\omega \to \omega' = \Lambda^{-1}d\Lambda + \Lambda^{-1}\omega\Lambda$, we find

$$dB \to dB' = H - \frac{\alpha'}{4}\Omega_3' = dB + \frac{\alpha'}{4}\left[d\,\mathrm{tr}(d\Lambda\Lambda^{-1}\wedge\omega) + \tfrac{1}{3}\,\mathrm{tr}(d\Lambda\Lambda^{-1}\wedge d\Lambda\Lambda^{-1}\wedge d\Lambda\Lambda^{-1})\right]\,, \tag{4.5}$$

or

$$B' = B + \frac{\alpha'}{4}[\mathrm{tr}(d\Lambda\Lambda^{-1}\wedge\omega) + B^{\mathrm{WZW}}(\Lambda)]\,, \tag{4.6}$$

where

$$dB^{\mathrm{WZW}}(\Lambda) = \tfrac{1}{3}\,\mathrm{tr}(d\Lambda\Lambda^{-1}\wedge d\Lambda\Lambda^{-1}\wedge d\Lambda\Lambda^{-1})\,. \tag{4.7}$$

This modification of the gauge-transformations of B is not part of the $O(D,D)$ covariant formalism described in the previous section. Therefore it is clear that the transformations need to be modified to account for the first α'-correction to the heterotic string in that formalism. To see how we should modify the transformations let us consider the infinitesimal version ($\Lambda = 1+\lambda$) of the above transformation which reads

$$\delta B = \frac{\alpha'}{4}\,\mathrm{tr}(d\lambda\wedge\omega)\,. \tag{4.8}$$

To generalize this to the $O(D,D)$ covariant form we note that a change in the transformation of B under Lorentz transformations implies a change in the transformation of the generalized vielbein under double Lorentz transformations

$$\delta E_A{}^M E_{BM} = \lambda_{AB} + \hat{\lambda}_{AB}\,, \tag{4.9}$$

where $\hat{\lambda}$ is of order α'. Since the lowest order transformation parameters λ_{AB} have non-zero components $\lambda_{\overline{ab}}$ and $\lambda_{\underline{ab}}$, any non-zero $\hat{\lambda}_{\overline{ab}}$ or $\hat{\lambda}_{\underline{ab}}$ can be absorbed into these. We may therefore take the components of $\hat{\lambda}$ to have only mixed projections $\hat{\lambda}_{\overline{a}\underline{b}} = -\hat{\lambda}_{\underline{b}\overline{a}}$. Looking at (4.8) we see that $\hat{\lambda}$ should involve a derivative of $\lambda_{\overline{ab}}$ or $\lambda_{\underline{ab}}$ and the doubled analog of the spin connection. In fact the closest analog of the spin connection is $F_{\underline{a}\overline{bc}}$ and $F_{\overline{a}\underline{bc}}$, as is easily seen from the first term in the transformations in (3.41). This suggests that we should take

$$\hat{\lambda}_{\overline{a}\underline{b}} = -\hat{\lambda}_{\underline{b}\overline{a}} = \frac{a}{2}\partial_{\underline{b}}\lambda^{\underline{cd}}F_{\overline{a}\underline{cd}} + \frac{b}{2}\partial_{\overline{a}}\lambda^{\overline{cd}}F_{\underline{b}\overline{cd}} = -\frac{a}{2}\,\mathrm{tr}(\partial_{\underline{b}}\lambda^{(-)}F_{\overline{a}}) - \frac{b}{2}\,\mathrm{tr}(\partial_{\overline{a}}\lambda^{(+)}F_{\underline{b}})\,, \tag{4.10}$$

for some a, b of order α'. Note that we suppressed the contracted indices in the last expression. Of course, for this to make sense, the corrected transformations (4.9) must close (to first order in α'). We find

$$[\delta', \delta] E_{\overline{a}}{}^M E_{\underline{b}M} = \lambda_{\overline{a}}{}^{\overline{c}} \hat{\lambda}'_{\overline{c}\underline{b}} - \lambda'_{\overline{a}}{}^{\overline{c}} \hat{\lambda}_{\overline{c}\underline{b}} + \hat{\lambda}_{\overline{a}\underline{c}} \lambda'^{\underline{c}}{}_{\underline{b}} - \hat{\lambda}'_{\overline{a}\underline{c}} \lambda^{\underline{c}}{}_{\underline{b}} + \delta' \hat{\lambda}_{\overline{a}\underline{b}} - \delta \hat{\lambda}'_{\overline{a}\underline{b}} + \mathcal{O}(\alpha'^2) \tag{4.11}$$

and plugging in the expression for $\hat{\lambda}$ we find

$$\begin{aligned}
&[\delta', \delta] E_{\overline{a}}{}^M E_{\underline{b}M} = \\
&\quad \frac{a}{2}\left[\mathrm{tr}(\partial_{\underline{b}}\lambda'^{(-)}\partial_{\overline{a}}\lambda^{(-)}) - \mathrm{tr}(\partial_{\underline{b}}\lambda^{(-)}\partial_{\overline{a}}\lambda'^{(-)}) + \mathrm{tr}(\partial_{\underline{b}}\lambda'^{(-)}[\lambda^{(-)}, F_{\overline{a}}]) - \mathrm{tr}(\partial_{\underline{b}}\lambda^{(-)}[\lambda'^{(-)}, F_{\overline{a}}])\right] \\
&\quad - \frac{b}{2}\left[\mathrm{tr}(\partial_{\overline{a}}\lambda^{(+)}\partial_{\underline{b}}\lambda'^{(+)}) - \mathrm{tr}(\partial_{\overline{a}}\lambda'^{(+)}\partial_{\underline{b}}\lambda^{(+)}) + \mathrm{tr}(\partial_{\overline{a}}\lambda^{(+)}[\lambda'^{(+)}, F_{\underline{b}}]) - \mathrm{tr}(\partial_{\overline{a}}\lambda'^{(+)}[\lambda^{(+)}, F_{\underline{b}}])\right] \\
&\quad + \mathcal{O}(\alpha'^2) \\
&\quad = 2E_{\overline{a}}{}^M E_{\underline{b}}{}^N \partial_{[M} Y_{N]} - \frac{a}{2}\,\mathrm{tr}(\partial_{\underline{b}}[\lambda^{(-)}, \lambda'^{(-)}] F_{\overline{a}}) - \frac{b}{2}\,\mathrm{tr}(\partial_{\overline{a}}[\lambda^{(+)}, \lambda'^{(+)}] F_{\underline{b}}) + \mathcal{O}(\alpha'^2)
\end{aligned} \tag{4.12}$$

with

$$Y_N = \frac{a}{2}\,\mathrm{tr}(\lambda^{(+)}\partial_N \lambda'^{(+)}) - \frac{b}{2}\,\mathrm{tr}(\lambda^{(+)}\partial_N \lambda'^{(+)})\,. \tag{4.13}$$

The first term is a generalized diffeomorphism with parameter Y_N and the second is precisely of the form of the corrected Lorentz transformation with parameter $[\lambda, \lambda']$. Therefore the transformations indeed close to the required order in α'.

We have found that it is possible to correct the double Lorentz transformations at order α'. The correction depends on two parameters, a and b. How does this correction compare to the one required for the heterotic string, (4.8)? Fixing the gauge $e^{(+)} = e^{(-)} = e$ in (3.32) one finds from the definition of the generalized fluxes (3.34) that

$$F_a{}^{bc} = -\frac{1}{\sqrt{2}}\omega_a^{(+)bc}\,, \quad F^a{}_{bc} = \frac{1}{\sqrt{2}}\omega^{(-)a}{}_{bc}\,, \qquad \omega_m^{(\pm)ab} = \omega_m{}^{ab} \pm \tfrac{1}{2} H_m{}^{ab} \tag{4.14}$$

and using these in the transformation (4.9) one finds[2]

$$\delta\bar{G}_{mn} = -\frac{a}{2}\partial_{(m}\lambda^{(-)cd}\omega^{(-)}{}_{n)cd} - \frac{b}{2}\partial_{(m}\lambda^{(+)}_{|cd|}\omega^{(+)cd}_{n)}\,, \tag{4.15}$$

$$\delta\bar{B}_{mn} = -\frac{a}{2}\partial_{[m}\lambda^{(-)cd}\omega^{(-)}{}_{n]cd} + \frac{b}{2}\partial_{[m}\lambda^{(+)}_{|cd|}\omega^{(+)cd}_{n]}\,. \tag{4.16}$$

Of course only the transformations with $\lambda^{(+)} = \lambda^{(-)} = -\lambda$ preserve our gauge choice $e^{(+)} = e^{(-)}$ and reduce to the usual Lorentz transformations,[3] but the more general case will be needed below. Note that we have denoted the fields G, B with a bar. The reason is that these fields, which are natural from the $O(D, D)$ covariant formulation, are not the ones we would usually consider since they are not Lorentz invariant. But we can define new fields

$$G_{mn} = \bar{G}_{mn} - \frac{a}{4}\omega_m^{(-)cd}\omega_{ncd}^{(-)} - \frac{b}{4}\omega_m^{(+)cd}\omega_{ncd}^{(+)}\,, \qquad B_{mn} = \bar{B}_{mn} + \frac{a+b}{4}\omega_{[m}{}^{cd} H_{n]cd}\,, \tag{4.17}$$

[2]Note that the components of F and λ are expressed relative to their defining index structures, F_{ABC} and $\lambda_A{}^B$.

[3]The sign of λ is fixed by requiring the usual lowest order transformation $\delta\omega^{ab} = d\lambda^{ab}$.

which transform as

$$\delta G_{mn} = 0\,, \qquad \delta B_{mn} = \frac{a-b}{2}\partial_{[m}\lambda^{cd}\omega_{n]cd}\,, \tag{4.18}$$

under local Lorentz transformations. We see that taking $a = -\alpha'$ and $b = 0$ we reproduce the correct heterotic string transformation (4.8), while taking $a = b$ we get the correct transformation for the bosonic string where B does not transform.

4.2 α'-corrected DFT action

We have seen that there is freedom to modify the double Lorentz transformations at order α' by adding the terms in (4.10). The correction has two free parameters, a and b, and to match the correction to the heterotic string we need to set one of them to zero while for the bosonic string we must set them equal. Modifying the double Lorentz transformations the lowest order action given by (3.43), (3.42) is no longer invariant, since its transformation will produce terms of order α'. However, one can show that the terms of order α' can be canceled by adding certain terms of order α' to the lowest order action [22]. The resulting DFT action to order α' takes the form

$$S = \int dX\, e^{-2d}\left(\mathcal{R} + a\mathcal{R}^{(-)} + b\mathcal{R}^{(+)}\right)\,, \tag{4.19}$$

where we have defined

$$\begin{aligned}\mathcal{R}^{(-)} = &-\left(\partial^{\overline{a}} - F^{\overline{a}}\right)\Big[\left(\partial^{\overline{b}} - F^{\overline{b}}\right)\left(F_{\overline{a}\underline{cd}}F_{\overline{b}}{}^{\underline{cd}}\right)\Big] - \tfrac{1}{2}\mathcal{R}_{\overline{ab}\underline{cd}}\mathcal{R}^{\overline{ab}\underline{cd}} + \partial^{\overline{a}}F^{\overline{b}}F_{\overline{a}\underline{cd}}F_{\overline{b}}{}^{\underline{cd}} \\ &+ F^{\overline{ab}C}F_{\overline{a}}{}^{\underline{de}}\partial_C F_{\overline{b}\underline{de}} - \tfrac{2}{3}F^{\overline{abc}}F_{\overline{a}\underline{d}}{}^{\underline{e}}F_{\overline{b}\underline{e}}{}^{\underline{f}}F_{\overline{c}\underline{f}}{}^{\underline{d}} + \left(F^{\underline{a}\overline{bc}}F_{\underline{a}\overline{bd}} + \tfrac{1}{2}F^{\overline{abc}}F_{\overline{abd}}\right)F_{\overline{c}\underline{ef}}F^{\overline{d}\underline{ef}}\,,\end{aligned} \tag{4.20}$$

where the first term is a total derivative and in the second we have introduced the object

$$\mathcal{R}_{\overline{ab}\underline{cd}} = 2\partial_{[\overline{a}}F_{\overline{b}]\underline{cd}} - F_{\overline{abe}}F^{\overline{e}}{}_{\underline{cd}} - 2F_{[\overline{a}|\underline{c}|}{}^{\underline{e}}F_{\overline{b}]\underline{ed}}\,, \tag{4.21}$$

which may be considered a doubled analog of the Riemann tensor[4] (although it is non-covariant under double Lorentz transformations). The expression for $\mathcal{R}^{(+)}$ is simply obtained by exchanging over- and underlined indices in $\mathcal{R}^{(-)}$. This α'-corrected DFT action can be shown to reproduce (after suitable field redefinitions) the first α'-correction to the bosonic string effective action (4.2) upon setting $a = b = -\alpha'$. Setting $a = -\alpha'$ and $b = 0$ it also reproduces the correction to the heterotic string effective action [22]. Note that for this to work one has to take into account the α' terms in the relation between $\bar{G}$, $\bar{B}$, the natural fields from the DFT point of view, and G, B, the usual Lorentz invariant fields in (4.17).

[4]Indeed, solving the section condition and fixing the gauge $e^{(+)} = e^{(-)}$ one finds that its components are

$$\mathcal{R}^{ab}{}_{cd} = \frac{1}{2}\left(R^{(-)ab}{}_{cd} + \omega^{(+)eab}\omega^{(-)}_{ecd}\right)\,.$$

4.3 α'-correction to YB deformations

The existence of this α'-corrected DFT action has interesting consequences for Yang-Baxter deformations of strings. Because the action (4.19) is expressed in terms of the generalized fluxes it follows that the equations of motion following from this action are also expressed in terms of these. But we have seen that (unimodular) Yang-Baxter deformations leave the generalized fluxes invariant and so it follows that the YB deformation maps solutions of the DFT equations to solutions, at least up to and including order α'.[5] It would seem therefore that the YB deformed background should solve the β-function equations also at order α' without any need of modifying it by terms of order α'. This is not correct however, in general the background will receive a correction of order α'. The reason is that while this correction is not needed in the DFT language the natural fields of that formulation, which we denoted $\bar{G}$ and $\bar{B}$, are not the usual Lorentz covariant G and B but rather related to these by (4.17). In addition we have seen that to go to the usual fields from the doubled formulation we must set the two vielbeins $e^{(+)}$ and $e^{(-)}$ equal, but since the YB deformation does not preserve this relation a compensating double Lorentz transformation (3.60) is needed. Since the fields do not transform covariantly under double Lorentz transformations this induces an α'-correction for G and B. In fact we can easily write down the explicit form of this correction as follows.

From (4.16) we see that setting $\lambda^{(+)} = 0$ we have

$$\delta\bar{G}_{mn} = -\frac{a}{2}\partial_{(m}\lambda^{(-)cd}\omega^{(-)}_{n)cd}\,, \qquad \delta\bar{B}_{mn} = -\frac{a}{2}\partial_{[m}\lambda^{(-)cd}\omega^{(-)}_{n]cd}\,. \tag{4.22}$$

Of course this is the correction one gets if one starts from the gauge $e^{(+)} = e^{(-)}$. But in our case we have $e^{(+)} \neq e^{(-)}$ and we want to perform the double Lorentz transformation which sets them equal. But if we think of it the other way around this is precisely the inverse of the transformation needed to go from $e^{(+)} = e^{(-)}$ to the $e^{(+)} \neq e^{(-)}$ we started from. Therefore the transformation we are looking for is as above but with the sign of the RHS changed. In order to find the finite transformation rather than the infinitesimal one we first note that (cf. (4.17))

$$\delta\left(\bar{G}_{mn} - \frac{a}{4}\omega_m^{(-)cd}\omega^{(-)}_{ncd}\right) = 0\,, \qquad \delta\left(\bar{B}_{mn} + \frac{a}{4}\omega_{[m}{}^{cd}H_{n]cd}\right) = -\frac{a}{2}\partial_{[m}\lambda^{(-)cd}\omega_{n]cd}\,. \tag{4.23}$$

The last transformation is now of the form (4.8), for which the finite transformation is given by (4.6). Therefore we find the finite transformation to be

$$\delta\bar{G}_{mn} = -\frac{a}{2}[\partial_{(m}\Lambda^{(-)}(\Lambda^{(-)})^{-1}]^{cd}\omega^{(-)}_{n)cd} - \frac{a}{4}(\partial_m\Lambda^{(-)})^{cd}(\partial_n\Lambda^{(-)})_{cd}\,, \tag{4.24}$$

$$\delta\bar{B}_{mn} = -\frac{a}{2}[\partial_{[m}\Lambda^{(-)}(\Lambda^{(-)})^{-1}]^{cd}\omega^{(-)}_{n]cd} + \frac{a}{4}B^{\mathrm{WZW}}_{mn}(\Lambda^{(-)})\,, \tag{4.25}$$

where

$$dB^{\mathrm{WZW}}(\Lambda) = \tfrac{1}{3}\,\mathrm{tr}(d\Lambda\Lambda^{-1}\wedge d\Lambda\Lambda^{-1}\wedge d\Lambda\Lambda^{-1})\,. \tag{4.26}$$

To find the α'-correction to YB deformations all we need to do is take $\Lambda^{(-)} = \tilde{\Lambda}$, defined in (3.60), and recall that we are performing the transformation in the opposite direction,

[5] The same conclusion can be shown to apply to non-abelian T-duality and its generalization known as Poisson-Lie T-duality [25, 26, 27].

going from $e^{(+)} \neq e^{(-)}$ to $e^{(+)} = e^{(-)}$, rather than the other way around, so the sign of the RHS is opposite. In addition we must take into account the relation between the barred fields and the usual unbarred ones (4.17). The correction then takes the form

$$\delta(G-B)_{mn} = \frac{\alpha'}{2}\omega_m^{(-)cd}\left(\omega_{ncd}^{(+)} + [\partial_n\tilde{\Lambda}\tilde{\Lambda}^{-1}]_{cd}\right) + \frac{\alpha'}{4}\partial_m\tilde{\Lambda}^{cd}\partial_n\tilde{\Lambda}_{cd} + \frac{\alpha'}{4}B_{mn}^{\mathrm{WZW}}(\tilde{\Lambda})\,, \quad (4.27)$$

where we have set $a = b = -\alpha'$ as appropriate for the bosonic string (for the heterotic string the correction vanishes) and also included a redefinition of the metric $G \to G - \frac{\alpha'}{4}H_m{}^{cd}H_{ncd}$ to go to the scheme of [28]. Finally, the correction to the dilaton is derived from the fact that the generalized dilaton d defined in (3.22) is uncorrected, since it does not transform under double Lorentz transformations. Therefore $\delta\Phi = \frac{1}{4}G^{mn}\delta G_{mn}$ (in a suitable scheme). Note that we must of course also take into account any possible α' corrections to the undeformed background, which have to be determined by other means.

Chapter 5

Conclusions & further developments

We introduced the Yang-Baxter deformations in the simple case of the PCM and showed that they preserve the integrabilty of the model. In the homogeneous case, when the R matrix involved solves the classical rather than the modified classical Yang-Baxter equation, we showed that the deformation can be constructed using non-abelian T-duality. Since the notion of NATD exists for more general 2d sigma models this allowed us to define a notion of YB deformation for bosonic strings in general (in backgrounds with isometries). We saw that these deformations indeed make sense in string theory, at least when R is unimodular, since they preserve the one-loop Weyl invariance. This was made possible by working in the $O(D, D)$ invariant formalism of DFT where the YB deformation greatly simplifies. Finally, we saw that these observations extend to first order in α', where one however finds that the YB deformation must be corrected in a particular way. Again this result relied crucially on the existence of an $O(D, D)$ invariant formalism.

There are many important developments in this subject which we did not have time to mention here. We end by listing some of these:

- We have considered mainly the homogeneous YB deformations here. The inhomogeneous ones are also interesting and were in fact the first to be considered. In that case there is a connection [29], via a generalization of T-duality known as Poisson-Lie T-duality [30], to another integrable deformation known as the λ-deformation [31].
- We have considered only the bosonic string here, but for applications in string theory the superstring is more relevant. Yang-Baxter deformations of the $AdS_5 \times S^5$ superstring were introduced in [32, 33] and a general definition for homogeneous deformations was given in [34]. The original inhomogeneous deformation violated unimodularity, but later it was realized how to construct unimodular examples [35].
- It appears to be possible to use DFT to describe also the correction at the next order in α', i.e. α'^2 [20, 36]. However, recently we showed that the DFT description breaks down at order α'^3 [15]. How to deal with Yang-Baxter deformations at that order is an open problem.
- It has been argued that in the context of the AdS/CFT holographic duality YB deformations should correspond to non-commutative deformations of the dual field

theory [37, 38], though no precise checks of this idea have been made.

- Recently these deformations have also been embedded in the general language of affine Gaudin models [39] and holomorphic $4d$ Chern-Simons theory [40].

Bibliography

[1] C. Klimcik, *"Yang-Baxter sigma models and dS/AdS T duality"*, JHEP 0212, 051 (2002), hep-th/0210095.

[2] C. Klimcik, *"On integrability of the Yang-Baxter sigma-model"*, J.Math.Phys. 50, 043508 (2009), arxiv:0802.3518.

[3] X. C. de la Ossa and F. Quevedo, *"Duality symmetries from nonAbelian isometries in string theory"*, Nucl. Phys. B403, 377 (1993), hep-th/9210021.

[4] B. Hoare and A. A. Tseytlin, *"Homogeneous Yang-Baxter deformations as non-abelian duals of the AdS_5 sigma-model"*, J. Phys. A49, 494001 (2016), arxiv:1609.02550.

[5] Y. Lozano, *"NonAbelian duality and canonical transformations"*, Phys. Lett. B 355, 165 (1995), hep-th/9503045.

[6] D. Osten and S. J. van Tongeren, *"Abelian Yang–Baxter deformations and TsT transformations"*, Nucl. Phys. B915, 184 (2017), arxiv:1608.08504.

[7] J. Polchinski, *"String theory. Vol. 1: An introduction to the bosonic string"*, Cambridge University Press (2007).

[8] A. Giveon and M. Rocek, *"On nonAbelian duality"*, Nucl. Phys. B 421, 173 (1994), hep-th/9308154.

[9] K. A. Meissner and G. Veneziano, *"Symmetries of cosmological superstring vacua"*, Phys. Lett. B 267, 33 (1991).

[10] A. Sen, *"O(d) x O(d) symmetry of the space of cosmological solutions in string theory, scale factor duality and two-dimensional black holes"*, Phys. Lett. B 271, 295 (1991).

[11] W. Siegel, *"Superspace duality in low-energy superstrings"*, Phys. Rev. D 48, 2826 (1993), hep-th/9305073.

[12] C. Hull and B. Zwiebach, *"Double Field Theory"*, JHEP 0909, 099 (2009), arxiv:0904.4664.

[13] O. Hohm, C. Hull and B. Zwiebach, *"Background independent action for double field theory"*, JHEP 1007, 016 (2010), arxiv:1003.5027.

[14] D. Geissbuhler, D. Marques, C. Nunez and V. Penas, *"Exploring Double Field Theory"*, JHEP 1306, 101 (2013), arxiv:1304.1472.

[15] S. Hronek and L. Wulff, *"$O(D,D)$ and the string α' expansion: an obstruction"*, JHEP 2104, 013 (2021), arxiv:2012.13410.

[16] E. Alvarez, L. Alvarez-Gaume and Y. Lozano, *"On nonAbelian duality"*, Nucl. Phys. B424, 155 (1994), hep-th/9403155.

[17] S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer and G. Veneziano, *"Remarks on nonAbelian duality"*, Nucl. Phys. B435, 147 (1995), hep-th/9409011.

[18] R. R. Metsaev and A. A. Tseytlin, *"Order alpha-prime (Two Loop) Equivalence of the String Equations of Motion and the Sigma Model Weyl Invariance Conditions: Dependence on the Dilaton and the Antisymmetric Tensor"*, Nucl. Phys. B293, 385 (1987).

[19] O. Hohm and B. Zwiebach, *"On the Riemann Tensor in Double Field Theory"*, JHEP 1205, 126 (2012), arxiv:1112.5296.

[20] W. H. Baron, E. Lescano and D. Marqués, *"The generalized Bergshoeff-de Roo identification"*, JHEP 1811, 160 (2018), arxiv:1810.01427.

[21] E. Bergshoeff and M. de Roo, *"Supersymmetric Chern-simons Terms in Ten-dimensions"*, Phys. Lett. B 218, 210 (1989).

[22] D. Marques and C. A. Nunez, *"T-duality and α'-corrections"*, JHEP 1510, 084 (2015), arxiv:1507.00652.

[23] J. Polchinski, *"String theory. Vol. 2: Superstring theory and beyond"*, Cambridge University Press (2007).

[24] M. B. Green and J. H. Schwarz, *"Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory"*, Phys. Lett. B 149, 117 (1984).

[25] R. Borsato and L. Wulff, *"Quantum Correction to Generalized T Dualities"*, Phys. Rev. Lett. 125, 201603 (2020), arxiv:2007.07902.

[26] F. Hassler and T. Rochais, *"α'-Corrected Poisson-Lie T-Duality"*, Fortsch. Phys. 68, 2000063 (2020), arxiv:2007.07897.

[27] T. Codina and D. Marques, *"Generalized Dualities and Higher Derivatives"*, JHEP 2010, 002 (2020), arxiv:2007.09494.

[28] C. M. Hull and P. K. Townsend, *"String Effective Actions From σ Model Conformal Anomalies"*, Nucl. Phys. B301, 197 (1988).

[29] B. Hoare and A. A. Tseytlin, *"On integrable deformations of superstring sigma models related to $AdS_n \times S^n$ supercosets"*, Nucl. Phys. B 897, 448 (2015), arxiv:1504.07213.

[30] C. Klimcik and P. Severa, *"Dual nonAbelian duality and the Drinfeld double"*, Phys.Lett. B351, 455 (1995), hep-th/9502122.

[31] K. Sfetsos, *"Integrable interpolations: From exact CFTs to non-Abelian T-duals"*, Nucl.Phys. B880, 225 (2014), arxiv:1312.4560.

[32] F. Delduc, M. Magro and B. Vicedo, *"An integrable deformation of the $AdS_5{\times}S^5$ superstring action"*, Phys.Rev.Lett. 112, 051601 (2014), arxiv:1309.5850.

[33] I. Kawaguchi, T. Matsumoto and K. Yoshida, *"Jordanian deformations of the AdS_5xS^5 superstring"*, JHEP 1404, 153 (2014), arxiv:1401.4855.

[34] R. Borsato and L. Wulff, *"Non-abelian T-duality and Yang-Baxter deformations of Green-Schwarz strings"*, JHEP 1808, 027 (2018), arxiv:1806.04083.

[35] B. Hoare and F. K. Seibold, *"Supergravity backgrounds of the η-deformed $AdS_2 \times S^2 \times T^6$ and $AdS_5 \times S^5$ superstrings"*, JHEP 1901, 125 (2019), arxiv:1811.07841.

[36] W. Baron and D. Marques, *"The generalized Bergshoeff-de Roo identification. Part II"*, JHEP 2101, 171 (2021), arxiv:2009.07291.

[37] T. Matsumoto and K. Yoshida, *"Integrability of classical strings dual for noncommutative gauge theories"*, JHEP 1406, 163 (2014), arxiv:1404.3657.

[38] S. J. van Tongeren, *"Yang–Baxter deformations, AdS/CFT, and twist-noncommutative gauge theory"*, Nucl. Phys. B904, 148 (2016), arxiv:1506.01023.

[39] B. Vicedo, *"On integrable field theories as dihedral affine Gaudin models"*, Int. Math. Res. Not. 2020, 4513 (2020), arxiv:1701.04856.

[40] F. Delduc, S. Lacroix, M. Magro and B. Vicedo, *"A unifying 2d action for integrable σ-models from 4d Chern-Simons theory"*, Lett. Math. Phys. 110, 1645 (2020), arxiv:1909.13824.

Reprints of articles

Published for SISSA by Springer

Received: *August 14, 2020*
Accepted: *September 10, 2020*
Published: *October 12, 2020*

Relaxing unimodularity for Yang-Baxter deformed strings

Stanislav Hronek and Linus Wulff

Department of Theoretical Physics and Astrophysics, Faculty of Science, Masaryk University, 611 37 Brno, Czech Republic

E-mail: 436691@mail.muni.cz, wulff@physics.muni.cz

Abstract: We consider so-called Yang-Baxter deformations of bosonic string sigma-models, based on an R-matrix solving the (modified) classical Yang-Baxter equation. It is known that a unimodularity condition on R is sufficient for Weyl invariance at least to two loops (first order in α'). Here we ask what the necessary condition is. We find that in cases where the matrix $(G+B)_{mn}$, constructed from the metric and B-field of the undeformed background, is degenerate the unimodularity condition arising at one loop can be replaced by weaker conditions. We further show that for non-unimodular deformations satisfying the one-loop conditions the Weyl invariance extends at least to two loops (first order in α'). The calculations are simplified by working in an $O(D,D)$-covariant doubled formulation.

Keywords: Bosonic Strings, Conformal Field Models in String Theory, String Duality

ArXiv ePrint: 2007.15663

https://doi.org/10.1007/JHEP10(2020)065

JHEP10(2020)065

Contents

1 Introduction

Yang-Baxter (YB) deformations of 2D σ-models were introduced by Klimčík in [1]. The name comes from the fact that the deformation is constructed using an R-matrix which solves the (modified) classical Yang-Baxter equation. It was later realized that these deformations preserve the classical integrability of the σ-model [2]. Delduc, Magro and Vicedo constructed the YB deformation for symmetric spaces in [3], and then for the $AdS_5 \times S^5$ superstring in [4], based on the Drinfeld-Jimbo R-matrix solving the modified classical YB equation. Shortly thereafter it was shown in [5] that essentially the same construction works also for R-matrices solving the ordinary (non-modified) classical YB equation. The latter are often referred to as homogenous YB deformations and have an interesting realization in terms of non-abelian T-duality [6–8].

Surprisingly it was found, starting with the paper [9], that the backgrounds corresponding to these deformed string σ-models did not always satisfy the equations of supergravity, but a certain generalization of these [10, 11]. When this is the case the deformed σ-model is only scale invariant, but not Weyl invariant, at one loop and cannot be interpreted as a consistent string. For supercoset models such as the $AdS_5 \times S^5$ superstring a condition was found on the R-matrix that leads to a viable, i.e. one-loop Weyl invariant, deformed string σ-model. The R-matrix should be unimodular, i.e. its trace with the Lie algebra structure constants should vanish, $R^{rs} f_{rs}{}^{t} = 0$ [12].

Subsequently, using the realization via non-abelian T-duality, homogeneous YB deformations were defined for a general Green-Schwarz superstring with isometries [13].[1] Interestingly, examples were found where a non-unimodular R-matrix nevertheless gave

[1]In the abelian case these deformations are equivalent to so-called TsT-transformations, consisting of T-duality, a coordinate shift and a T-duality back [14].

rise to a good (super)gravity background [15, 16]. Therefore, while the unimodularity condition is sufficient, it is not necessary to solve the one-loop Weyl invariance conditions, i.e. the background (super)gravity equations.

Here we will determine the precise conditions for (bosonic) YB deformations to respect one-loop Weyl-invariance. We find that, at least for deformations of symmetric spaces, the only exceptions to the unimodularity condition occur when the matrix $(G+B)_{mn}$, where G, B are the metric and B-field of the undeformed background, is degenerate.[2,3] In that case, the unimodularity condition is no longer necessary and is replaced by weaker conditions which we give. This is consistent with the examples found in [15, 16] since the $AdS_3 \times S^3$ background considered there has degenerate $G+B$.

We then go on to analyze what happens at two loops, i.e. when we include the first α'-correction to the (super)gravity equations. We find that the conditions at two loops are weaker and only a subset of the one-loop conditions are needed.

These calculations are simplified enormously by working with the $O(D,D)$-covariant formulation known as Double Field Theory (DFT). In DFT a manifestly $O(D,D)$-covariant formulation is achieved by doubling the coordinates to $X^M = (\tilde{x}_m, x^m)$. One then imposes a "section condition" which effectively removes half of them, leaving the right number of physical coordinates. Here we will work only with the standard choice of section, $X^M = (0, x^m)$ or $\partial_M = (0, \partial_m)$, and therefore the coordinates are not doubled. However, the tangent space is effectively doubled and there are two copies of the Lorentz group. Therefore there are two sets of vielbeins $e^{(+)}$ and $e^{(-)}$ which transform independently under each Lorentz group factor. Fixing the gauge $e^{(+)} = e^{(-)} = e$ breaks the doubled Lorentz group down to its diagonal, which becomes the standard Lorentz group. With this gauge fixing the action and equations of motion of the doubled formulation reduce to those of standard (super)gravity. The reason the doubled formulation is useful is that the YB deformation becomes equivalent to a coordinate dependent $O(D,D)$-transformation which is easy to analyze. In fact the so-called generalized fluxes, the basic fields of the so-called flux formulation we are using [18], transform very simply under the YB deformation. The 3-form flux is invariant while the 1-form acquires a shift. This shift vanishes in the unimodular case and the generalized fluxes are simply invariant, from which one can immediately conclude that such YB deformations preserve Weyl invariance at least to two loops [19]. In the present case, we are interested in non-unimodular R-matrices and we have to take the shift into account. Provided that this shift satisfies certain conditions, which we determine, the Weyl-invariance is preserved at least up to two loops. It is interesting that it is possible to shift the 1-form generalized flux in certain ways and still preserve the equations of the doubled formulation including the first α'-correction. This should have an interpretation in gauged DFT [20], but we will not pursue this here.

In [19] the doubled formulation was used to determine the first α'-correction to the

[2]In the homogeneous case we prove this only for rank $R < 8$ for technical reasons.

[3]Gauge-transformations of B, which could affect this, are severely restricted by the fact that B is required to be invariant under the isometries involved in the deformation. This is required in the homogeneous case [13]. In the inhomogeneous case with a WZ-term [17] it is not required and our analysis is incomplete in that case.

deformed background for unimodular R. This correction arises because the fields of the doubled formulation are not Lorentz-covariant once α'-corrections are included and a double Lorentz transformation is needed to go to the gauge $e^{(+)} = e^{(-)} = e$ and reduce to the standard (super)gravity fields, thus leading to a correction to the background.[4] Our analysis here shows that no additional corrections are needed in the non-unimodular case, so the correction to the deformed background is still given by the expressions found in [19].

The outline of this paper is as follows. First we review the elements we need of the flux formulation of DFT and how the α'-correction to the double Lorentz transformations determine the action to the first order in α'. In section 3 we derive the conditions for a YB deformation to lead to a (super)gravity background, i.e. the conditions needed for one-loop Weyl-invariance. The situation at two loops is analyzed in section 4 where we find weaker conditions than at one loop. We end with some conclusions.

2 Doubled (flux) formulation

The $O(D, D)$-covariant formulation of (super)gravity used in DFT [22–24] turns out to be very powerful for the kinds of questions we are interested in here. In particular we will work with a frame-like formulation of DFT [25–27] where the structure group consists of two copies of the Lorentz group $O(1, D-1)\times O(D-1, 1)$. In particular we use the so-called flux formulation of [18, 28] where the first α'-correction to the bosonic and heterotic string can also be nicely incorporated. We will always assume that the section condition is solved in the standard way $\partial_M = (0, \partial_m)$ so that we are really just working with a rewriting of (super)gravity.

The starting point is to introduce a generalized (inverse) vielbein parametrized as

$$E_A{}^M = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{(+)a}{}_m - e^{(+)an} B_{nm} & e^{(+)am} \\ -e^{(-)}_{am} - e_a^{(-)n} B_{nm} & e_a^{(-)m} \end{pmatrix} . \tag{2.1}$$

Here $e^{(\pm)}$ are two sets of vielbeins for the metric G_{mn} which transform independently as $\Lambda^{(\pm)}e^{(\pm)}$ under the two Lorentz-group factors. To go to the standard supergravity picture one fixes a gauge $e^{(+)} = e^{(-)} = e$, leaving only one copy of the Lorentz-group. The dilaton Φ is encoded in the generalized dilaton d defined as

$$e^{-2d} = e^{-2\Phi}\sqrt{-G}\,. \tag{2.2}$$

There are two constant metrics, the $O(D, D)$-metric η^{AB} and the generalized metric $\mathcal{H}^{AB}$ which take the form

$$\eta^{AB} = \eta_{AB} = \begin{pmatrix} \bar\eta & 0 \\ 0 & -\bar\eta \end{pmatrix} , \qquad \mathcal{H}^{AB} = \begin{pmatrix} \bar\eta & 0 \\ 0 & \bar\eta \end{pmatrix} , \tag{2.3}$$

where $\bar\eta = (-1, 1, \ldots, 1)$ is the usual Minkowski metric. The flat tangent space indices $A, B, \ldots$ are raised and lowered with η^{AB}, η_{AB}. The generalized vielbein is used to convert

[4] The correction agrees with what is found by a much more involved calculation using standard (super)gravity [21].

JHEP10(2020)065

between these indices and coordinate indices $M, N, \ldots$. In particular we have the usual expressions for the $O(D,D)$-metric and the generalized metric in a coordinate basis

$$\eta^{MN} = E_A{}^M \eta^{AB} E_B{}^N = \begin{pmatrix} 0 & \delta_m{}^n \\ \delta^m{}_n & 0 \end{pmatrix}, \tag{2.4}$$

$$\mathcal{H}^{MN} = E_A{}^M \mathcal{H}^{AB} E_B{}^N = \begin{pmatrix} G_{mn} - B_{mk} G^{kl} B_{ln} & B_{mk} G^{kn} \\ -G^{mk} B_{kn} & G^{mn} \end{pmatrix}. \tag{2.5}$$

We also define

$$\partial_A = E_A{}^M \partial_M \,, \tag{2.6}$$

where $\partial_M = (0, \partial_m)$ is the ordinary derivative.

The basic fields of the flux formulation are the generalized fluxes. These are constructed from the generalized vielbein as

$$\mathcal{F}_{ABC} = 3\partial_{[A} E_B{}^M E_{C]M} \,, \qquad \mathcal{F}_A = \partial^B E_B{}^M E_{AM} + 2\partial_A d \,. \tag{2.7}$$

The importance of these objects comes from the fact that they transform as scalars under generalized diffeomorphisms implemented by the generalized Lie derivative defined as

$$\mathcal{L}_X Y^M = X^N \partial_N Y^M + (\partial^M X_N - \partial_N X^M) Y^N \,. \tag{2.8}$$

The generalized diffeomorphisms contain the usual diffeomorphisms and B-field gauge-transformations. The generalized fluxes satisfy the following Bianchi identities

$$4\partial_{[A} \mathcal{F}_{BCD]} = 3\mathcal{F}_{[AB}{}^E \mathcal{F}_{CD]E} \,, \qquad 2\partial_{[A} \mathcal{F}_{B]} = -(\partial^C - \mathcal{F}^C) \mathcal{F}_{ABC} \,. \tag{2.9}$$

Note also that

$$[\partial_A, \partial_B] = \mathcal{F}_{ABC}\, \partial^C \,. \tag{2.10}$$

The bosonic/heterotic[5] string low-energy effective action can be cast in doubled form as

$$S = \int dX\, e^{-2d} \mathcal{R} \,, \tag{2.11}$$

where the generalized Ricci scalar is defined as[6]

$$\mathcal{R} = -4\partial^A \mathcal{F}_A^{(-)} + 2\mathcal{F}^A \mathcal{F}_A^{(-)} - \mathcal{F}_{ABC}^{(-)} \mathcal{F}^{(-)ABC} - \frac{1}{3} \mathcal{F}_{ABC}^{(--)} \mathcal{F}^{(--)ABC} \,. \tag{2.12}$$

Here we have defined certain projections of the generalized fluxes using the natural projection operators

$$P_\pm = \frac{1}{2} (\eta \pm \mathcal{H}) \,, \tag{2.13}$$

[5]Setting the gauge fields and fermions of the heterotic string to zero.

[6]The last two terms are often written instead as $\frac{1}{4} \mathcal{F}_{ACD} \mathcal{F}_B{}^{CD} \mathcal{H}^{AB} - \frac{1}{12} \mathcal{F}_{ABC} \mathcal{F}_{DEF} \mathcal{H}^{AD} \mathcal{H}^{BE} \mathcal{H}^{CF} - \frac{1}{6} \mathcal{F}_{ABC} \mathcal{F}^{ABC}$. In terms of the generalized metric we have instead

$$\begin{aligned} \mathcal{R} = {} & 4\partial_M(\mathcal{H}^{MN} \partial_N d) - \partial_M \partial_N \mathcal{H}^{MN} - 4\mathcal{H}^{MN} \partial_M d \partial_N d \\ & + \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_K \mathcal{H}_{LN} \,. \end{aligned}$$

as follows

$$\mathcal{F}_A^{(\pm)} = (P_\pm)_A{}^B \mathcal{F}_B \,, \tag{2.14}$$

and

$$\mathcal{F}_{ABC}^{(\pm)} = (P_\mp)_A{}^D (P_\pm)_B{}^E (P_\pm)_C{}^F \mathcal{F}_{DEF} \,, \qquad \mathcal{F}_{ABC}^{(\pm\pm)} = (P_\pm)_A{}^D (P_\pm)_B{}^E (P_\pm)_C{}^F \mathcal{F}_{DEF} \,. \tag{2.15}$$

Setting $e^{(+)} = e^{(-)} = e$ in the generalized vielbein (2.1) this can be shown to reduce to the correct low-energy effective (super)gravity action.

We will be interested in whether certain transformations of the generalized fluxes map a solution to another solution, so we will need the equations of motion following from the action (2.11). These can be easily found using the variations of the generalized fluxes with respect to the generalized vielbein and dilaton

$$\delta_E \mathcal{F}_{ABC} = 3\partial_{[A} \delta E_{BC]} + 3\delta E_{[A}{}^D \mathcal{F}_{BC]D} \,, \quad \delta_E \mathcal{F}_A = \partial^B \delta E_{BA} + \delta E_A{}^B \mathcal{F}_B \,, \quad \delta_d \mathcal{F}_A = 2\partial_A \delta d \,, \tag{2.16}$$

where $\delta E_{AB} = \delta E_A{}^M E_{BM}$ is anti-symmetric by construction. The equations of motion become

$$\mathcal{R} = 0 \,, \qquad \partial_A^{(+)} \mathcal{F}_B^{(-)} + (\partial^C - \mathcal{F}^C) \mathcal{F}_{ABC}^{(-)} - \mathcal{F}_{CDA}^{(+)} \mathcal{F}^{(-)DC}{}_B = 0 \,. \tag{2.17}$$

Here we have defined the projected derivatives $\partial_A^{(\pm)} = (P_\pm)_A{}^B \partial_B$. The second equation of motion can equivalently be written with the opposite projections by exchanging + and − superscripts. Setting $e^{(+)} = e^{(-)} = e$ they reduce to correct (super)gravity equations of motion.

The action (2.11) is invariant under three important symmetries. The first is generalized diffeomorphisms, which encode regular diffeomorphisms and B-field gauge transformations. In the flux formulation we are working with here the generalized diffeomorphism invariance is manifest since the fluxes and the derivative ∂_A transform as scalars. The second symmetry is that of global $O(D,D)$-transformations

$$X^M \to X^N h_N{}^M \,, \qquad E_A{}^M \to E_A{}^N h_N{}^M \qquad \text{with} \qquad h_M{}^N \in O(D,D) \tag{2.18}$$

and $h_M{}^N$ constant. Again the action is manifestly invariant under these transformations since the fluxes are invariant. In our case we are always imposing the standard section condition $\partial_M = (0, \partial_m)$ so this symmetry is (partially) broken.

Finally, the most important symmetry for us will be the invariance under double Lorentz transformations

$$\delta E_A{}^M E_{BM} = \delta E_{AB} = \lambda_{AB} \qquad \text{with} \qquad (P_+)_A{}^C (P_-)_B{}^D \lambda_{CD} = 0 \,. \tag{2.19}$$

The parameters of the infinitesimal double Lorentz transformation λ_{AB} commute with the projectors $P_\pm$ so their non-trivial components are $\lambda_{AB}^{(+)}$ and $\lambda_{AB}^{(-)}$, corresponding to the two copies of the Lorentz group. These two copies rotate the two vielbeins $e^{(\pm)}$ in (2.1) independently. The (double) Lorentz invariance of the action (2.11) is not manifest. It can be verified with a bit of algebra using the variations of the fluxes (2.16). In particular it follows from these expressions that under a double Lorentz transformation

$$\delta\mathcal{F}^{(\pm)}_{ABC} = \lambda_A^{(\mp)D}\mathcal{F}^{(\pm)}_{DBC} + 2\lambda_{[B}^{(\pm)D}\mathcal{F}^{(\pm)}_{|AD|C]} + \partial_A^{(\mp)}\lambda^{(\pm)}_{BC}\,, \tag{2.20}$$

which, except for the projections, is precisely the transformation of a connection. Indeed, suppressing the last two indices we have[7]

$$\delta\mathcal{F}^{(\pm)}_M = \partial_M^{(\mp)}\lambda^{(\pm)} + [\lambda^{(\pm)},\mathcal{F}^{(\pm)}_M]\,, \tag{2.21}$$

so that $\mathcal{F}^{(\pm)}_M$ behave very much like connections. In fact, fixing the double Lorentz transformations by setting $e^{(+)} = e^{(-)} = e$ the non-zero components of $\mathcal{F}^{(\pm)}$ are [28]

$$\mathcal{F}_M^{(+)ab} = \frac{1}{2}\begin{pmatrix} G^{mn}\omega_n^{(+)ab} \\ -(1-BG)_m{}^n\omega_n^{(+)ab}\end{pmatrix}\,, \qquad \mathcal{F}^{(-)}_{Mab} = \frac{1}{2}\begin{pmatrix} G^{mn}\omega^{(-)}_{nab} \\ (1+BG)_m{}^n\omega^{(-)}_{nab}\end{pmatrix}\,, \tag{2.22}$$

where $\omega_m^{(\pm)cd} = \omega_m{}^{cd} \pm \frac{1}{2}H_m{}^{cd}$. These expressions will be useful later.

A very important point is that the double Lorentz transformations receive α'-corrections. In fact, this is a good thing since it allows us to derive the first α'-correction to the action (2.11) from the knowledge of the correction to the Lorentz transformations. We will now see how this works.

2.1 The first α'-correction

At the first order in α' the double Lorentz transformations get corrected to [28]

$$\delta E_{AB} = \lambda_{AB} + a\,\mathrm{tr}\left(\partial^{(-)}_{[A}\lambda\mathcal{F}^{(-)}_{B]}\right) - b\,\mathrm{tr}\left(\partial^{(+)}_{[A}\lambda\mathcal{F}^{(+)}_{B]}\right)\,, \tag{2.23}$$

where $a = b = -\alpha'$ for the bosonic string and $a = -\alpha'$, $b = 0$ for the heterotic string ($a = b = 0$ for type II). The correction involves the connection-like objects $\mathcal{F}^{(\pm)}_{ABC}$ (note the trace over the last two indices) and is therefore of the form of a Green-Schwarz transformation.

The knowledge of the correction to the Lorentz transformation can be used to find the α'-correction to the action [28], as we will now review. For simplicity we will set $b = 0$ in the derivation and restore b at the end. The variation (2.23) is then of the form $\delta = \delta^0 + a\delta^1$ and a short calculation gives for the projections of the generalized fluxes appearing in the lowest order action (2.11), (2.12)

$$\delta^1\mathcal{F}^{(-)}_A = -\frac{1}{2}(\partial^B - \mathcal{F}^B)\,\mathrm{tr}\left(\partial^{(-)}_A\lambda\mathcal{F}^{(-)}_B\right)\,, \tag{2.24}$$

$$\delta^1\mathcal{F}^{(---)}_{ABC} = \frac{3}{2}\,\mathrm{tr}\left(\partial^{(-)}_{[A}\lambda\mathcal{F}^{(-)D}\right)\mathcal{F}^{(-)}_{D|BC]} \tag{2.25}$$

and

$$\delta^1\mathcal{F}^{(-)}_{ABC} = (P_-)_{[C}{}^D\,\mathrm{tr}\left(\partial^{(-)}_{B]}\lambda\mathcal{R}^{(-)}_{AD}\right) + \frac{1}{2}\,\mathrm{tr}\left(\mathcal{F}^{(-)}_A\partial^D\lambda\right)\mathcal{F}^{(-)}_{DBC} + \frac{1}{2}\,\mathrm{tr}\left(\partial^{(-)}_{[B}\lambda\mathcal{F}^{(-)D}\right)\mathcal{F}^{(+)}_{C]AD}\,. \tag{2.26}$$

[7]We will try to be clear about when we suppress the last two indices to avoid possible confusion with the generalized flux with one index $\mathcal{F}_A$.

In the last expression we have defined the 'curvature' of the 'connection' $\mathcal{F}^{(-)}_{ABC}$ as (suppressing the last two indices which are projected by P_-)

$$\mathcal{R}^{(-)}_{AB} = 2\partial_{[A}\ \mathcal{F}^{(-)}_{B]} - (P_+)_{[B}{}^{D}\mathcal{F}_{A]DE}\mathcal{F}^{(-)E} - [\mathcal{F}^{(-)}_{A}, \mathcal{F}^{(-)}_{B}]\,. \tag{2.27}$$

This object will be useful later. In particular when we project the indices A and B with P_+ we have, writing $\bar{\mathcal{R}}^{(-)}_{AB} = (P_+)_A{}^C(P_+)_B{}^D\mathcal{R}^{(-)}_{CD}$, (again the last two indices are suppressed)

$$\delta^0\bar{\mathcal{R}}^{(-)}_{AB} = 2\lambda^{(+)C}_{[A}\bar{\mathcal{R}}^{(-)}_{|C|B]} + [\lambda^{(-)}, \mathcal{R}^{(-)}_{AB}] + \mathcal{F}^{(+)}_{CAB}\partial^C\lambda^{(-)} - \partial^C\lambda^{(+)}_{AB}\mathcal{F}^{(-)}_{C}\,, \tag{2.28}$$

which apart from the last two terms is the expected transformation of a curvature.

At lowest order in α' the action is Lorentz invariant. At the next order we find

$$\begin{aligned}\delta^1\mathcal{R} =& -4(\partial^A - \mathcal{F}^A)\delta^1\mathcal{F}^{(-)}_{A} - 2\partial^B\mathcal{F}^A\,\mathrm{tr}\left(\partial^{(-)}_{A}\lambda\mathcal{F}^{(-)}_{B}\right)\\ &-\frac{2}{3}\mathcal{F}^{(--)ABC}\delta^1\mathcal{F}^{(--)}_{ABC} - 2\mathcal{F}^{(-)ABC}\delta^1\mathcal{F}^{(-)}_{ABC}\,.\end{aligned} \tag{2.29}$$

Using the expressions for the δ^1-variations (2.24), (2.25) and (2.26) as well as the Bianchi identity for $\mathcal{F}_A$ (2.9), (2.10) and the section condition this becomes

$$\begin{aligned}\delta^1\mathcal{R} = \delta^0&\left(-\partial^A\left[(\partial^B - \mathcal{F}^B)\,\mathrm{tr}\left(\mathcal{F}^{(-)}_{A}\mathcal{F}^{(-)}_{B}\right)\right] + (\partial^A - \mathcal{F}^A)\left[\mathcal{F}^B\,\mathrm{tr}\left(\mathcal{F}^{(-)}_{A}\mathcal{F}^{(-)}_{B}\right)\right]\right)\\ &- \mathcal{F}^{ABC}\,\mathrm{tr}\left(\partial_A\partial^{(+)}_{B}\lambda\mathcal{F}^{(-)}_{C}\right) + \mathcal{F}^{ABC}\,\mathrm{tr}\left(\partial^{(+)}_{A}\lambda\partial_B\mathcal{F}^{(-)}_{C}\right) - 2\mathcal{F}^{ABC}\,\mathrm{tr}\left(\partial_C\lambda\partial_A\mathcal{F}^{(-)}_{B}\right)\\ &+ 2\mathcal{F}^{(-)ABC}\,\mathrm{tr}\left(\partial_B\lambda\mathcal{R}^{(-)}_{CA}\right) + \mathcal{F}^{ABC}\partial_C\lambda_{AD}\,\mathrm{tr}\left(\mathcal{F}^{(-)D}\mathcal{F}^{(-)}_{B}\right)\\ &+ \partial^B\lambda^{AC}\partial_A\,\mathrm{tr}\left(\mathcal{F}^{(-)}_{C}\mathcal{F}^{(-)}_{B}\right) + \mathcal{F}^{ABC}\mathcal{F}_{BCD}\,\mathrm{tr}\left(\partial^D\lambda\mathcal{F}^{(-)}_{A}\right)\\ &- \mathcal{F}^{(--)ABC}\mathcal{F}^{(-)}_{DBC}\,\mathrm{tr}\left(\partial_A\lambda\mathcal{F}^{(-)D}\right) - \mathcal{F}^{(-)ABC}\mathcal{F}^{(-)}_{DBC}\,\mathrm{tr}\left(\partial^D\lambda\mathcal{F}^{(-)}_{A}\right)\\ &- \mathcal{F}^{(-)ABC}\mathcal{F}^{(+)}_{CAD}\,\mathrm{tr}\left(\partial_B\lambda\mathcal{F}^{(-)D}\right).\end{aligned} \tag{2.30}$$

We must now find terms of order α' whose lowest order Lorentz transformation cancels the terms on the r.h.s. . The first term on the second line must be canceled by the variation of a term of the form $\mathcal{F}^{ABC}\,\mathrm{tr}\left(\partial_A\mathcal{F}^{(-)}_{B}\mathcal{F}^{(-)}_{C}\right)$ and we find

$$\begin{aligned}\delta^1\mathcal{R} = \delta^0&\left(-\partial^A\left[(\partial^B - \mathcal{F}^B)\,\mathrm{tr}\left(\mathcal{F}^{(-)}_{A}\mathcal{F}^{(-)}_{B}\right)\right] + (\partial^A - \mathcal{F}^A)\left[\mathcal{F}^B\,\mathrm{tr}\left(\mathcal{F}^{(-)}_{A}\mathcal{F}^{(-)}_{B}\right)\right]\right)\\ &- \delta^0\left[\mathcal{F}^{ABC}\,\mathrm{tr}\left(\partial_A\mathcal{F}^{(-)}_{B}\mathcal{F}^{(-)}_{C}\right)\right] - \mathcal{F}^{ABC}\,\mathrm{tr}\left(\partial^{(-)}_{C}\lambda\bar{\mathcal{R}}^{(-)}_{AB}\right) + \partial^C\lambda^{AB}\,\mathrm{tr}\left(\mathcal{F}^{(-)}_{C}\bar{\mathcal{R}}^{(-)}_{AB}\right)\\ &+ 2\partial^C\lambda^{AB}\,\mathrm{tr}\left(\mathcal{F}^{(-)}_{A}\mathcal{F}^{(-)}_{B}\mathcal{F}^{(-)}_{C}\right) + 2\mathcal{F}^{ABC}\,\mathrm{tr}\left(\mathcal{F}^{(-)}_{A}\mathcal{F}^{(-)}_{B}\partial^{(+)}_{C}\lambda\right)\\ &+ \mathcal{F}^{(++)}_{ABE}\partial^C\lambda^{AB}\,\mathrm{tr}\left(\mathcal{F}^{(-)}_{C}\mathcal{F}^{(-)E}\right) + 2\mathcal{F}^{ABE}\partial_B\lambda_{CA}\,\mathrm{tr}\left(\mathcal{F}^{(-)C}\mathcal{F}^{(-)}_{E}\right)\\ &+ \mathcal{F}^{ABC}\mathcal{F}_{DBC}\,\mathrm{tr}\left(\partial^{(+)D}\lambda\mathcal{F}^{(-)}_{A}\right) - \mathcal{F}^{(-)ABC}\mathcal{F}^{(-)}_{DBC}\,\mathrm{tr}\left(\partial^D\lambda\mathcal{F}^{(-)}_{A}\right),\end{aligned} \tag{2.31}$$

where we used the definition of the 'curvature' in (2.27). Using (2.28) we see that the last two terms on the second line come from the variation of $\mathrm{tr}\left(\bar{\mathcal{R}}^{(-)AB}\bar{\mathcal{R}}^{(-)}_{AB}\right)$ and the remaining terms are also easy to write as the variation of something. When the dust has settled one finds, reinstating b, that the corrected action

$$S = \int dX\, e^{-2d} \left(\mathcal{R} + a\mathcal{R}^{(-)} + b\mathcal{R}^{(+)} \right) \tag{2.32}$$

is invariant under Lorentz transformations up to and including order α' where

$$\begin{aligned} \mathcal{R}^{(-)} = & \, \partial^A \left[(\partial^B - \mathcal{F}^B) \operatorname{tr} \left(\mathcal{F}^{(-)}_A \mathcal{F}^{(-)}_B \right) \right] - (\partial^B - \mathcal{F}^B) \left[\mathcal{F}^A \operatorname{tr} \left(\mathcal{F}^{(-)}_A \mathcal{F}^{(-)}_B \right) \right] \\ & + \frac{1}{2} \operatorname{tr} \left(\bar{\mathcal{R}}^{(-)AB} \bar{\mathcal{R}}^{(-)}_{AB} \right) + \frac{1}{6} \mathcal{F}^{ABC} \mathcal{C}^{(-)}_{ABC} \, . \end{aligned} \tag{2.33}$$

In the last term we have introduced the 'Chern-Simons' form

$$\mathcal{C}^{(-)}_{ABC} = 6 \operatorname{tr} \left(\mathcal{F}^{(-)}_{[A} \partial_B \mathcal{F}^{(-)}_{C]} \right) + 3 (\mathcal{F}^{(-)}_{D[AB} - \mathcal{F}_{D[AB}) \operatorname{tr} \left(\mathcal{F}^{(-)}_{C]} \mathcal{F}^{(-)D} \right) - 4 \operatorname{tr} \left(\mathcal{F}^{(-)}_{[A} \mathcal{F}^{(-)}_B \mathcal{F}^{(-)}_{C]} \right) . \tag{2.34}$$

The expression for $\mathcal{R}^{(+)}$ is obtained by reversing the projections in an obvious way. These expressions agree with the ones written in [29] but are much more compact.

3 Yang-Baxter deformations and one-loop Weyl invariance

Yang-Baxter deformations are closely related to a generalization of T-duality known as Poisson-Lie (PL) T-duality. In particular homogeneous YB deformations can be constructed using non-abelian T-duality [6, 7]. It is therefore not surprising that they have a natural formulation in terms of DFT. In the flux formulation we are working with they are described as a coordinate dependent $O(D,D)$-transformation [13, 30, 31]

$$E_A{}^M \rightarrow \tilde{E}_A{}^M = E_A{}^N (1 + \Theta)_N{}^M \, . \tag{3.1}$$

The only non-zero components of $\Theta_N{}^M$ are $\Theta^{mn} = k^m_r k^n_s R^{rs}$ where k^m_r are Killing vectors belonging to some Lie algebra $\mathfrak{g}$ indexed by $(r, s, t, \ldots)$ and R^{rs} is a constant anti-symmetric matrix satisfying, in the homogeneous case, the classical YB equation

$$[RX, RY] - R([RX, Y] + [X, RY]) = 0 \, , \quad \forall X, Y \in \mathfrak{g} \, , \tag{3.2}$$

which implies the 'Jacobi identity' for Θ[8]

$$\Theta^{N[K} \partial_N \Theta^{LM]} = 0 \, . \tag{3.3}$$

If we start from a symmetric space σ-model we can also define the inhomogeneous deformation [3] where R satisfies the modified classical YB equation

$$[RX, RY] - R([RX, Y] + [X, RY]) = [X, Y] \, , \quad \forall X, Y \in \mathfrak{g} \, . \tag{3.4}$$

[8]Conversely, if we don't impose any condition on R, this condition follows by requiring that we get a (super)gravity solution [32].

The canonical solution is the Drinfeld-Jimbo R-matrix defined to annihilate elements of the Cartan subalgebra and to multiply generators corresponding to positive(negative) roots by $+i(-i)$. We can define again $\Theta^{mn} = k_r^m k_s^n R^{rs}$ which also satisfies (3.3).[9]

Note that letting R be multiplied by a small parameter, usually called η, these become deformations of the original background. It is not hard to show, using the definitions (2.7), that these deformations preserve the form of the generalized fluxes up to a shift of $\mathcal{F}_A$ [19]

$$\tilde{\mathcal{F}}_{ABC} = \mathcal{F}_{ABC}\,, \qquad \tilde{\mathcal{F}}_A = \mathcal{F}_A - 2K_A\,. \tag{3.5}$$

In addition derivatives of $\mathcal{F}_{ABC}$ are invariant, e.g. $\tilde{\partial}_A \tilde{\mathcal{F}}_{BCD} = \partial_A \mathcal{F}_{BCD}$. Because of the shift this is in general not true for $\mathcal{F}_A$, instead

$$\tilde{\partial}_A \tilde{\mathcal{F}}_B = \partial_A \mathcal{F}_B - 2\partial_A K_B - 2E_A{}^N \Theta_N{}^M \partial_M K_B\,. \tag{3.6}$$

The shift of $\mathcal{F}_A$ is given by a certain distinguished Killing vector namely $K^M = (0, K^m)$ with

$$K^m = \nabla_n \Theta^{mn} = \nabla_n k_r^m k_s^n R^{rs} = -\tfrac{1}{2} R^{rs} f_{rs}{}^t k_t^m\,, \tag{3.7}$$

where the third step involves using the algebra of the Killing vectors. This shift vanishes precisely when R is unimodular, i.e. when $R^{rs} f_{rs}{}^t = 0$.[10] In this case the generalized fluxes and their derivatives are invariant under the deformation and this directly implies that the deformation preserves Weyl-invariance at least up to order α' (2 loops) [19]. If we drop the unimodularity condition we will generically get a scale-invariant but not Weyl-invariant σ-model at one loop. This is reflected in the background solving the generalized supergravity equations [10, 11] instead of the usual ones, the extra Killing vector appearing in these equations being given by K^m.

Here we want to ask what happens if you don't require unimodularity but still require the deformed model to preserve one-loop Weyl invariance.[11] We will argue that, at least in the case of symmetric spaces, it is possible to find such non-unimodular R-matrices (at least of low enough rank to be interesting) only if the combination of metric and B-field of the original model $G \pm B$ is a degenerate matrix. An example where this happens is for $AdS_3 \times S^3$ and indeed in that case several non-unimodular R-matrices that lead to (super)gravity solutions have been found [15, 16].

The requirement that the equations of motion (2.17) remain invariant under the deformation, which is equivalent to preservation of one-loop Weyl-invariance, becomes, us-

[9]This was first noted in special examples in [33]. We thank S. van Tongeren for pointing this out to us. The fact that the r.h.s. in the modified YB equation does not contribute can be seen as follows. For a symmetric space $\mathfrak{g}$ is generated by P_a, M_{ab} with commutators of the form $[P,P] \sim M$, $[M,P] \sim P$ and $[M,M] \sim M$. The Killing vectors are given by $k_r{}^m = \ell_a{}^m (\hat{P}\mathrm{Ad}_g)^a{}_r$ (see for example [13]), where $\ell_a{}^m$ are inverse vielbeins of the left-invariant one-forms and $\hat{P}$ projects on the Lie algebra generators P_a. Now since the structure constants are Ad-invariant and since they have no component corresponding to three P_a generators it follows that the r.h.s. in the modified YB equation does not contribute.

[10]It is easy to see that the Drinfeld-Jimbo R-matrix of the inhomogeneous deformation is not unimodular.

[11]This corresponds to having a solution of the generalized supergravity equations which also solves the standard supergravity equations. Such 'trivial' solutions were analyzed in [34].

JHEP10(2020)065

ing (3.5) and (3.6)

$$\partial_A^{(+)} K_B^{(-)} + (P_+)_A{}^C E_C{}^N \Theta_N{}^M \partial_M K_B^{(-)} - K^C \mathcal{F}_{ABC}^{(-)} = 0\,, \tag{3.8}$$

$$\partial^A K_A^{(-)} + E_A{}^N \Theta_N{}^M \partial_M K^{(-)A} - K^A \mathcal{F}_A^{(-)} + K^A K_A^{(-)} = 0\,. \tag{3.9}$$

Since we should think of Θ as being multiplied by a small deformation parameter these equations contain terms of first and second order in this parameter (note that K (3.7) is of first order). These contributions then need to vanish separately.

3.1 First order terms

At the lowest order in the deformation we find the conditions

$$\partial_A^{(+)} K_B^{(-)} - K^C \mathcal{F}_{ABC}^{(-)} = 0\,, \qquad \partial^A K_A^{(-)} - K^A \mathcal{F}_A^{(-)} = 0\,. \tag{3.10}$$

Using the form of the generalized vielbein (2.1) with $e^{(+)} = e^{(-)} = e$, the fact that $K^M = (0, K^m)$ and the form of $\mathcal{F}_{ABC}^{(-)}$ in (2.22) the first equation becomes

$$\nabla_a[(1+B)_b{}^c K_c] - \frac{1}{2} H_{abc}(1+B)^c{}_d K^d = 0\,. \tag{3.11}$$

Symmetrizing in a, b and using the fact that K is Killing we find that $\tilde{K} = i_K B$ is also a Killing vector. Anti-symmetrizing we find, using $\mathcal{L}_K B = 0$, that

$$dK + i_{\tilde{K}} H = 0\,. \tag{3.12}$$

This equation implies that H is invariant under $\tilde{K}$ since $\mathcal{L}_{\tilde{K}} H = d i_{\tilde{K}} H = -ddK = 0$. We also have the same equation with K and $\tilde{K}$ exchanged since $d\tilde{K} = d i_K B = -i_K H$ from the invariance of the B-field under isometries, which we have assumed here.[12] From the dilaton equation we get, using the fact that K and $\tilde{K}$ are Killing vectors, the condition

$$\tilde{K}^m \partial_m \Phi = 0\,, \tag{3.13}$$

i.e. the dilaton is invariant under the isometry generated by $\tilde{K}$. To summarize, the conditions we find at this order are that $\tilde{K} = i_K B$ generates isometries of the background fields G, H, Φ and satisfies (3.12).

For our later discussion of two-loop conformal invariance it will be useful to express these conditions in terms of the generalized fluxes. The fact that K and $\tilde{K}$ generate symmetries of the original background implies that under YB deformations

$$\tilde{\mathcal{F}}_A^{(\pm)} \tilde{\partial}^A (\text{something invariant}) = \mathcal{F}_A^{(\pm)} \partial^A (\text{something invariant})\,. \tag{3.14}$$

In addition we have

$$\begin{aligned} K^A \mathcal{F}_{ABC}^{(-)} &= \frac{1}{2}(K^m + \tilde{K}^m)\omega_{mbc}^{(-)} \delta_B^b \delta_C^c \\ &= -\frac{1}{2}(\nabla_b K_c + \nabla_b \tilde{K}_c)\delta_B^b \delta_C^c - \frac{1}{4}(K^m + \tilde{K}^m) H_{mbc} \delta_B^b \delta_C^c \\ &= 0\,, \end{aligned} \tag{3.15}$$

[12]This seems to be required in the construction of the general homogeneous deformations [13]. In the inhomogeneous case this should be relaxed [17], but we will not try to do this here since it would take us too far afield.

where we used invariance of the vielbein under $K, \tilde{K}$ which implies $i_K\omega_{ab} = -\nabla_a K_b$ and similarly for $\tilde{K}$ as well as the equation (3.12) and the same with K and $\tilde{K}$ exchanged. The same is true with the opposite projection and therefore we have

$$\tilde{\mathcal{F}}^A \tilde{\mathcal{F}}^{(\pm)}_{ABC} = \mathcal{F}^A \mathcal{F}^{(\pm)}_{ABC} \,. \tag{3.16}$$

3.2 Second order terms

At second order in the deformation the conditions (3.8) and (3.9) read

$$(P_+)_A{}^C E_C{}^N \Theta_N{}^M \partial_M K^{(-)}_B = 0 \,, \qquad E_A{}^N \Theta_N{}^M \partial_M K^{(-)A} + K^A K^{(-)}_A = 0 \,. \tag{3.17}$$

We need to evaluate

$$\begin{aligned} E_A{}^N \Theta_N{}^M \partial_M K_B &= E_A{}^N E_B{}^L \Theta_N{}^M \partial_M K_L + E_A{}^N \Theta_N{}^M \partial_M E_B{}^L K_L \\ &= R^{rs} E_A{}^N E_B{}^M k_{rN} \left(k_s^L \partial_L K_M - K^L \partial_L k_{sM}\right) \,, \end{aligned} \tag{3.18}$$

where we used the fact that $\Theta^{mn} = k_r^m k_s^n R^{rs}$ and the isometry of the generalized vielbein, i.e. its generalized Lie derivative (2.8) along k_r vanishes, in the second step. Now we use the form of K^m in (3.7) and the algebra of the Killing vectors to reduce this to

$$E_A{}^N \Theta_N{}^M \partial_M K_B = -\frac{1}{2} k_{rA} k_{wB} R^{rs} f_{sv}{}^w R^{tu} f_{tu}{}^v \,. \tag{3.19}$$

Using the Jacobi identity and the (modified) classical YB equation this expression can be seen to be symmetric in the indices A and B. The second order conditions now become

$$(G-B)_{an} k_r^n (G+B)_{bm} k_w^m R^{rs} f_{sv}{}^w R^{tu} f_{tu}{}^v = 0 \,, \qquad K^2 + \tilde{K}^2 = 0 \,. \tag{3.20}$$

The first condition can be expressed as

$$k_r^n k_w^m R^{rs} f_{sv}{}^w R^{tu} f_{tu}{}^v = v_+^m v_+^n + v_-^m v_-^n \,, \tag{3.21}$$

where $v_\pm$ are zero-eigenvectors of $G \pm B$, i.e. $(G \pm B)v_\pm = 0$. When $G \pm B$ is degenerate precisely one such vector $v_\pm$ exists (up to rescaling). When $G \pm B$ is non-degenerate, for example if B vanishes, then the r.h.s. is zero and we get the condition

$$k_r^n k_w^m R^{rs} f_{sv}{}^w R^{tu} f_{tu}{}^v = 0 \,. \tag{3.22}$$

This condition is very strong and in fact it seems to imply the unimodularity condition, at least for deformations of symmetric spaces. In that case the condition becomes (see footnote 9)

$$(\hat{P}\mathrm{Ad}_g)^a{}_r (\hat{P}\mathrm{Ad}_g)^b{}_w R^{rs} f_{sv}{}^w R^{tu} f_{tu}{}^v = 0 \,, \tag{3.23}$$

and taking $g = e^{\epsilon^a P_a}$ and expanding in ϵ this leads to

$$R^{rs} f_{sv}{}^w R^{tu} f_{tu}{}^v = 0 \,. \tag{3.24}$$

It is easy to see from the form of the Drinfeld-Jimbo R-matrix that this rules out the inhomogeneous deformations. For the homogeneous deformations R is invertible on the subalgebra where it is defined and this condition is equivalent to the condition that the distinguished Lie algebra element $R^{rs} f_{rs}{}^t T_t$ must lie in the center of the algebra.[13] While we have not found a general proof that this implies unimodularity one can easily verify that this is true for R-matrices of rank< 8. In the rank 2 case this is trivial to see. For rank 4 the relevant algebras are classified in [36] and it is easy to check that only unimodular examples satisfy the condition. For rank 6 the relevant algebras are classified in [37] (nilpotent algebras are automatically unimodular) and again only unimodular ones satisfy the condition. In addition we note that for AdS_5, corresponding to the isometry group $\mathrm{SO}(2,4)$, the maximum rank of R is 8 [12], however it is easy to see that the 8-dimensional algebras in question have a trivial center and can therefore not lead to any exception to the unimodularity condition. This rules out non-unimodular deformations of AdS_n with $n \leq 5$ if $G + B$ is invertible.

Therefore we conclude that for deformations of symmetric spaces non-unimodular R-matrices can lead to one-loop Weyl invariant σ-models only if $G \pm B$ of the undeformed model is degenerate (with the caveat that we checked this only up to rank 6). In that case they must satisfy (3.20) as well as the conditions we found at first order, namely that $\tilde{K} = i_K B$ generates isometries of G, H, Φ and equation (3.12).[14] Examples of such backgrounds were found in [15, 16].

We will now turn to the question of what happens at two loops, i.e. including the first α'-correction to the (super)gravity equations of motion. We will find that the conditions at two loops as actually weaker. We will only need to satisfy the conditions we found at first order in the deformation to solve also the two-loop equations.

4 Two-loop Weyl invariance

Here we will show that the α'-correction to the equations of motion can be cast in a form that is manifestly invariant under non-unimodular YB deformations satisfying the one-loop Weyl invariance conditions of the previous section. In fact our calculation will be more general. We will assume only that the following remain invariant under the transformation in question

$$\begin{aligned} &\mathcal{F}_{ABC}\,, && \partial_{A_1}\cdots\partial_{A_n}(\text{anything invariant})\,,\\ &\mathcal{F}^A\mathcal{F}^{(\pm)}_{ABC}\,, && \mathcal{F}^{(\pm)}_A\partial^A(\text{anything invariant})\,. \end{aligned} \tag{4.1}$$

However, $\mathcal{F}_A$ and its derivatives need not be invariant. As we have seen this is true for any YB deformation that is one-loop Weyl invariant (3.14), (3.16) (it is trivially true for unimodular deformations since in that case also $\mathcal{F}_A$ is invariant under the deformation).

[13] It also implies that the algebra can be constructed as a so-called symplectic double extension of a lower-dimensional symplectic, or quasi-Frobenius, Lie algebra [35]. The question is then if the symplectic double extension of a unimodular Lie algebra is always unimodular, in which case this condition would imply unimodularity.

[14] For general inhomogeneous deformations with WZ-term a more careful analysis, where the condition of invariance of B is dropped, is required.

JHEP10(2020)065

To get the equations of motion at order α' we must vary the corrected action (2.32) using the expressions for the variations of the fluxes in (2.16). The variation with respect to the generalized dilaton is easy and gives just the vanishing of the Lagrangian itself

$$\mathcal{R} + a\mathcal{R}^{(-)} + b\mathcal{R}^{(+)} = 0\,. \tag{4.2}$$

In the following we will set $b = 0$ to simplify the calculations. In the end our results will apply also for $b \neq 0$. Displaying only the order α'-terms that are not trivially invariant under the YB deformation we have from (2.33)

$$\mathcal{R}^{(-)} = -2\partial^A\left[\mathcal{F}^B\,\mathrm{tr}\left(\mathcal{F}^{(-)}_A\mathcal{F}^{(-)}_B\right)\right] + \mathcal{F}^A\mathcal{F}^B\,\mathrm{tr}\left(\mathcal{F}^{(-)}_A\mathcal{F}^{(-)}_B\right) + \ldots \tag{4.3}$$

where the ellipsis denotes terms involving only $\mathcal{F}_{ABC}$, which are trivially invariant. Using the invariance of the expressions in (4.1) we see that the r.h.s. is invariant. Therefore the dilaton equation remains satisfied to order α' for such deformations.

Varying the action (2.32) with respect to the generalized vielbein using (2.16) the terms involving $\mathcal{F}_A$, i.e. the first two terms, in $\mathcal{R}^{(-)}$ (2.33) give the following contributions to the equations of motion

$$\begin{aligned}
&2(\partial^C - \mathcal{F}^C)\left[(\partial^D\mathcal{F}^{(+)}_A + \partial^{(+)}_A\mathcal{F}^D)\mathcal{F}^{(-)}_{DCB}\right] - (\partial_C - \mathcal{F}_C)\left[(\partial^D\mathcal{F}^{(+)C} + \partial^{(+)C}\mathcal{F}^D)\mathcal{F}^{(-)}_{DAB}\right]\\
&\quad + (\partial^{(-)}_A\mathcal{F}^C - \partial^C\mathcal{F}^{(+)}_A)\,\mathrm{tr}\left(\mathcal{F}^{(-)}_C\mathcal{F}^{(-)}_B\right) - (\partial^C\mathcal{F}^{(+)}_A + \partial^{(+)}_A\mathcal{F}^C)\,\mathrm{tr}\left(\mathcal{F}^{(-)}_C\mathcal{F}^{(--)}_B\right)\\
&\quad + 2(\partial^C\mathcal{F}^D + \partial^D\mathcal{F}^C)\mathcal{F}^{(+)E}{}_{CA}\mathcal{F}^{(-)}_{DEB} - (A \leftrightarrow B) + \ldots\,,
\end{aligned} \tag{4.4}$$

where we suppress terms that are manifestly invariant, i.e. constructed form the invariant combinations in (4.1). The variation of the $\mathcal{R}^2_{AB}$-term in (2.33) gives rise to the terms

$$\begin{aligned}
&4\partial^C\left[\partial^{(+)}_B\mathcal{F}^D\mathcal{F}^{(-)}_{DCA}\right] + 4\mathcal{F}^C(\partial^D - \mathcal{F}^D)\bar{\mathcal{R}}^{(-)}_{DBCA} - 4\partial^C\left[\mathcal{F}^D\mathcal{F}^{(++)}_{DBE}\mathcal{F}^{(-)E}{}_{CA}\right]\\
&\quad + 2\partial^{(+)}_A\mathcal{F}^C\,\mathrm{tr}\left(\mathcal{F}^{(-)}_C\mathcal{F}^{(--)}_B\right) - 4\partial^{(+)C}\mathcal{F}^D\mathcal{F}^{(+)E}{}_{CA}\mathcal{F}^{(-)}_{DEB}\\
&\quad + 2\mathcal{F}^C\mathcal{F}^{(++)}_{ACD}\,\mathrm{tr}\left(\mathcal{F}^{(-)D}\mathcal{F}^{(--)}_B\right) + 4\mathcal{F}_C\mathcal{F}^{(++)CEF}\mathcal{F}^{(+)D}{}_{EA}\mathcal{F}^{(-)}_{FDB}\\
&\quad - 2\mathcal{F}^C\mathcal{F}^{(++)DE}{}_A\bar{\mathcal{R}}^{(-)}_{DECB} - 4\mathcal{F}^C\mathcal{F}^{(-)DE}{}_B\bar{\mathcal{R}}^{(-)}_{ADEC}\\
&\quad + 4\mathcal{F}_C\mathcal{F}^{(-)DEC}\bar{\mathcal{R}}^{(-)}_{ADEB} - (A \leftrightarrow B) + \ldots\,,
\end{aligned} \tag{4.5}$$

where we have noted that using the definition (2.27) we have

$$\mathcal{F}^A\bar{\mathcal{R}}^{(-)}_{ABCD} = \partial^{(+)}_B\mathcal{F}^A\mathcal{F}^{(-)}_{ACD} - \mathcal{F}^A\mathcal{F}^{(++)}_{ABE}\mathcal{F}^{(-)E}{}_{CD} + \ldots\,. \tag{4.6}$$

Finally the variation of the $\mathcal{C}_{ABC}$-term in (2.33) gives

$$\begin{aligned}
&\partial_A\mathcal{F}^C\,\mathrm{tr}\left(\mathcal{F}^{(-)}_B\mathcal{F}^{(-)}_C\right) - \mathcal{F}^{(+)}_C\mathcal{F}^{CDE}\mathcal{R}^{(-)}_{DEAB} - 2\mathcal{F}^C\mathcal{F}^{(++)DE}{}_B\bar{\mathcal{R}}^{(-)}_{DECA}\\
&\quad - 4\mathcal{F}^C\mathcal{F}^{(+)DE}{}_B\mathcal{R}^{(-)}_{DECA} - \partial_C\left[\mathcal{F}_D\mathcal{F}^{(++)CDE}\mathcal{F}^{(-)}_{EAB}\right]\\
&\quad + 2(\partial^C - \mathcal{F}^C)\left[\mathcal{F}_D\mathcal{F}^{(++)DE}{}_A\mathcal{F}^{(-)}_{ECB}\right] + 2\mathcal{F}_C\partial^D\mathcal{F}^{(+)}_{DBE}\mathcal{F}^{(-)EC}{}_A
\end{aligned}$$

$$
\begin{aligned}
&+2\mathcal{F}_C\partial^D\mathcal{F}^{(++)}_{DBE}\mathcal{F}^{(-)EC}{}_A-\mathcal{F}^{(+)}_C\partial_D\mathcal{F}^{CDE}\mathcal{F}^{(-)}_{EAB}-\mathcal{F}_C\mathcal{F}_D\mathcal{F}^{(+)CDE}\mathcal{F}^{(-)}_{EAB}\\
&-2\mathcal{F}^C\mathcal{F}^D\mathcal{F}^{(+)}_{CEA}\mathcal{F}^{(-)E}{}_{DB}+2\mathcal{F}^C\mathcal{F}^{(+)}_{ACD}\,\mathrm{tr}\left(\mathcal{F}^{(-)D}\mathcal{F}^{(-)}_B\right)+\mathcal{F}^C\mathcal{F}^{(++)}_{ACD}\,\mathrm{tr}\left(\mathcal{F}^{(-)D}\mathcal{F}^{(-)}_B\right)\\
&-\mathcal{F}^C\mathcal{F}^{(++)}_{ACD}\,\mathrm{tr}\left(\mathcal{F}^{(-)D}\mathcal{F}^{(--)}_B\right)+2\mathcal{F}_C\mathcal{F}^{(++)CDE}\mathcal{F}_{DA}{}^F\mathcal{F}^{(-)}_{EBF}\\
&+\mathcal{F}_C\mathcal{F}^{(+)EFC}\mathcal{F}^{(+)}_{EFD}\mathcal{F}^{(-)D}{}_{AB}+2\mathcal{F}^C\mathcal{F}^{(+)}_{EFB}\mathcal{F}^{(+)EFD}\mathcal{F}^{(-)}_{DCA}\\
&-(A\leftrightarrow B)+\dots\,. \qquad (4.7)
\end{aligned}
$$

Now we need to add together these three potentially non-invariant contributions to the equations of motion.

Using the Bianchi identity for $\mathcal{F}_A$ (2.9) and noting also that the second term in (4.4) can be written

$$
\begin{aligned}
&2\partial^{(+)}_C\left(\partial^{(C}\mathcal{F}^{D)}\mathcal{F}^{(-)}_{DAB}\right)\\
&\quad=\partial^C\left(\mathcal{F}^D\mathcal{F}^{(++)}_{CDE}\mathcal{F}^{(-)E}{}_{AB}-2\partial^{(-)}_C\mathcal{F}^D\mathcal{F}^{(-)}_{DAB}\right)+2\mathcal{F}^C\partial_C\mathcal{F}^D\mathcal{F}^{(-)}_{DAB}+\dots\\
&\quad=\partial^C\left(\mathcal{F}^D[\mathcal{F}^{(++)}_{CDE}+2\mathcal{F}^{(+)}_{CDE}]\mathcal{F}^{(-)E}{}_{AB}\right)+2\mathcal{F}^C\partial_C\mathcal{F}^D\mathcal{F}^{(-)}_{DAB}+\dots \qquad (4.8)
\end{aligned}
$$

we find, after a bit of algebra, that all terms involving only $\mathcal{F}^{(+)}_A$ can be eliminated leaving the terms

$$
\begin{aligned}
&8\mathcal{F}^C\partial^D\partial^{(+)}_{[A}\mathcal{F}^{(-)}_{D]CB}-4\mathcal{F}^C\partial^D[\mathcal{F}^{(-)E}_{AC}\mathcal{F}^{(-)}_{DEB}]-4\mathcal{F}^C\partial^D[\mathcal{F}^{(-)E}_{AB}\mathcal{F}^{(-)}_{DCE}]\\
&\quad-4\mathcal{F}^C\mathcal{F}^{(++)E}_{AD}\partial^D\mathcal{F}^{(-)}_{ECB}-8\mathcal{F}^C\mathcal{F}^{(-)DE}{}_B\partial^{(+)}_{[A}\mathcal{F}^{(-)}_{D]EC}+8\mathcal{F}_C\mathcal{F}^{(-)DEC}\partial^{(+)}_{[A}\mathcal{F}^{(-)}_{D]EB}\\
&\quad+4\mathcal{F}^C\partial^D[\mathcal{F}^{(+)}_{DEA}\mathcal{F}^{(-)E}{}_{CB}]-8\mathcal{F}^C\mathcal{F}^D\partial^{(+)}_{[A}\mathcal{F}^{(-)}_{D]CB}-4\mathcal{F}^C\partial^{(+)}_A\mathcal{F}^D\mathcal{F}^{(-)}_{DCB}\\
&\quad-4\mathcal{F}^C\mathcal{F}^D\mathcal{F}^{(+)}_{CEA}\mathcal{F}^{(-)E}{}_{DB}+4\mathcal{F}^C\mathcal{F}^D\mathcal{F}^{(-)E}_{AC}\mathcal{F}^{(-)}_{DEB}+4\mathcal{F}_C\mathcal{F}_D\mathcal{F}^{(-)}_{ABE}\mathcal{F}^{(-)CDE}\\
&\quad-8\mathcal{F}_C\mathcal{F}^{(-)DEC}\mathcal{F}^{(++)F}_{AD}\mathcal{F}^{(-)}_{FEB}-8\mathcal{F}_C\mathcal{F}^{(-)DEC}\mathcal{F}^{(-)F}_{AE}\mathcal{F}^{(-)}_{DFB}\\
&\quad-4\mathcal{F}^C\mathcal{F}^{(-)}_{DEC}\mathcal{F}^{(-)DEF}\mathcal{F}^{(-)}_{ABF}-4\mathcal{F}^C\mathcal{F}^{(-)}_{AFC}\mathcal{F}^{(-)}_{DEB}\mathcal{F}^{(-)DEF}-4\mathcal{F}^C\mathcal{F}^{(-)}_{FCB}\mathcal{F}^{(+)}_{DEA}\mathcal{F}^{(+)DEF}\\
&\quad-2\mathcal{F}_C\left(\partial^{(-)C}\mathcal{F}^{(+)D}+(\partial_E-\mathcal{F}_E)\mathcal{F}^{(+)CDE}-\mathcal{F}^{(-)EFC}\mathcal{F}^{(+)D}_{FE}\right)\mathcal{F}^{(-)}_{DAB}\\
&\quad+2\left(\partial^{(+)}_C\mathcal{F}^{(-)}_A-\mathcal{F}^E\mathcal{F}^{(-)}_{CAE}\right)\mathrm{tr}\left(\mathcal{F}^{(-)}_B\mathcal{F}^{(-)C}\right)-(A\leftrightarrow B)+\dots \qquad (4.9)
\end{aligned}
$$

The last two terms drop out using the lowest order equations of motion (2.17). We now rewrite the first term as

$$
\begin{aligned}
&8\mathcal{F}^D\partial^C\partial^{(-)}_{[B}\mathcal{F}^{(+)}_{D]AC}-8\mathcal{F}^D\partial^C\left(\partial^{(+)}_{[A}\mathcal{F}^{(-)}_{C]BD}+\partial^{(-)}_{[B}\mathcal{F}^{(+)}_{D]AC}\right)\\
&=-8\mathcal{F}^D\partial^{(-)}_{[D}\left(\partial^{(-)}_{B]}\mathcal{F}^{(+)}_A+(\partial^C-\mathcal{F}^C)\mathcal{F}^{(+)}_{B]AC}-\mathcal{F}^{(-)}_{EFB}\mathcal{F}^{(+)FE}{}_A\right)-4\mathcal{F}^D\mathcal{F}^{(--)}_{BDE}\partial^E\mathcal{F}^{(+)}_A\\
&\quad-4\mathcal{F}^D\mathcal{F}^{(-)}_{EBD}\partial^E\mathcal{F}^{(+)}_A+8\mathcal{F}^D\mathcal{F}^{(+)}_{[D|AC|}\partial^{(-)}_{B]}\mathcal{F}^{(+)C}-8\mathcal{F}^C\mathcal{F}^D\partial^{(+)}_{[A}\mathcal{F}^{(-)}_{C]BD}\\
&\quad+4\mathcal{F}_D\mathcal{F}^{(+)}_{FEA}\partial^{(-)}_B\mathcal{F}^{(-)EFD}+8\mathcal{F}_D\mathcal{F}^{(-)EFD}\partial^{(+)}_{[A}\mathcal{F}^{(-)}_{E]BF}+4\mathcal{F}^D\mathcal{F}^{(+)E}_{BC}\partial^C\mathcal{F}^{(+)}_{DAE}\\
&\quad-4\mathcal{F}^D\mathcal{F}^{(-)E}{}_{CB}\partial^C\mathcal{F}^{(+)}_{DAE}-8\mathcal{F}^D(\partial^C-\mathcal{F}^C)\left(\partial^{(+)}_{[A}\mathcal{F}^{(-)}_{C]BD}+\partial^{(-)}_{[B}\mathcal{F}^{(+)}_{D]AC}\right)\\
&\quad-8\mathcal{F}_E\mathcal{F}^{(-)CDE}\left(\partial^{(+)}_{[A}\mathcal{F}^{(-)}_{C]BD}+\partial^{(-)}_{[B}\mathcal{F}^{(+)}_{D]AC}\right)+\dots \qquad (4.10)
\end{aligned}
$$

The first term vanishes by the lowest order equations of motion (2.17). In the last two terms we can use the Bianchi identity for $\mathcal{F}_{ABC}$ (2.9), which implies in particular that

$$
\begin{aligned}
2\partial^{(+)}_{[A}\mathcal{F}^{(-)}_{C]BD}+2\partial^{(-)}_{[B}\mathcal{F}^{(+)}_{D]AC}=&\mathcal{F}^{(-)}_{AB}{}^{E}\mathcal{F}^{(-)}_{CED}-\mathcal{F}^{(+)}_{BA}{}^{E}\mathcal{F}^{(+)}_{DCE}+\mathcal{F}^{(-)}_{AD}{}^{E}\mathcal{F}^{(-)}_{CBE}\\
&-\mathcal{F}^{(+)}_{DA}{}^{E}\mathcal{F}^{(+)}_{BEC}+\mathcal{F}^{(++)}_{AC}{}^{E}\mathcal{F}^{(-)}_{EBD}+\mathcal{F}^{(+)E}{}_{AC}\mathcal{F}^{(--)}_{BDE}\,. \qquad (4.11)
\end{aligned}
$$

After a bit of algebra we are left with

$$
\begin{aligned}
&8\mathcal{F}^{D}\mathcal{F}^{(+)}_{[D|A|}{}^{C}\left(\partial^{(-)}_{B]}\mathcal{F}^{(+)}_{C}+(\partial^{E}-\mathcal{F}^{E})\mathcal{F}^{(+)}_{B]CE}-\mathcal{F}^{(-)EF}{}_{B]}\mathcal{F}^{(+)}_{FEC}\right)\\
&\quad-4\mathcal{F}^{D}\mathcal{F}^{(--)}_{BD}{}^{E}\left(\partial^{(-)}_{E}\mathcal{F}^{(+)}_{A}+(\partial^{C}-\mathcal{F}^{C})\mathcal{F}^{(+)}_{EAC}-\mathcal{F}^{(-)}_{CFE}\mathcal{F}^{(+)FC}{}_{A}\right)\\
&\quad-4\mathcal{F}_{E}\mathcal{F}^{(-)CDE}\left(\mathcal{F}^{(-)}_{AD}{}^{F}\mathcal{F}^{(-)}_{CFB}-\mathcal{F}^{(+)}_{DA}{}^{F}\mathcal{F}^{(+)}_{BFC}+\mathcal{F}^{(++)}_{AC}{}^{F}\mathcal{F}^{(-)}_{FDB}+\mathcal{F}^{(+)F}{}_{AC}\mathcal{F}^{(--)}_{BDF}\right)\\
&\quad-8\mathcal{F}^{D}\mathcal{F}^{(-)CE}{}_{B}\left(\partial^{(+)}_{[A}\mathcal{F}^{(-)}_{C]ED}+\partial^{(-)}_{[E}\mathcal{F}^{(+)}_{D]AC}\right)+4\mathcal{F}^{D}\mathcal{F}^{(--)}_{BDE}\mathcal{F}^{(-)CEF}\mathcal{F}^{(+)}_{FCA}\\
&\quad-4\mathcal{F}^{C}\mathcal{F}^{(-)}_{AFC}\mathcal{F}^{(-)}_{DEB}\mathcal{F}^{(-)DEF}+4\mathcal{F}^{C}\mathcal{F}^{(+)DE}{}_{A}\left(\partial_{E}\mathcal{F}^{(--)}_{BCD}-\partial^{(-)}_{B}\mathcal{F}^{(-)}_{ECD}+\partial_{D}\mathcal{F}^{(-)}_{ECB}\right)\\
&\quad-4\mathcal{F}^{C}\mathcal{F}^{(-)}_{FCB}\mathcal{F}^{(+)}_{DEA}\mathcal{F}^{(+)DEF}-(A\leftrightarrow B)+\ldots\,. \qquad (4.12)
\end{aligned}
$$

The first two terms vanish by the lowest order equations of motion and the remaining terms cancel using the Bianchi identity for $\mathcal{F}_{ABC}$. This completes the proof that the α'-correction to the equations of motion can be cast in a manifestly invariant form provided that the expressions in (4.1) are invariant. In particular this implies that if a YB deformation preserves Weyl invariance at one-loop it also preserves it at two loops.

5 Conclusions

We have analyzed the conditions for a YB deformation of the bosonic/heterotic string sigma-model to be Weyl-invariant at one loop, i.e. for the corresponding background to be a (super)gravity solution. When $(G+B)_{mn}$ of the undeformed background is invertible one finds no solution in the inhomogeneous case (although our analysis for the YB model with WZ-term is not quite complete). For a homogeneous deformation of a symmetric space one finds that the distinguished Lie algebra element $R^{rs}f_{rs}{}^{t}T_{t}$ must belong to the center of the algebra. We showed that, at least for rank $R<8$, this in fact implies the usual unimodularity condition $R^{rs}f_{rs}{}^{t}=0$ of [12]. When $(G+B)_{mn}$ of the undeformed background is non-invertible instead the unimodularity condition is replaced by the weaker conditions (3.12), (3.20) together with the condition that $\tilde{K}=i_K B$ generate isometries of the undeformed background G,H,Φ. This is consistent with what has been seen in specific examples [15, 16] and the conditions we find agree with those coming from an analysis of generalized supergravity, see appendix E of [16], when specifying to YB deformations. We have also seen that when these conditions are satisfied the deformation in fact preserves Weyl-invariance at least to two loops, i.e. the background solves the low-energy effective string equations including the first α'-correction.

Interestingly, while in the case of unimodular deformations the fact that the two-loop equations are satisfied is trivial in the doubled formulation we are using, this is not the

JHEP10(2020)065

case for non-unimodular ones due to the shift of $\mathcal{F}_A$ by the generalized Killing vector K_A. In fact it took quite a bit of work to show that the equations of motion can be cast in a form where it is easy to see that they are invariant under the deformation. It would be interesting to understand if one can improve the formulation so that the invariance is manifest also in the non-unimodular case and, if so, what this implies for the structure of higher-derivative corrections. Perhaps the natural starting point to analyzing this question is the gauged version of DFT [20].

For unimodular YB deformations the first α'-correction to the deformed background was derived in [19], also by using the doubled formulation. The same correction is valid also for the non-unimodular examples discussed here.

It would be interesting to extend our analysis to the general case of inhomogeneous YB deformation with WZ-term by relaxing the requirement that B is invariant under the isometries. The conditions must become essentially the same in that case since they are mostly fixed by the generalized supergravity analysis. It would be interesting to understand if there exist any non-unimodular Weyl-invariant examples in that case. It seems unlikely to be the case since R is much more constrained than in the homogeneous case.

Finally, it would be interesting to extend the present analysis to the case of Poisson-Lie T-duality, for which the first α'-correction was recently found [38–40] using essentially the same approach as for YB.

Acknowledgments

We thank R. Borsato, D. Marqués and S. van Tongeren for interesting discussions and R. Borsato for comments on the manuscript. The work of LW is supported by the grant "Integrable Deformations" (GA20-04800S) from the Czech Science Foundation (GAČR).

References

[1] C. Klimčík, *Yang-Baxter σ-models and dS/AdS T duality*, JHEP **12** (2002) 051 [hep-th/0210095] [INSPIRE].

[2] C. Klimčík, *On integrability of the Yang-Baxter σ-model*, J. Math. Phys. **50** (2009) 043508 [arXiv:0802.3518] [INSPIRE].

[3] F. Delduc, M. Magro and B. Vicedo, *On classical q-deformations of integrable σ-models*, JHEP **11** (2013) 192 [arXiv:1308.3581] [INSPIRE].

[4] F. Delduc, M. Magro and B. Vicedo, *An integrable deformation of the* $AdS_5 \times S^5$ *superstring action*, Phys. Rev. Lett. **112** (2014) 051601 [arXiv:1309.5850] [INSPIRE].

[5] I. Kawaguchi, T. Matsumoto and K. Yoshida, *Jordanian deformations of the* $AdS_5 \times S^5$ *superstring*, JHEP **04** (2014) 153 [arXiv:1401.4855] [INSPIRE].

[6] B. Hoare and A.A. Tseytlin, *Homogeneous Yang-Baxter deformations as non-abelian duals of the* AdS_5 *σ-model*, J. Phys. A **49** (2016) 494001 [arXiv:1609.02550] [INSPIRE].

[7] R. Borsato and L. Wulff, *Integrable deformations of T-dual σ models*, *Phys. Rev. Lett.* **117** (2016) 251602 [arXiv:1609.09834] [INSPIRE].

[8] R. Borsato and L. Wulff, *On non-abelian T-duality and deformations of supercoset string σ-models*, *JHEP* **10** (2017) 024 [arXiv:1706.10169] [INSPIRE].

[9] G. Arutyunov, R. Borsato and S. Frolov, *Puzzles of η-deformed $AdS_5 \times S^5$*, *JHEP* **12** (2015) 049 [arXiv:1507.04239] [INSPIRE].

[10] G. Arutyunov, S. Frolov, B. Hoare, R. Roiban and A.A. Tseytlin, *Scale invariance of the η-deformed $AdS_5 \times S^5$ superstring, T-duality and modified type-II equations*, *Nucl. Phys. B* **903** (2016) 262 [arXiv:1511.05795] [INSPIRE].

[11] L. Wulff and A.A. Tseytlin, *κ-symmetry of superstring σ-model and generalized* 10*d supergravity equations*, *JHEP* **06** (2016) 174 [arXiv:1605.04884] [INSPIRE].

[12] R. Borsato and L. Wulff, *Target space supergeometry of η and λ-deformed strings*, *JHEP* **10** (2016) 045 [arXiv:1608.03570] [INSPIRE].

[13] R. Borsato and L. Wulff, *Non-abelian T-duality and Yang-Baxter deformations of Green-Schwarz strings*, *JHEP* **08** (2018) 027 [arXiv:1806.04083] [INSPIRE].

[14] D. Osten and S.J. van Tongeren, *Abelian Yang-Baxter deformations and TsT transformations*, *Nucl. Phys. B* **915** (2017) 184 [arXiv:1608.08504] [INSPIRE].

[15] J.-I. Sakamoto and Y. Sakatani, *Local β-deformations and Yang-Baxter σ-model*, *JHEP* **06** (2018) 147 [arXiv:1803.05903] [INSPIRE].

[16] R. Borsato and L. Wulff, *Marginal deformations of WZW models and the classical Yang-Baxter equation*, *J. Phys. A* **52** (2019) 225401 [arXiv:1812.07287] [INSPIRE].

[17] F. Delduc, M. Magro and B. Vicedo, *Integrable double deformation of the principal chiral model*, *Nucl. Phys. B* **891** (2015) 312 [arXiv:1410.8066] [INSPIRE].

[18] D. Geissbuhler, D. Marques, C. Núñez and V. Penas, *Exploring double field theory*, *JHEP* **06** (2013) 101 [arXiv:1304.1472] [INSPIRE].

[19] R. Borsato, A. Vilar López and L. Wulff, *The first α'-correction to homogeneous Yang-Baxter deformations using $O(d,d)$*, *JHEP* **07** (2020) 103 [arXiv:2003.05867] [INSPIRE].

[20] M. Graña and D. Marques, *Gauged double field theory*, *JHEP* **04** (2012) 020 [arXiv:1201.2924] [INSPIRE].

[21] R. Borsato and L. Wulff, *Two-loop conformal invariance for Yang-Baxter deformed strings*, *JHEP* **03** (2020) 126 [arXiv:1910.02011] [INSPIRE].

[22] G. Aldazabal, D. Marques and C. Núñez, *Double field theory: a pedagogical review*, *Class. Quant. Grav.* **30** (2013) 163001 [arXiv:1305.1907] [INSPIRE].

[23] O. Hohm, D. Lüst and B. Zwiebach, *The spacetime of double field theory: review, remarks, and outlook*, *Fortsch. Phys.* **61** (2013) 926 [arXiv:1309.2977] [INSPIRE].

[24] D.S. Berman and D.C. Thompson, *Duality symmetric string and M-theory*, *Phys. Rept.* **566** (2014) 1 [arXiv:1306.2643] [INSPIRE].

[25] W. Siegel, *Superspace duality in low-energy superstrings*, *Phys. Rev. D* **48** (1993) 2826 [hep-th/9305073] [INSPIRE].

[26] W. Siegel, *Two vierbein formalism for string inspired axionic gravity*, *Phys. Rev. D* **47** (1993) 5453 [hep-th/9302036] [INSPIRE].

[27] O. Hohm and S.K. Kwak, *Frame-like geometry of double field theory*, *J. Phys. A* **44** (2011) 085404 [arXiv:1011.4101] [INSPIRE].

[28] D. Marques and C.A. Núñez, *T-duality and α'-corrections*, *JHEP* **10** (2015) 084 [arXiv:1507.00652] [INSPIRE].

[29] W.H. Baron, J.J. Fernandez-Melgarejo, D. Marques and C. Núñez, *The odd story of α'-corrections*, *JHEP* **04** (2017) 078 [arXiv:1702.05489] [INSPIRE].

[30] T. Araujo et al., *Yang-Baxter σ-models, conformal twists, and noncommutative Yang-Mills theory*, *Phys. Rev. D* **95** (2017) 105006 [arXiv:1702.02861] [INSPIRE].

[31] J.-i. Sakamoto, Y. Sakatani and K. Yoshida, *Homogeneous Yang-Baxter deformations as generalized diffeomorphisms*, *J. Phys. A* **50** (2017) 415401 [arXiv:1705.07116] [INSPIRE].

[32] I. Bakhmatov, E. Ó Colgáin, M.M. Sheikh-Jabbari and H. Yavartanoo, *Yang-Baxter deformations beyond coset spaces (a slick way to do TsT)*, *JHEP* **06** (2018) 161 [arXiv:1803.07498] [INSPIRE].

[33] T. Araujo, E. Ó Colgáin, J. Sakamoto, M.M. Sheikh-Jabbari and K. Yoshida, *I in generalized supergravity*, *Eur. Phys. J. C* **77** (2017) 739 [arXiv:1708.03163] [INSPIRE].

[34] L. Wulff, *Trivial solutions of generalized supergravity vs non-abelian T-duality anomaly*, *Phys. Lett. B* **781** (2018) 417 [arXiv:1803.07391] [INSPIRE].

[35] A. Medina and P. Revoy, *Lattices in symplectic Lie groups*, *J. Lie Theor.* **17** (2007) 27.

[36] G.P. Ovando, *Four dimensional symplectic Lie algebras*, *Beit. Alg. Geom.* **47** (2006) 419 [math/0407501].

[37] R. Campoamor-Stursberg, *Symplectic forms on six-dimensional real solvable Lie algebras I*, *Alg. Colloq.* **16** (2009) 253 [math/0507499].

[38] R. Borsato and L. Wulff, *Quantum correction to generalized T-dualities*, arXiv:2007.07902 [INSPIRE].

[39] F. Hassler and T. Rochais, *α'-corrected Poisson-Lie T-duality*, arXiv:2007.07897 [INSPIRE].

[40] T. Codina and D. Marques, *Generalized dualities and higher derivatives*, arXiv:2007.09494 [INSPIRE].

JHEP10(2020)065

Quantum Correction to Generalized T Dualities

Riccardo Borsato *

Instituto Galego de Física de Altas Enerxías (IGFAE), Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain

Linus Wulff †

Department of Theoretical Physics and Astrophysics, Masaryk University, 611 37 Brno, Czech Republic

(Received 22 July 2020; accepted 29 September 2020; published 13 November 2020)

Poisson-Lie duality is a generalization of Abelian and non-Abelian T duality, and it can be viewed as a map between solutions of the low-energy effective equations of string theory, i.e., at the (super) gravity level. We show that this fact extends to the next order in α' (two loops in σ-model perturbation theory) provided that the map is corrected. The α' correction to the map is induced by the anomalous Lorentz transformations of the fields that are necessary to go from a doubled $O(D, D)$-covariant formulation to the usual (super)gravity description.

DOI: 10.1103/PhysRevLett.125.201603

Introduction.—The notion of T duality [1,2] is central in string theory. It says that a closed string on a background with Abelian isometries has another description as a string on a dual background. In the simplest case of T duality on a circle, the duality acts by inverting the radius of the circle. More generally, backgrounds may have non-Abelian isometry groups, and at least at the classical level there is indeed a generalization to a non-Abelian version of T duality [3]. Unlike in the Abelian case, non-Abelian T duality (NATD) does not generically preserve the isometries of the background, and it is therefore not obvious how to invert the transformation. This problem was overcome by Klimčík and Ševera in [4,5]. They realized that the map can be made invertible by relaxing the notion of isometry. One requires the background to have instead so-called Poisson-Lie (PL) symmetry, namely to possess vector fields v_i, with $[v_i, v_j] = -f_{ij}{}^k v_k$, under which the metric and B field of the σ-model transform as

$$\mathcal{L}_{v_i} M_{mn} = -\tilde{f}^{jk}{}_i v_j{}^p v_k{}^q M_{mp} M_{qn}, \tag{1}$$

where $M_{mn} = G_{mn} - B_{mn}$ and $\tilde{f}^{jk}{}_i$ are structure constants of a dual Lie algebra. This more general notion of symmetry allows to define a dual background (see below). This construction became known as "Poisson-Lie T duality" since the group structure underlying it is that of a PL group. The σ models on the original and dual backgrounds are classically equivalent being related by a canonical transformation [6]. When the dual structure constants $\tilde{f}$ vanish, v_i generate standard isometries, and one recovers (N)ATD.

At the world sheet quantum level, i.e., including string α' corrections, things are more subtle. While Abelian T duality remains a symmetry of the world sheet conformal field theory to all orders in α', it was quickly realized that NATD cannot be a symmetry at the quantum level [7]. At best it can map one world sheet conformal field theory to another—inequivalent—one. It can therefore be used to generate new string backgrounds from old ones. Except for an anomaly when dualizing nonunimodular groups [8,9], this has been shown to work to zeroth order in α', i.e., at the low-energy (supergravity) level of the string effective equations, which corresponds to one loop order in σ-model perturbation theory. Similar results are known for PL duality, see, e.g., [10,11]. It has been a long-standing problem whether PL and NATD can be extended beyond this lowest order.

Here, we show that PL duality can be extended to order α', i.e., two loops in the σ-model perturbation theory, provided that the map is corrected. A special case of our results gives the corrections to NATD. When specifying to the Abelian case we recover the results of [12].

To find this correction we exploit a powerful formulation of the string effective equations inspired from double field theory (DFT). It has long been known that the bosonic string compactified on a d torus has an $O(d, d)$ T duality symmetry [13]. DFT is a field theory where this symmetry is made manifest form the start [14–18] and is therefore well suited to working with T duality. This is achieved by doubling the dimension of the physical manifold, and by imposing a "section condition" which effectively eliminates the dependence on half of the coordinates, giving the correct dimension in the end ($D = 26$ for the bosonic string and $D = 10$ for the superstring). Here, we always work with the standard choice of section, so that the background depends only on the physical coordinates. In this formulation it is rather the dimension of the *tangent* space that is doubled, and we have two copies of the Lorentz group instead of one [19]. The standard Lorentz group is the diagonal of the doubled one, and under this breaking the equations of DFT reduce to the standard string effective equations, at lowest order in α'. A crucial point is that at the quantum level it is impossible to preserve both the $O(D,D)$ and the Lorentz covariance of the fields [20–23]. If we insist on fields which transform nicely under T duality and $O(D,D)$, they must transform noncovariantly under Lorentz transformations [24] (see [25] for another manifestation of this fact). The fact that the Lorentz transformation of the fields receives corrections at order α' makes the discussion of the Lorentz invariance of the theory nontrivial. But this can be turned into a virtue rather than a shortcoming. In fact the α' correction to the Lorentz transformation fixes the correction to the DFT action [24]. Remarkably, this α'-corrected $O(D,D)$-covariant action correctly reproduces the α' corrections to the bosonic and heterotic string effective actions [24].

Our strategy is to use the rewriting of PL duality in the doubled language, where it takes a natural form, see, e.g., [26–30]. The basic fields of the formulation we use, the "generalized fluxes," turn out to be invariant under PL duality. Since the string effective equations, including the first α' correction, can be written in terms of the generalized fluxes [24,31], at least to this order PL duality maps solutions of the doubled equations to solutions. At the standard (super)gravity level there are in fact explicit corrections to the PL duality rules. They arise from the noncovariance of the doubled fields under the double Lorentz transformation needed to gauge fix down to the diagonal subgroup and to go to the standard (nondoubled) description. See Fig. 1 for a summary. This strategy was used in [32] to find the α' correction to the "homogeneous Yang-Baxter deformations" (related to NATD [33,34]) and

$$
\begin{tikzcd}
E,d \arrow[rr, leftrightarrow, "\text{Double PL map}"] \arrow[d, "\Lambda^{(+)}=\Lambda"'] & & \tilde{E},\tilde{d} \arrow[d, "\Lambda^{(+)}=\tilde{\Lambda}"] \\
E',d' \arrow[d, "\text{gauge fix}"'] & & \tilde{E}',\tilde{d}' \arrow[d, "\text{gauge fix}"] \\
G,B,\Phi \arrow[rr, leftrightarrow, dashed, "\text{PL duality}"] & & \tilde{G},\tilde{B},\tilde{\Phi}
\end{tikzcd}
$$

FIG. 1. Starting with the PL duality map for the doubled fields (E,d), the map for the standard (super)gravity fields (G,B,Φ) is obtained after a double Lorentz transformation $(\Lambda^{(+)},\Lambda^{(-)}) = (\Lambda,1)$ to set $e^{(+)} = e^{(-)}$, thus breaking the double Lorentz group down to its diagonal subgroup. The α' corrections to the PL duality map follow from the anomalous Lorentz transformations of the fields.

it works for any $O(D,D)$ transformation leaving the generalized fluxes invariant.

Poisson-Lie duality.—In PL duality, $f_{ij}{}^k$ and $\tilde{f}^{ij}{}_k$ are interpreted as structure constants of Lie groups denoted by G and $\tilde{G}$. These are combined into a "Drinfel'd double" $\mathcal{D}$ whose Lie algebra is generated by $T_I = (T_i, \tilde{T}^i)$, where T_i are generators of $\mathrm{Lie}(G)$, and $\tilde{T}^i$ of $\mathrm{Lie}(\tilde{G})$. Obviously $\mathrm{Lie}(G)$ and $\mathrm{Lie}(\tilde{G})$ are subalgebras of $\mathcal{D}$ but there are also mixed commutation relations

$$
\begin{aligned}
&[T_i, T_j] = f_{ij}{}^k T_k, \qquad [\tilde{T}^i, \tilde{T}^j] = \tilde{f}^{ij}{}_k \tilde{T}^k, \\
&[T_i, \tilde{T}^j] = \tilde{f}^{jk}{}_i T_k - f_{ik}{}^j \tilde{T}^k.
\end{aligned} \tag{2}
$$

Importantly, $\mathcal{D}$ is endowed with the invariant symmetric bilinear form $\langle T_I, T_J \rangle$ defined by

$$\langle T_i, T_j \rangle = \langle \tilde{T}^i, \tilde{T}^j \rangle = 0, \qquad \langle T_i, \tilde{T}^j \rangle = \delta_i^j. \tag{3}$$

Having introduced $\mathcal{D}$ we can now present PL duality as an invertible map between an "original" background (specified by a metric G_{mn}, a Kalb-Ramond field B_{mn}, and a dilaton Φ) and another "dual" background (with fields $\tilde{G}_{mn}$, $\tilde{B}_{mn}$, and $\tilde{\Phi}$). We split the coordinates of the original background as $x^m = (y^\sigma, x^\mu)$, where y^σ are coordinates on the group G to be dualized, and x^μ are coordinates that play the role of spectators under the dualization. Similarly, for the dual background we have $\tilde{x}^m = (\tilde{y}^\sigma, x^\mu)$ with $\tilde{y}^\sigma$ coordinates on $\tilde{G}$. The y and $\tilde{y}$ dependence is in fact encoded in the group elements $g(y) \in G$ and $\tilde{g}(\tilde{y}) \in \tilde{G}$ featuring below. To present the map between the original and dual backgrounds we first need the fact that the condition (1) implies that $M_{mn} \equiv G_{mn} - B_{mn}$ is of the form

$$M = U\dot{M}(1 + \Pi\dot{M})^{-1}U^T, \tag{4}$$

where we suppressed matrix indices for readability. The matrix $U_m{}^r$ depends only on y and it is of block form with nonvanishing components $U_\mu{}^\nu = \delta_\mu{}^\nu$ and $U_\sigma{}^i = u_\sigma{}^i$, the latter being the components of the Maurer-Cartan form $u = g^{-1}vg = g^{-1}dg = dy^\sigma u_\sigma{}^i T_i$ [35]. The matrix Π^{rs} depends only on y and its only nontrivial components are

$$\Pi^{ij} = \langle Ad_g^{-1} \circ \mathsf{P} \circ Ad_g \tilde{T}^i, \tilde{T}^j \rangle, \tag{5}$$

where $Ad_g X = gXg^{-1}$ and P is the projector on $\mathrm{Lie}(G)$. Notice that in general $\Pi \neq 0$ thanks to the mixed commutation relation of $\mathcal{D}$ if $\tilde{f}^{ij}{}_k \neq 0$. The map between M_{mn} and $\tilde{M}_{mn}$ is achieved by relating both of them to $\dot{M}_{rs}$, a matrix depending only on spectators x^μ and on which no other condition is imposed [36]. The dual background $\tilde{M}_{mn}$ is obtained by [37]

$$\tilde{M} = \tilde{U}[(\dot{M} + \tilde{\Pi})P + \bar{P}]^{-1}(\dot{M}\,\bar{P} + P)\tilde{U}^T, \tag{6}$$

where $\tilde{U}_\mu{}^\nu = \delta_\mu{}^\nu$, $\tilde{U}_\sigma{}^i = \tilde{u}_{\sigma j}\delta^{ji}$ and P, $\bar{P}$ project on indices i, j and μ, ν, respectively. As previously $\tilde{u} = \tilde{g}^{-1}d\tilde{g} = d\tilde{y}^\sigma \tilde{u}_{\sigma i}\tilde{T}^i$ is a Maurer-Cartan form, and now $\tilde{\Pi}_{ij} = \langle Ad_{\tilde{g}}^{-1}\circ\tilde{\mathsf{P}}\circ Ad_{\tilde{g}}T_i, T_j\rangle$, where $\tilde{\mathsf{P}}$ projects on Lie($\tilde{G}$). Finally, the dilatons of the two backgrounds are related by [38]

$$\exp(-2\Phi)\frac{(\det G)^{1/2}}{\det u} = \exp(-2\tilde{\Phi})\frac{(\det \tilde{G})^{1/2}}{\det \tilde{u}}. \qquad (7)$$

Taking $\tilde{G}$ Abelian ($\tilde{f}^{ij}{}_k = 0$) implies $\Pi = 0$ and Eq. (4) simplifies to $M = U\dot{M}U^T$, encoding the usual consequences of having isometries for M. Then parametrizing the Abelian group as $\tilde{g} = \exp(\tilde{y}_i\tilde{T}^i)$ with $\tilde{y}_i = \tilde{y}^\sigma\delta_{\sigma i}$ it follows that $\tilde{u}_{\sigma i} = \delta_{\sigma i}$, $\tilde{\Pi}_{ij} = \tilde{y}_k f_{ij}{}^k$, and from (6) and (7) we recover the rules of NATD in the presence of spectators [39]. Even simpler is the case when also G is Abelian ($f_{ij}{}^k = 0$) so that M is invariant under dim(G) $U(1)$ isometries. Then also $\tilde{\Pi} = 0$ and Eq. (6) implements dim (G) factorized T dualities, reducing to the celebrated Buscher rules when only one isometry is dualized.

Double formulation.—The nonlinear maps in (4) and (6) admit a much simpler and linear formulation in the doubled language, where one works with matrices $\mathcal{O}_M{}^N$ of dimension $2D \times 2D$. These are elements of the group $O(D, D)$, meaning that $\mathcal{O}_M{}^P\mathcal{O}_N{}^Q\eta_{PQ} = \eta_{MN}$ where

$$\eta_{MN} = \begin{pmatrix} 0 & \delta^m{}_n \\ \delta_m{}^n & 0 \end{pmatrix}. \qquad (8)$$

In fact let us construct the (inverse) "generalized vielbein" which we parametrize as

$$E_A{}^M = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{(+)an}M_{nm} & e^{(+)am} \\ -e_a^{(-)n}M_{nm} & e_a^{(-)m} \end{pmatrix}, \qquad (9)$$

where A is a flat index and M curved. We use similar parametrizations for $\tilde{E}_A{}^M$ and $\dot{E}_A{}^R$, adding tildes and dots. Above, $e^{(\pm)}$ are two possible vielbeins for the metric G_{mn}. They are not necessarily equal and in general they are related by a nontrivial Lorentz transformation. Each of them transform under only one of the two copies of the Lorentz group [distinguished by the (+) and (−)] arising in the doubled formulation. The generalized vielbein is one of the main ingredients of the "framelike formulation" of DFT, and it will be important for our derivation of the "unimodularity condition" (18) and the α' corrections to PL duality. It is straightforward to check that the relations (4) and (6) are equivalent to the relations

$$E = \dot{E}(1+\Pi)\mathcal{U}, \qquad \tilde{E} = \dot{E}(1+\tilde{\Pi})\tilde{\mathcal{U}}, \qquad (10)$$

where we suppressed indices. In our notation all dotted quantities only depend on x^μ. The nonvanishing components of $\Pi_R{}^S$ and $\tilde{\Pi}_R{}^S$ are again only Π^{ij} and $\tilde{\Pi}_{ij}$ and the antisymmetry properties $\Pi^{ij} = -\Pi^{ji}$ and $\tilde{\Pi}_{ij} = -\tilde{\Pi}_{ji}$ imply that $(1+\Pi)$, $(1+\tilde{\Pi})$ are elements of $O(D, D)$. The matrices $\mathcal{U}, \tilde{\mathcal{U}}$ are also elements of $O(D, D)$ with $\mathcal{U}_i{}^\sigma = u_i{}^\sigma$, $\mathcal{U}^i{}_\sigma = u_\sigma{}^i$, $\tilde{\mathcal{U}}^{i\sigma} = \tilde{u}^{i\sigma}$, $\tilde{\mathcal{U}}_{i\sigma} = \tilde{u}_{\sigma i}$, $\mathcal{U}_\mu{}^\nu = \mathcal{U}^\mu{}_\nu = \tilde{\mathcal{U}}_\mu{}^\nu = \tilde{\mathcal{U}}^\mu{}_\nu = \delta_\mu{}^\nu$. We are using a notation so that u_i^σ and $\tilde{u}^{i\sigma}$ are the inverses of $u_\sigma{}^i$ and $\tilde{u}_{\sigma i}$, respectively. To match (4) and (6) with (10) one finds that the (+) and (−) vielbeins must transform differently

$$e_a^{(\pm)m} = \dot{e}_a^{(\pm)s}O_{(\pm)s}{}^r(U^{-1})_r{}^m,$$
$$\tilde{e}_a^{(\pm)m} = \dot{e}_a^{(\pm)s}\tilde{O}_{(\pm)s}{}^r(\tilde{U}^{-1})_r{}^m, \qquad (11)$$

where

$$O_{(+)} = 1 + \dot{M}\Pi, \qquad \tilde{O}_{(+)} = \bar{P} + (\tilde{\Pi} + \dot{M})P,$$
$$O_{(-)} = 1 - \dot{M}^T\Pi, \qquad \tilde{O}_{(-)} = \bar{P} + (\tilde{\Pi} - \dot{M}^T)P. \qquad (12)$$

In both cases the (+) and (−) vielbeins are then related by Lorentz transformations as $e_a^{(-)m} = \Lambda_a{}^b e_b^{(+)m}$ and $\tilde{e}_a^{(-)m} = \tilde{\Lambda}_a{}^b\tilde{e}_b^{(+)m}$ where

$$\Lambda = \dot{e}^{-1}O_{(-)}O_{(+)}^{-1}\dot{e}, \qquad \tilde{\Lambda} = \dot{e}^{-1}\tilde{O}_{(-)}\tilde{O}_{(+)}^{-1}\dot{e}, \qquad (13)$$

if we fix $\dot{e} = \dot{e}^{(+)} = \dot{e}^{(-)}$. Finally, the transformation (7) is translated into

$$d + \frac{1}{2}\log\det u = \dot{d} = \tilde{d} + \frac{1}{2}\log\det\tilde{u}, \qquad (14)$$

where $d, \dot{d}, \tilde{d}$ are called "generalized dilatons" and are parametrized as in $d = \Phi - \frac{1}{4}\log\det G$ [41].

PL duality as a map between string backgrounds.—The double formulation is very useful because in this language it is very simple to prove that the PL duality transformation is a solution generating technique in string theory, at least to leading and subleading order in the α' expansion. From $E_A{}^M$ and d one can construct the generalized fluxes

$$\mathcal{F}_{ABC} = 3E_{[A}{}^M\partial_M E_B{}^N E_{C]N},$$
$$\mathcal{F}_A = E^{BM}\partial_M E_B{}^N E_{AN} + 2E_A{}^M\partial_M d, \qquad (15)$$

that are the dynamical fields of the framelike formulation of DFT. In fact the DFT equations of motion can be written only in terms of the above fluxes and their flat derivatives $\partial_A\mathcal{F} = E_A{}^M\partial_M\mathcal{F}$, both at leading and subleading order in the α' expansion [31]. Under the transformation (10) we have

$$\mathcal{F}_{ABC} = 3\dot{E}_{[A}{}^A\partial_\mu\dot{E}_B{}^N\dot{E}_{C]N}$$
$$+ 3\dot{E}_{[A}{}^i\dot{E}_B{}^j\dot{E}_{C]k}f_{ij}{}^k + 3\dot{E}_{[A}{}^i\dot{E}_{Bj}\dot{E}_{C]k}\tilde{f}^{jk}{}_i, \qquad (16)$$

$$\mathcal{F}_A = \dot{E}^{B\mu}\partial_\mu \dot{E}_B{}^N \dot{E}_{AN} + 2\dot{E}_A{}^\mu \partial_\mu \dot{d} + \dot{E}_A{}^i f_{ij}{}^j - \dot{E}_{Ai}(\tilde{f}^{ij}{}_j + f_{jk}{}^i \Pi^{jk}). \tag{17}$$

For the reader's convenience we give the details of the computation in the Supplemental Material [37]. The results for the dual background are analogous upon exchanging tilded and untilded quantities, and appropriately raising or lowering i, j, k indices. Because of the symmetry of (16) under this transformation, it immediately follows that $\mathcal{F}_{ABC} = \tilde{\mathcal{F}}_{ABC}$. Equation (17) instead is not symmetric under this transformation, but it becomes symmetric if we impose the tracelessness of the structure constants

$$f_{ij}{}^j = 0, \qquad \tilde{f}^{ij}{}_j = 0, \tag{18}$$

as detailed in the Supplemental Material. When this "unimodularity condition" holds we have simply

$$\mathcal{F}_A = \dot{E}^{B\mu}\partial_\mu \dot{E}_B{}^N \dot{E}_{AN} + 2\dot{E}_A{}^\mu \partial_\mu \dot{d}, \tag{19}$$

and $\mathcal{F}_A = \tilde{\mathcal{F}}_A$ immediately follows. Notice that not only both fluxes but also their flat derivatives are invariant under the PL transformation. In fact, since they only depend on spectators x^μ it follows that $E_A{}^M \partial_M \mathcal{F} = E_A{}^\mu \partial_\mu \mathcal{F} = \tilde{E}_A{}^\mu \partial_\mu \tilde{\mathcal{F}} = \tilde{E}_A{}^M \tilde{\partial}_M \tilde{\mathcal{F}}$.

If we start from a string background, or in other words given a model with $E_A{}^M$ and d of the PL form (10) and (14) that satisfies the doubled equations of motion to zeroth and first order in α', we conclude that the dual model given by $\tilde{E}_A{}^M$ and $\tilde{d}$ also satisfies the same equations, at least when (18) holds. This observation extends to higher orders under the assumption that there exists a formulation of the string effective action in terms of the generalized fluxes and their flat derivatives [31] also at higher orders in α'. This in turn should be true as long as it is possible to make diffeomorphisms, B-field gauge transformations, and $O(D,D)$ symmetry manifest.

This is a proof that PL duality is a solution generating technique in string theory at least when both structure constants are traceless, as found already to lowest order in [42]. When $\tilde{G}$ is Abelian this condition reduces to the unimodularity condition for NATD [8,9].

α' corrections.—So far, we have shown that in the doubled formulation the PL duality transformation works and remains uncorrected at least to order α'. Note that the assumption is that the DFT equations are satisfied without the need of correcting the $O(D,D)$ form (10) of the PL transformation, and therefore only $\dot{M}$ and $\dot{d}$ in (10) and (14) can depend on α'.

The description of the two models in terms of standard (i.e., nondoubled) fields (G, B, Φ) and $(\tilde{G}, \tilde{B}, \tilde{\Phi})$ is different, and the PL duality transformation between these does receive α' corrections. The reason is that when going from a doubled to a standard (super)gravity formulation we must first perform a double Lorentz transformation to set the two vielbeins $e^{(+)}$ and $e^{(-)}$ equal [24]. At order α' the fields of the doubled formulation transform noncovariantly under local Lorentz transformations, and this induces extra α' corrections also for the standard fields. The situation is illustrated in Fig. 1. Because of the noncovariance even under the *diagonal* of the double Lorentz group, we say that the reduction from the doubled to the standard formulation picks a specific noncovariant "scheme," which we call the scheme of DFT. To translate our results into the covariant schemes of [43–46] one must implement α'-dependent field redefinitions. We provide a dictionary [47] in the Supplemental Material [37].

The first α' correction to M_{mn} induced by the compensating double Lorentz transformation with $\Lambda^{(+)}$ and $\Lambda^{(-)}$ is [48]

$$a\Delta^{(-)}_{\Lambda^{(-)}} M^{(\mathrm{DFT})}_{nm} + b\Delta^{(+)}_{\Lambda^{(+)}} M^{(\mathrm{DFT})}_{mn}, \tag{20}$$

where $a = b = -\alpha'$ for the bosonic string and $a = -\alpha$, $b = 0$ for the heterotic string (and $a = b = 0$ for type II). The finite form of the anomalous transformations is [37]

$$\Delta^{(\pm)}_\Lambda M^{(\mathrm{DFT})}_{mn} = \frac{1}{2}\mathrm{tr}(\partial_m \Lambda \Lambda^{-1} \omega^{(\pm)}_n) - B^{\mathrm{WZW},(\Lambda)}_{mn} + \frac{1}{4}\mathrm{tr}(\partial_m \Lambda \Lambda^{-1} \partial_n \Lambda \Lambda^{-1}), \tag{21}$$

where $\omega^{(\pm)b}_{ma} = \omega_{ma}{}^b \pm \frac{1}{2} H_{ma}{}^b$ and ω is the spin connection for the vielbein e after the diagonal gauge fixing. The WZW-like contribution to B is defined by

$$dB^{\mathrm{WZW},(\Lambda)} = -\frac{1}{12}\mathrm{tr}(d\Lambda\Lambda^{-1} d\Lambda\Lambda^{-1} d\Lambda\Lambda^{-1}). \tag{22}$$

The α' corrections to the original model can be obtained, for example, after choosing $e = e^{(-)}$ and doing the double Lorentz transformation on $e^{(\pm)}$ to achieve the diagonal gauge with $(\Lambda^{(+)}, \Lambda^{(-)}) = (\Lambda, 1)$ and Λ given in (13) [49]. Then the correction is

$$\Delta M^{(\mathrm{DFT})} = b\Delta^{(+)}_\Lambda M^{(\mathrm{DFT})} + \alpha' U(1 + \dot{M}\Pi)^{-1} \Delta\dot{M} (1 + \Pi\dot{M})^{-1} U^T, \tag{23}$$

where the second term comes when expanding (4) with the α' corrections $\dot{M} \to \dot{M} + \alpha' \Delta\dot{M}$. Notice that for the heterotic string ($b = 0$) the PL map is uncorrected in the DFT scheme in the gauge $e = e^{(-)}$ [50]. For the dual background the same reasoning applies, and choosing $\tilde{e} = \tilde{e}^{(-)}$

$$\Delta\tilde{M}^{(\mathrm{DFT})} = b\Delta^{(+)}_{\tilde{\Lambda}}\tilde{M}^{(\mathrm{DFT})} + \alpha'\tilde{U}(\dot{M}P + \tilde{\Pi} + \bar{P})^{-1}\Delta\dot{M}(-P\tilde{U}^{-1}\tilde{M} + \bar{P}\tilde{U}^{T}). \tag{24}$$

The transformation of the dilatons follows from the fact that the generalized dilaton (14) is not anomalous under Lorentz [24] and that the parametrization in terms of standard metric and dilaton holds to α' order. Then

$$\Delta\Phi^{(\mathrm{DFT})} = \alpha'\Delta\dot{d} + \frac{1}{4}G^{mn}\Delta G^{(\mathrm{DFT})}_{mn},$$
$$\Delta\tilde{\Phi}^{(\mathrm{DFT})} = \alpha'\Delta\dot{d} + \frac{1}{4}\tilde{G}^{mn}\Delta\tilde{G}^{(\mathrm{DFT})}_{mn}, \tag{25}$$

where we allowed an α' correction $\dot{d} \to \dot{d} + \alpha'\Delta\dot{d}$.

We refer to the Supplemental Material for an example of a computation of such α' corrections, and for Refs. [51,52]. When specifying the map to a single $U(1)$ T duality, Eqs. (24) and (25) reproduce the rules written in [12] by Kaloper and Meissner, as proved in [32].

Conclusions.—In this Letter, we employed the framelike formulation of DFT to show that [when the conditions in (18) hold] PL duality is a map between solutions of the low-energy effective string equations at least to first order in α' and quite possibly to all orders. We did this for a two-parameter family of theories interpolating between the bosonic and the heterotic string (when the gauge fields and fermions of the latter are set to zero). It would be interesting to generalize these results to the case in which G, for example, is replaced by the coset G/H.

The importance of Eqs. (23), (24), and (25) is twofold. First, they provide the necessary quantum corrections to the PL duality transformation rules in order to extend the map to order α'. Second, they imply that the form of the α' corrections of backgrounds admitting PL symmetry is strongly constrained by the PL symmetry itself [53]. In particular, Eqs. (23) and (25) can be interpreted as an efficient way to compute α' corrections for PL symmetric backgrounds, since the only unknowns are $\Delta\dot{M}$ and $\Delta\dot{d}$, and they can be found by imposing the order α' equations of motion. This is much simpler than trying to compute the corrections directly for M and Φ. It would be interesting to see if, when considering nonconformal σ models, the α' corrections that we find preserve the form of the β functions.

The work of R. B. is supported by the fellowship of “la Caixa Foundation” (ID 100010434) with code LCF/BQ/PI19/11690019, by Agencia Estatal de Investigación-Spain (FPA2017-84436-P and Unidad de Excelencia María de Maetzu MDM-2016-0692), by Xunta de Galicia-Consellería de Educación (Centro singular de investigación de Galicia accreditation 2019–2022, ED431C-2017/07 and ED431G2019/05), and by FEDER. The work of L. W. is supported by the grant “Integrable Deformations” (GA20-04800S) from the Czech Science Foundation (GACR).

Note added.—When this work was being written up we learned of the closely related independent work [54], and shortly after of [55].

*riccardo.borsato@usc.es
†wulff@physics.muni.cz

[1] T. H. Buscher, Phys. Lett. B **194**, 59 (1987).
[2] T. Buscher, Phys. Lett. B **201**, 466 (1988).
[3] X. C. de la Ossa and F. Quevedo, Nucl. Phys. **B403**, 377 (1993).
[4] C. Klimcik and P. Severa, Phys. Lett. B **351**, 455 (1995).
[5] C. Klimcik and P. Severa, Phys. Lett. B **372**, 65 (1996).
[6] K. Sfetsos, Nucl. Phys. **B517**, 549 (1998).
[7] A. Giveon and M. Rocek, Nucl. Phys. **B421**, 173 (1994).
[8] E. Alvarez, L. Alvarez-Gaume, and Y. Lozano, Nucl. Phys. **B424**, 155 (1994).
[9] S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer, and G. Veneziano, Nucl. Phys. **B435**, 147 (1995).
[10] A. Alekseev, C. Klimcik, and A. A. Tseytlin, Nucl. Phys. **B458**, 430 (1996).
[11] E. Tyurin and R. von Unge, Phys. Lett. B **382**, 233 (1996).
[12] N. Kaloper and K. A. Meissner, Phys. Rev. D **56**, 7940 (1997).
[13] A. Giveon, M. Porrati, and E. Rabinovici, Phys. Rep. **244**, 77 (1994).
[14] W. Siegel, Phys. Rev. D **47**, 5453 (1993).
[15] W. Siegel, Phys. Rev. D **48**, 2826 (1993).
[16] G. Aldazabal, D. Marques, and C. Nunez, Classical Quantum Gravity **30**, 163001 (2013).
[17] D. S. Berman and D. C. Thompson, Phys. Rep. **566**, 1 (2015).
[18] O. Hohm, D. Lüst, and B. Zwiebach, Fortschr. Phys. **61**, 926 (2013).
[19] O. Hohm and S. K. Kwak, J. Phys. A **44**, 085404 (2011).
[20] O. Hohm and B. Zwiebach, J. High Energy Phys. 05 (2012) 126.
[21] O. Hohm, W. Siegel, and B. Zwiebach, J. High Energy Phys. 02 (2014) 065.
[22] O. Hohm and B. Zwiebach, J. High Energy Phys. 11 (2014) 075.
[23] O. Hohm and B. Zwiebach, J. High Energy Phys. 01 (2015) 012.
[24] D. Marqués and C. A. Nuñez, J. High Energy Phys. 10 (2015) 084.
[25] C. Eloy, O. Hohm, and H. Samtleben, Phys. Rev. Lett. **124**, 091601 (2020).
[26] F. Hassler, Phys. Lett. B **807**, 135455 (2020).
[27] B. Jurco and J. Vysoky, J. Geom. Phys. **130**, 1 (2018).
[28] S. Demulder, F. Hassler, and D. C. Thompson, J. High Energy Phys. 02 (2019) 189.
[29] Y. Sakatani, Prog. Theor. Exp. Phys. **2019**, 073B04 (2019).
[30] A. Catal-Ozer, J. High Energy Phys. 08 (2019) 115.
[31] W. H. Baron, J. J. Fernandez-Melgarejo, D. Marques, and C. Nunez, J. High Energy Phys. 04 (2017) 078.

[32] R. Borsato, A. V. López, and L. Wulff, J. High Energy Phys. 07 (2020) 103.

[33] B. Hoare and A. A. Tseytlin, J. Phys. A **49**, 494001 (2016).

[34] R. Borsato and L. Wulff, Phys. Rev. Lett. **117**, 251602 (2016).

[35] We will use indices r, s where $r=(i,\mu)$ to distinguish them from indices m, n where $m=(\sigma,\mu)$.

[36] We obviously have to assume invertibility of the inverse matrices in (4) and (6).

[37] See the Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.125.201603 for a rewriting of this formula more familiar to the usual form of the (N)ATD rules. In the case of no spectators (and assuming invertibility of $\dot{M}$) Eq. (4) reduces to $M=u(\dot{M}^{-1}+\Pi)^{-1}u^T$ and Eq. (6) to $\tilde{M}=\tilde{u}(\dot{M}+\tilde{\Pi})^{-1}\tilde{u}^T$, which are more symmetric and more familiar in the PL duality literature. Also, we review how to obtain the finite form of the transformation used in (21) from the infinitesimal one of [24], as explained in [32].

[38] R. Von Unge, J. High Energy Phys. 07 (2002) 014.

[39] Compared to [40] here $\tilde{y}_i=-\nu_i$ and we swap $\tilde{e}^{(\pm)}$ appearing later.

[40] R. Borsato and L. Wulff, J. High Energy Phys. 08 (2018) 027.

[41] Because $\dot{d}$ depends only on x^μ it follows that $\partial_\sigma\Phi=\frac{1}{2}\partial_\sigma\log\det M-\partial_\sigma\log\det u$ and similarly for $\tilde{\Phi}$.

[42] A. Bossard and N. Mohammedi, Nucl. Phys. **B619**, 128 (2001).

[43] E. Bergshoeff and M. de Roo, Phys. Lett. B **218**, 210 (1989).

[44] E. A. Bergshoeff and M. de Roo, Nucl. Phys. **B328**, 439 (1989).

[45] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. **B293**, 385 (1987).

[46] C. M. Hull and P. K. Townsend, Nucl. Phys. **B301**, 197 (1988).

[47] R. Borsato and L. Wulff, J. High Energy Phys. 03 (2020) 126.

[48] Notice the first transposition.

[49] Notice that it is not possible to set directly $e^{(+)}=e^{(-)}$ without the help of a Lorentz transformation, as this would be incompatible with the assumption that $\dot{e}^{(\pm)}$ depend only on x^μ. Therefore we used the gauge $\dot{e}=\dot{e}^{(+)}=\dot{e}^{(-)}$.

[50] Alternatively, one may choose $e=e^{(+)}$ and the anomalous terms would be $a\Delta^{(-)}_{\Lambda^{-1}}M^{(\mathrm{DFT})}_{nm}$.

[51] A. Eghbali, Phys. Rev. D **99**, 026001 (2019).

[52] A. Eghbali, R. Naderi, and A. Rezaei-Aghdam, arXiv:2002.00675.

[53] This is under the assumption that the PL symmetry is not anomalous.

[54] F. Hassler and T. Rochais, arXiv:2007.07897.

[55] T. Codina and D. Marques, arXiv:2007.09494.

Published for SISSA by Springer

Received: *April 23, 2020*
Accepted: *June 27, 2020*
Published: *July 16, 2020*

The first α'-correction to homogeneous Yang-Baxter deformations using $O(d,d)$

Riccardo Borsato,[a] Alejandro Vilar López[a] and Linus Wulff[b]

[a] *Instituto Galego de Física de Altas Enerxías (IGFAE), Universidade de Santiago de Compostela, Santiago de Compostela, Spain*

[b] *Department of Theoretical Physics and Astrophysics, Masaryk University, 611 37 Brno, Czech Republic*

E-mail: riccardo.borsato@usc.es, alejandrovilar.lopez@usc.es, wulff@physics.muni.cz

Abstract: We use the $O(d,d)$-covariant formulation of supergravity familiar from Double Field Theory to find the first α'-correction to (unimodular) homogeneous Yang-Baxter (YB) deformations of the bosonic string. A special case of this result gives the α'-correction to TsT transformations. In a suitable scheme the correction comes entirely from an induced anomalous double Lorentz transformation, which is needed to make the two vielbeins obtained upon the YB deformation equal. This should hold more generally, in particular for abelian and non-abelian T-duality, as we discuss.

Keywords: Bosonic Strings, String Duality

ArXiv ePrint: 2003.05867

https://doi.org/10.1007/JHEP07(2020)103

JHEP07(2020)103

Contents

1 Introduction

Yang-Baxter deformations were originally constructed as deformations of the Principal Chiral Model and (super)coset sigma models with the interesting property that they preserve integrability [1–4]. The deformations are built using an R-matrix which solves the classical Yang-Baxter equation (CYBE)

$$[RX, RY] - R([RX, Y] + [X, RY]) = c^2[X, Y]\,, \qquad \forall X, Y \in \mathfrak{g}\,, \tag{1.1}$$

where $c = 0$ gives the standard CYBE equation and $c \neq 0$ corresponds to the modified CYBE. We will consider only the $c = 0$ case here, for which the deformed models are often called homogeneous YB models. It was shown in [5, 6] that homogeneous deformations can be generated using non-abelian T-duality. One simply adds a closed, non-degenerate, B-field defined on a subalgebra of the isometry algebra and dualizes on that subalgebra.[1] This construction means that these deformations can be defined for a general sigma model as long as it admits isometries that can be dualized. In particular the YB deformation of the Green-Schwarz superstring was constructed in [7]. A special case of this deformation

[1] From this construction one obtains a deformation of non-abelian T-duality, but it is possible to show that a local field redefinition (i.e. a diffeomorphism in target space) plus a shift of the B-field permit to rewrite the result as a homogeneous YB deformation. See [5, 6] for more details.

JHEP07(2020)103

is when the isometries are abelian and in that case the deformed model is simply a T-duality-shift-T-duality (TsT) transformation [8], which are usually called β-shifts or β-transformations in the context of $O(d,d)$.

Just as in non-abelian T-duality [9, 10], these models may in principle have a Weyl anomaly. When the anomaly is present the target space fields do not solve the standard supergravity equations but a generalization of these [11, 12]. Similar to the non-abelian T-duality case this anomaly is absent if one requires the R-matrix to satisfy a unimodularity condition [13]. This is the case we consider here although unimodularity is not a necessary condition to avoid a Weyl anomaly [14–16].

The realization of homogeneous YB models using T-duality makes it natural to try to describe these models using the $O(d,d)$-covariant language of Double Field Theory (DFT), as was done starting with the work of [17]. In fact the YB deformations take the form of a so-called β-transformation [14, 18] in $O(d,d)$ language. This language is particularly useful since, as we will show in this paper, unimodular homogeneous YB deformations leave the generalized fluxes — the basic building blocks in the $O(d,d)$-covariant formalism — invariant (see also [18, 19]). With this observation it becomes very simple to prove that the deformed model solves the low-energy field equations, since those have an $O(d,d)$-covariant formulation in terms of the generalized fluxes. In fact the same is true for the first α'-correction to these equations as shown in [20]. Therefore it is also straightforward to argue that YB-deformed bosonic strings[2] are Weyl invariant at least up to two loops. The fact that all higher derivative corrections should respect the $O(d,d)$ structure suggest that this should even be true to all orders in α', although a complete proof that the string effective action can be written only in terms of generalized fluxes is not known to the authors.

Naively this argument may seem to suggest that the YB-deformed backgrounds should not receive any α'-corrections beyond those coming from the intrinsic α'-dependence of the original (i.e. undeformed) background. But this is at odds with the results of [21], where non-trivial corrections were found working to second order in the expansion in the deformation parameter. When focusing on the class of TsT transformations, it is also at odds with the fact that abelian T-duality is known to receive α'-corrections, as was shown in various works starting from [22–28], which would be expected to lead to corrections to TsT. As we will explain in more detail in the rest of the paper, the resolution is that while in the doubled formalism there is indeed no correction, corrections appear when one wants to go from the doubled formalism to a standard (super)gravity formulation. In order to do that one has to fix the double Lorentz gauge-invariance in such a way that the two vielbeins that naturally exist in the doubled formulation are set equal. This requires a certain double Lorentz transformation and — given that the fields of the doubled formulation have an anomalous transformation[3] under double Lorentz transformations [33] — this induces an

[2]Note that while here we consider only the bosonic string for definiteness, very similar results hold for the heterotic string. In fact they can both be treated at the same time by introducing parameters that interpolate between the two as in for example [20]. Note that the relevant equations for the target space fields for the bosonic string are the type II supergravity equations with RR fields set to zero. Therefore we will often loosely refer to them as the (super)gravity equations.

[3]The fact that manifest $O(d,d)$ symmetry requires the fields to transform non-covariantly was clarified in the works [29–32].

JHEP07(2020)103

extra α'-correction to the deformed background whose form we determine. A special case of our formula gives the α'-correction to TsT transformations.

This discussion naturally connects also to the identification of α'-corrections to abelian T-duality transformations, as mentioned above. We will discuss also this and comment on the comparison to the results of [26]. Starting from the corrections to T-duality we will be able to provide an independent way to obtain α'-corrections to TsT transformations, which does not make use of the double $O(d,d)$ formulation.

The outline of the paper is as follows. First we give a very brief introduction to the concepts needed from the $O(d,d)$-covariant formulation as used in DFT. In section 3 we describe what Yang-Baxter deformations are in this language and show that they leave the generalized fluxes invariant. The α'-correction to these deformations induced by the compensating anomalous Lorentz transformation is described in section 4. Section 5 focuses on abelian T-duality and TsT transformations and we show that the results agree with those obtained using the $O(d,d)$-covariant formulation. We end with some concluding comments.

2 $O(d,d)$ covariant formulation of supergravity

We will take inspiration from DFT and use the $O(d,d)$ covariant formulation of (super)gravity. In particular we will work with the so-called frame-like formulation of DFT [34–36] where the structure group is taken as two copies of the Lorentz group $O(1,d-1)\times O(d-1,1)$. More details and references can be found in the reviews [37–39]. However, unlike in DFT, we will always assume that the section condition is solved in the standard way $\partial_M=(0,\partial_m)$ so that we are really just working with a rewriting of supergravity. Here we will actually consider only the NSNS sector as appropriate for the bosonic string.

In the frame-like formulation one writes the generalized metric in terms of generalized (inverse) vielbeins

$$\mathcal{H}^{MN}=E_A{}^M\mathcal{H}^{AB}E_B{}^N\,,\tag{2.1}$$

where $\mathcal{H}^{AB}$ is block diagonal with the usual Minkowski metric $\bar\eta=(-1,1,\dots,1)$ in each block. Coordinate indices are raised and lowered with the $O(d,d)$ metric

$$\eta^{MN}=\eta_{MN}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\tag{2.2}$$

and flat indices with the metric

$$\eta^{AB}=\eta_{AB}=\begin{pmatrix}\bar\eta&0\\0&-\bar\eta\end{pmatrix}.\tag{2.3}$$

The generalized metric can be parameterized in the form

$$\mathcal{H}^{MN}=\begin{pmatrix}G_{mn}-B_{mk}G^{kl}B_{ln}&B_{mk}G^{kn}\\-G^{mk}B_{kn}&G^{mn}\end{pmatrix},\tag{2.4}$$

in terms of the usual metric G and B-field. We take the generalized (inverse) vielbein to be

$$E_A{}^M=\frac{1}{\sqrt2}\begin{pmatrix}e^{(+)a}{}_m-e^{(+)an}B_{nm}&e^{(+)am}\\-e^{(-)}_{am}-e^{(-)n}_a B_{nm}&e^{(-)m}_a\end{pmatrix}.\tag{2.5}$$

Here $e^{(\pm)}$ are two sets of vielbeins which transform independently as $\Lambda^{(\pm)}e^{(\pm)}$ under the two Lorentz-group factors. To go to the standard supergravity picture one fixes a gauge $e^{(+)}=e^{(-)}=e$ leaving only one copy of the Lorentz-group.

An important object is the so-called generalized Weitzenböck connection, defined in terms of the generalized vielbeins as

$$\Omega_{ABC} = E_A{}^M \partial_M E_B{}^N E_{CN} \,. \tag{2.6}$$

From this the generalized fluxes are constructed as

$$\mathcal{F}_{ABC} = 3\Omega_{[ABC]} \,, \qquad \mathcal{F}_A = \Omega^B{}_{BA} + 2E_A{}^M \partial_M \hat{d} \,, \tag{2.7}$$

where $\hat{d}$ is the generalized dilaton related to the standard one as $e^{-2\hat{d}} = e^{-2\Phi}\sqrt{-G}$. The importance of these objects comes from the fact that the generalized fluxes are scalars under generalized diffeomorphisms. This follows from the fact that a generalized diffeomorphism is implemented by the generalized Lie derivative which acts on a vector field as

$$\mathcal{L}_X Y^M = X^N \partial_N Y^M + (\partial^M X_N - \partial_N X^M) Y^N \,. \tag{2.8}$$

The NSNS sector supergravity equations, or bosonic string low-energy effective equations, can be expressed in terms of the generalized fluxes only. To do this we first introduce the projectors

$$P_\pm = \frac{1}{2}\left(\eta \pm \mathcal{H}\right) \,. \tag{2.9}$$

Defining the following projections of the generalized fluxes

$$\mathcal{F}^{(\pm)}_{ABC} = (P_\mp)_A{}^D (P_\pm)_B{}^E (P_\pm)_C{}^F \mathcal{F}_{DEF} \,, \qquad \mathcal{F}^{(\pm)}_A = (P_\pm)_A{}^B \mathcal{F}_B \,, \tag{2.10}$$

they take the form[4]

$$(P_+)_A{}^C (P_-)_B{}^D \left[\partial_C \mathcal{F}_D - (\mathcal{F}^E - \partial^E)\mathcal{F}^{(-)}_{CDE} + \frac{1}{4}\mathcal{F}_C{}^{EF}\mathcal{F}_{DEF} - \frac{1}{4}(\mathcal{F}^2)_{CD}\right] = 0 \,, \tag{2.11}$$

$$\mathcal{R} = -4\partial_A \mathcal{F}^{(-)A} + 2\mathcal{F}_A \mathcal{F}^{(-)A} + \frac{1}{4}\mathcal{F}_A{}^{CD}\mathcal{F}_{BCD}\mathcal{H}^{AB} - \frac{1}{12}\mathcal{F}^2 - \frac{1}{6}\mathcal{F}_{ABC}\mathcal{F}^{ABC} = 0 \,, \tag{2.12}$$

where $(\mathcal{F}^2)_{AB} = \mathcal{F}_{ACD}\mathcal{H}^{CE}\mathcal{H}^{DF}\mathcal{F}_{BEF}$ and $\mathcal{F}^2 = \mathcal{H}^{AB}(\mathcal{F}^2)_{AB}$. The last line defines the generalized Ricci scalar and these equations of motion can be derived from the action

$$S = \int dX \, e^{-2\hat{d}} \mathcal{R} \,. \tag{2.13}$$

Let us emphasize again that for us this is just a convenient rewriting of the usual bosonic string effective action and equations of motion at lowest order in α'.

[4]Note that eq. (3.78) in [37] is not correct, since for example the P_+P_+ projection does not vanish.

JHEP07(2020)103

3 Yang-Baxter deformations in $O(d,d)$ language

We first need to show how to write YB deformations in $O(d,d)$ language at leading order in α', which will be needed later when discussing their α'-corrections. Under a YB deformation we have (e.g. [7, 40])[5]

$$G - B \to \tilde{G} - \tilde{B} = (G-B)(1+\Theta(G-B))^{-1}\,. \tag{3.1}$$

The transformation of the dilaton is such that the generalized dilaton $\hat{d}$ is invariant. The transformation of G and B is equivalent to the following transformation of the generalized metric (2.4)

$$\mathcal{H} \to \tilde{\mathcal{H}} = h^T \mathcal{H} h\,, \qquad h_M{}^N = \delta_M{}^N + \Theta_M{}^N\,, \tag{3.2}$$

where

$$\Theta_M{}^N = \begin{pmatrix} 0 & \Theta^{mn} \\ 0 & 0 \end{pmatrix}, \qquad \Theta^{mn} = k_r^m k_s^n R^{rs}, \tag{3.3}$$

where k_r^m are Killing vectors of the undeformed background[6] and R^{rs} is a constant anti-symmetric matrix satisfying (1.1) with $c=0$ (r,s are Lie algebra indices). Later we will show that if we just impose that R is constant and anti-symmetric, the additional property of satisfying the CYBE (1.1) will have a natural interpretation. The generalized vielbein (2.5) then transforms as

$$E_A{}^M \to \tilde{E}_A{}^M = E_A{}^N h_N{}^M\,. \tag{3.4}$$

Note that the two sets of vielbeins in (2.5) transform differently, namely

$$\tilde{e}_a^{(\pm)m} = e_a^{(\pm)n}\left[\delta_n^m - (B_{nk} \mp G_{nk})\Theta^{km}\right]. \tag{3.5}$$

This means that if we start from an undeformed background in a gauge such that $e^{(+)} = e^{(-)} = e$, we will need to accompany the YB deformation by a generalized double Lorentz transformation. We will keep $\tilde{e}^{(+)}$ invariant and transform $\tilde{e}^{(-)}$ by

$$(\Lambda^{(-)})_a{}^b = \tilde{\Lambda}_a{}^b = [1+(G-B)\Theta]_a{}^c([1-(B+G)\Theta]^{-1})_c{}^b\,, \tag{3.6}$$

in order to preserve the gauge $\tilde{e}^{(+)} = \tilde{e}^{(-)}$. At the (super)gravity level this is of no concern since all objects transform covariantly, but when one considers α'-corrections this transformation becomes important due to anomalous transformations of the fields, as we will discuss in the next section.

Note that this is a reformulation of YB deformations in the form of an $O(d,d)$ transformation, in fact h has the form of a so-called β-transformation or β-shift. It is not a standard $O(d,d)$ transformation, such as the ones under which the DFT action is invariant,

[5]We use a tilde to denote quantities after doing the deformation. We absorb the deformation parameter (usually denoted by η) into Θ to simplify the expressions.

[6]The assumption is that the Lie derivatives along k_r^m of the metric, the B-field and the dilaton of the original background vanish. One could in principle relax the isometry condition on B by demanding only that the Lie derivative of H vanishes, but we will not consider this generalization here.

JHEP07(2020)103

though. This is first of all because Θ^{mn} is (in general) not constant and second, and more importantly, because Θ^{mn} depends on the background itself since it is constructed using Killing vectors. This is therefore *not* a symmetry but a map of a background to another background, which is in fact a deformation of the first if we take Θ to be multiplied by a small parameter.

It follows from the transformation of the generalized vielbein that the generalized Weitzenböck connection (2.6) transforms as

$$\begin{aligned}\tilde{\Omega}_{ABC} &= E_A{}^L h_L{}^M \partial_M E_B{}^N E_{CN} + E_A{}^L h_L{}^M \partial_M h_K{}^N (h^{-1})_N{}^P E_B{}^K E_{CP} \\ &= \Omega_{ABC} + E_A{}^L \Theta_L{}^M \partial_M E_B{}^N E_{CN} + E_A{}^L h_L{}^M \partial_M \Theta_{KN} E_B{}^K E_C{}^N\,, \end{aligned} \tag{3.7}$$

where we used the fact that any expression with two Θ's contracted (with η_{MN}) vanishes. Now we use the fact that k_r generate isometries, i.e. the generalized Lie derivative of $E_A{}^M$ and $\hat{d}$ along k_r vanish[7]

$$k_r^L \partial_L E_A{}^M + (\partial^M k_{rL} - \partial_L k_r^M) E_A{}^L = 0\,, \qquad k_r^K \partial_K \hat{d} = \frac{1}{2}\partial_K k_r^K\,. \tag{3.8}$$

Using this fact one finds that the change of the generalized flux $\mathcal{F}_{ABC}$ is proportional to the YB equation for R in the form

$$\Theta^{M[K}\partial_M \Theta^{LN]} = 0\,. \tag{3.9}$$

Therefore $\mathcal{F}_{ABC}$ is invariant under a YB deformation (see also [18, 19][8]). For $\mathcal{F}_A$ we find

$$\tilde{\mathcal{F}}_A = \mathcal{F}_A - \Theta_L{}^K \partial_K E_A{}^L - \partial_K \Theta_L{}^K E_A{}^L + 2E_A{}^N \Theta_N{}^M \partial_M \hat{d}\,, \tag{3.10}$$

and using (3.8) we find

$$\tilde{\mathcal{F}}_A = \mathcal{F}_A + E_A{}^M \Delta\mathcal{F}_M\,, \qquad \Delta\mathcal{F}_M = \begin{pmatrix} -2\nabla_n \Theta^{mn} \\ 0 \end{pmatrix}. \tag{3.11}$$

Therefore $\mathcal{F}_A$ is invariant precisely when the R-matrix is unimodular, since $\nabla_n\Theta^{mn} \propto f_{rs}^t R^{rs}$, and $f_{rs}^t R^{rs} = 0$ is the unimodularity condition of [13].

We have therefore shown that the generalized fluxes are invariant under unimodular YB deformations. In fact their derivatives are also invariant since for example

$$\partial_A \mathcal{F}_B = E_A{}^M \partial_M \mathcal{F}_B \to \partial_A \mathcal{F}_B - E_A{}^N \Theta_N{}^M \partial_M \mathcal{F}_B = \partial_A \mathcal{F}_B\,, \tag{3.12}$$

because $k_r^M \partial_M \mathcal{F}_B = \mathcal{L}_{k_r} \mathcal{F}_B = 0$ by isometry.

Since the (NSNS sector) supergravity equations of motion can be cast in terms of the generalized fluxes and their derivatives, this is enough to conclude that they are invariant

[7]Recall that $\hat{d}$ is a density rather than a scalar, hence the non-zero r.h.s. in the second equation. Note also that we are assuming the vielbeins and not just the metric to be invariant. This assumption was also made in [7], whose derivation we rely on, but it should be possible to relax it. We comment more on this in the next section.

[8]This is however at odds with [41].

under unimodular YB deformations. In other words such YB deformations map SUGRA solutions to SUGRA solutions. Moreover, also the first α'-correction to the bosonic string equations can be cast in terms of the generalized fluxes and their derivatives, and therefore our argument shows that in fact the YB deformation preserves Weyl invariance at least to two loops.[9] In fact one would expect that all α'-corrections to the equations can be expressed in $O(d,d)$ covariant form, which probably means they can be written only in terms of the generalized fluxes and their derivatives. If this is the case then our argument implies that YB deformations of the bosonic string preserve Weyl-invariance to all loops, i.e. they map a consistent bosonic string to another consistent bosonic string to all orders in α'.

4 The α'-correction to YB deformations

Our general argument above has shown that YB deformations preserve two-loop Weyl invariance for the bosonic string. In fact they seem to require no additional α'-corrections to the background besides those that are induced from the corrections to the original background. Here we want to understand how this fits with the results of [21] where additional α'-corrections were found for YB deformations. The resolution is that the additional α'-corrections are indeed absent in the $O(d,d)$ covariant approach, but when one goes down to a standard supergravity formulation one has to fix the double Lorentz symmetry by fixing $e^{(+)} = e^{(-)} = e$. The double Lorentz transformation required to do this induces, via the *anomalous* transformation of the generalized vielbein at order α', additional α'-corrections to the YB deformed model. Let us now see how this works.

It was shown in [33] that at order α' the generalized vielbein acquires an anomalous transformation under (double) Lorentz transformations. The transformation of the vielbein is given by[10]

$$\delta E_A{}^M = -\lambda_A{}^B E_B{}^M + \alpha' \hat{\delta}_\lambda E_A{}^M , \quad \hat{\delta}_\lambda E_A{}^M = \left(\partial^{(-)}_{[A} \lambda_C{}^D \mathcal{F}^{(-)}_{B]D}{}^C - \partial^{(+)}_{[A} \lambda_C{}^D \mathcal{F}^{(+)}_{B]D}{}^C \right) E^{BM} , \tag{4.1}$$

where $\lambda_C{}^D$ are parameters of an infinitesimal double Lorentz transformation and the second term is the anomalous piece. Note that we have defined the projected derivatives $\partial^{(\pm)}_A = (P_\pm)_A{}^B \partial_B$. After fixing the gauge $e^{(+)} = e^{(-)} = e$ the non-zero components of $\mathcal{F}^{(\pm)}$ are [33]

$$\mathcal{F}^{(+)ab}_M = \frac{1}{2} \begin{pmatrix} G^{mn} \omega^{(+)ab}_n \\ -(1-BG)_m{}^n \omega^{(+)ab}_n \end{pmatrix} , \qquad \mathcal{F}^{(-)}_{Mab} = \frac{1}{2} \begin{pmatrix} G^{mn} \omega^{(-)}_{nab} \\ (1+BG)_m{}^n \omega^{(-)}_{nab} \end{pmatrix} , \tag{4.2}$$

where $\omega^{(\pm)cd}_m = \omega_m{}^{cd} \pm \frac{1}{2} H_m{}^{cd}$, and the spin-connection is related to the vielbein and the Christoffel symbols Γ^p_{mn} as

$$\omega_{mc}{}^d = e_c{}^n \partial_m e_n{}^d - \Gamma^p_{mn} e_c{}^n e_p{}^d . \tag{4.3}$$

[9] The action was written in terms of the fluxes in [20]. But the variation of the generalized fluxes are again expressed in terms of the generalized fluxes which shows that the equations of motion are also expressed in this way, which is all we need.

[10] We are specifying here to the case of the bosonic string by setting $a = b = -\alpha'$ in the formulas of [33].

This leads to the anomalous infinitesimal transformations[11]

$$\hat{\delta}\bar{G}_{mn} = -\frac{1}{2}\partial_{(m}\lambda^{(+)cd}\omega^{(+)}_{n)cd} - \frac{1}{2}\partial_{(m}\lambda^{(-)cd}\omega^{(-)}_{n)cd}\,, \tag{4.4}$$

$$\hat{\delta}\bar{B}_{mn} = \frac{1}{2}\partial_{[m}\lambda^{(+)cd}\omega^{(+)}_{n]cd} - \frac{1}{2}\partial_{[m}\lambda^{(-)cd}\omega^{(-)}_{n]cd}\,. \tag{4.5}$$

Of course, after fixing the gauge $e^{(+)} = e^{(-)} = e$ only the transformations with $\lambda^{(+)} = \lambda^{(-)} = \lambda$ remain and the anomalous Lorentz transformations of the fields become

$$\hat{\delta}\bar{G}_{mn} = -\partial_{(m}\lambda^{cd}\omega_{n)cd}\,, \tag{4.6}$$

$$\hat{\delta}\bar{B}_{mn} = \frac{1}{2}\partial_{[m}\lambda^{cd}H_{n]cd}\,. \tag{4.7}$$

We see from these expressions that we can define new fields that transform non-anomalously by[12]

$$G^{(\mathrm{MT})}_{mn} = \bar{G}_{mn} + \alpha'\left(\frac{1}{2}\omega_{mcd}\omega_n{}^{cd} + \frac{3}{8}H_{mkl}H_n{}^{kl}\right)\,, \tag{4.8}$$

$$B^{(\mathrm{MT})}_{mn} = \bar{B}_{mn} + \frac{\alpha'}{2}H_{cd[m}\omega_{n]}{}^{cd}\,. \tag{4.9}$$

The explicit non-covariant terms are constructed such as to cancel the anomalous Lorentz transformations. Notice that the above redefinitions also fix the *finite* form of the anomalous Lorentz transformations of $\bar{G}, \bar{B}$.

4.1 Compensating anomalous transformation

When we are dealing with the YB deformation it is crucial to remember the compensating double Lorentz transformation needed to make $\tilde{e}^{(+)} = \tilde{e}^{(-)}$ given by (3.6). Setting $\lambda^{(+)} = 0$ and $\lambda^{(-)} = \tilde{\lambda}$ in (4.5) we find that this induces an extra transformation of the fields at order α' given by[13]

$$\hat{\delta}\bar{G}_{mn} = -\frac{1}{2}\partial_{(m}\tilde{\lambda}^{cd}\tilde{\omega}^{(-)}_{n)cd}\,, \qquad \hat{\delta}\bar{B}_{mn} = -\frac{1}{2}\partial_{[m}\tilde{\lambda}^{cd}\tilde{\omega}^{(-)}_{n]cd}\,. \tag{4.10}$$

We now need the finite form of the transformation since we are doing a finite transformation $\tilde{\Lambda} = e^{\tilde{\lambda}}$ given by (3.6). To find it we use the same strategy as above. We redefine G and B by terms involving the spin connection in such a way that the new fields do not have

[11]The bar on the fields is to emphasize that these are the fields coming from the doubled formulation and which have an anomalous Lorentz transformation. Below we will define unbarred fields that transform covariantly.

[12]We have included an extra shift of G_{mn} by H^2_{mn} to go to the scheme of Metsaev and Tseytlin (MT), see [33] and appendix A.

[13]In (4.5) we assumed $e^{(+)} = e^{(-)} = e$ and we were doing a double Lorentz transformation from that starting point. Here we can use the same logic, assuming that we start from the gauge $\tilde{e}^{(+)} = \tilde{e}^{(-)} = \tilde{e}$ for a YB deformation and go back to the situation where $\tilde{e}^{(+)} = \tilde{e}$ and $\tilde{e}^{(-)} = \tilde{\Lambda}^T\tilde{e}$ as in (3.5). In this way we construct the inverse of the anomalous transformation we want. We remind that $\omega^{(\pm)cd}_m = \omega_m{}^{cd} \pm \frac{1}{2}H_m{}^{cd}$, so that the $(\pm)$ on the torsionful spin-connection should not be confused with the $(\pm)$ on the two vielbeins coming from DFT. Setting $\lambda^{(+)} = 0$ means that for the deformed model we take $\tilde{e} = \tilde{e}^{(+)}$.

any anomalous transformation. From this one can then read off the finite form of the transformation.

For $\bar{G}_{mn}$ this is easily done by noting that $\bar{G}_{mn} + \frac{\alpha'}{4}\tilde{\omega}_m^{(-)cd}\tilde{\omega}_{ncd}^{(-)}$ is invariant under the above transformation and so the finite transformation for $\bar{G}$ is[14]

$$\delta_{\rm comp}\bar{G}_{mn} = -\frac{1}{2}[\tilde{\Lambda}\partial_{(m}\tilde{\Lambda}^T]^{cd}\tilde{\omega}_{n)cd}^{(-)} + \frac{1}{4}[\tilde{\Lambda}\partial_m\tilde{\Lambda}^T]^{cd}[\tilde{\Lambda}\partial_n\tilde{\Lambda}^T]_{cd}\,. \tag{4.11}$$

For $\bar{B}_{mn}$ things are more subtle because a similar term $\frac{1}{4}\tilde{\omega}_{[m}^{(-)cd}\tilde{\omega}_{n]cd}^{(-)}$ vanishes by anti-symmetry. The part involving H in $\omega^{(-)}$ can be integrated as before, while the part involving ω can be found by the following trick. Consider the anomalous transformation of $H = dB$ instead. One finds that H transforms like the Chern-Simons form for ω

$$\hat{\delta}\bar{H} = -\frac{1}{4}\delta\mathrm{CS}(\omega) = -\frac{1}{4}\delta\mathrm{tr}\left(\omega d\omega + \frac{2}{3}\omega\omega\omega\right)\,. \tag{4.12}$$

The finite transformation of the CS form is

$$\delta\mathrm{CS}(\omega) = d(\tilde{\Lambda}d\tilde{\Lambda}^T\omega) - \frac{1}{3}\mathrm{tr}(\tilde{\Lambda}^T d\tilde{\Lambda}\tilde{\Lambda}^T d\tilde{\Lambda}\Lambda^T d\tilde{\Lambda})\,. \tag{4.13}$$

This implies that the transformation of B can be taken to be

$$\delta_{\rm comp}\bar{B}_{mn} = -\frac{1}{2}[\tilde{\Lambda}\partial_{[m}\tilde{\Lambda}^T]^{cd}\tilde{\omega}_{n]cd}^{(-)} + B_{mn}^{\rm WZW}\,, \tag{4.14}$$

where $B^{\rm WZW}$ is defined by

$$dB^{\rm WZW} = -\frac{1}{12}\mathrm{tr}(\tilde{\Lambda}^T d\tilde{\Lambda}\tilde{\Lambda}^T d\tilde{\Lambda}\tilde{\Lambda}^T d\tilde{\Lambda})\,. \tag{4.15}$$

Now that we have found the pieces induced by the compensating double Lorentz transformation we are ready to write the α'-correction to the YB-transformed metric and B-field.

4.2 The correction to Yang-Baxter deformations

Putting everything together the α'-correction to the YB-deformed background in the scheme of Hull and Townsend is[15]

$$\delta(\tilde{G}-\tilde{B})_{mn}^{\rm (HT)} = \frac{1}{2}\tilde{\omega}_{mcd}^{(-)}\left(\tilde{\omega}_n^{(+)cd} - [\tilde{\Lambda}\partial_n\tilde{\Lambda}^T]^{cd}\right) + \frac{1}{4}\partial_m\tilde{\Lambda}^{cd}\partial_n\tilde{\Lambda}_{cd} - B_{mn}^{\rm WZW} + \delta'(\tilde{G}-\tilde{B})_{mn}\,, \tag{4.16}$$

$$\delta\tilde{\Phi}^{\rm (HT)} = \frac{1}{4}\tilde{G}^{kl}\delta\tilde{G}_{kl} + \frac{1}{48}(\tilde{H}^2 - H^2)\,. \tag{4.17}$$

The correction to the dilaton follows from the fact that in the HT scheme when $\Phi' = \Phi^{\rm (HT)} - \frac{1}{48}\alpha' H^2$ the combination $e^{-2\Phi'}\sqrt{-G}$ is invariant under YB deformations, up

[14] Recall that we are computing minus the anomalous transformation we are after.

[15] See appendix A for the field redefinitions connecting all schemes. Here we set the parameter q of Hull and Townsend to zero.

to order α' included[16] [21]. The term $\delta'(\tilde{G}-\tilde{B})$ takes into account the scheme-change of the undeformed background[17]

$$\delta(G-B)_{mn} = -\frac{1}{2}\omega_m^{(-)\,cd}\omega_{ncd}^{(+)}\,, \tag{4.18}$$

needed to relate the HT scheme to the $O(d,d)$ covariant scheme (see appendix A) and it takes the form

$$\delta'(\tilde{G}-\tilde{B})_{mn} = \left[(1+(G-B)\Theta)^{-1}\delta(G-B)(1+\Theta(G-B))^{-1}\right]_{mn}\,. \tag{4.19}$$

Note that in addition to this, one has the α'-corrections to the original background, which will need to be included in (3.1) and will therefore induce a term of the same form — where now $\delta(G-B)$ is the correction to the original background.

It is important to stress that our derivation assumes that the B-field and vielbein of the undeformed background are invariant under the isometries generated by the Killing vectors entering Θ. When there is no gauge where this is possible, equations (4.16) and (4.17) do not necessarily lead to a background solving the α'-corrected supergravity equations. See however the next subsection.

The spin connection for the YB deformed background entering these expressions is computed using the vielbein $\tilde{e}=\tilde{e}^{(+)}$ defined in (3.5) and is given by

$$\tilde{\omega}_m{}^{ab}(\tilde{e}^{(\pm)}) = \omega_m{}^{ab} + \nabla_m[(B\mp G)\Theta]^{[a|k|}([1-(B\mp G)\Theta]^{-1})_k{}^{b]} - \tilde{e}^{(\pm)[a|k|}\tilde{e}^{(\pm)b]l}\nabla_k\tilde{G}_{lm}\,. \tag{4.20}$$

To see that (4.16) and (4.17) reproduces the results found in [21] one sets $B=0$ and expands to order Θ^2 obtaining

$$\delta\tilde{G}_{mn} = -\nabla_m\Theta^{cd}\nabla_c\Theta_{dn} - \nabla_n\Theta^{cd}\nabla_c\Theta_{dm} + \mathcal{O}(\Theta^4) \tag{4.21}$$

$$\delta\tilde{B}_{mn} = 2\partial_{[m}\left(\omega_{n]}{}^{cd}\Theta_{cd}\right) - \Theta_{cd}R_{mn}{}^{cd} + \mathcal{O}(\Theta^3) \tag{4.22}$$

$$\delta\tilde{\Phi} = \frac{1}{16}\nabla^m\Theta_{cd}\nabla_m\Theta_{cd} - \frac{3}{8}\nabla^m\Theta^{cd}\nabla_c\Theta_{dm} + \mathcal{O}(\Theta^4)\,, \tag{4.23}$$

which, up to a diffeomorphism and B-field gauge transformation, is the same as in [21]. Note that one has to use the fact that the isometry of the vielbein implies that

$$i_k\omega^{ab} = -\nabla^a k^b\,. \tag{4.24}$$

It is worth noting that in the case of a single TsT transformation the correction simplifies. Recall that, given two isometric coordinates y_1, y_2, a TsT transformation is implemented by the sequence of T-duality $y_1 \to T(y_1)$ followed by a shift $y_2 \to y_2 - \eta T(y_1)$ and by another T-duality $T(y_1)\to y_1$. It is understood as a special case of YB with $\Theta = \eta k_1 \wedge k_2$, where $k_i = \partial_{y_i}$ are Killing vectors. The above correction simplifies in the TsT case since $B^{\rm WZW}$ vanishes. This follows by noting that $\tilde{\Lambda} = 1 + 2\Theta([1-(B+G)\Theta]^{-1})$ which means that when Θ has rank 2 the Lorentz transformation is only non-trivial in a 2×2 block. In this block it is e^λ with λ an anti-symmetric 2×2 matrix. Since such a matrix only has one independent component, the r.h.s. of (4.15) vanishes.

[16] Notice that Φ' is in fact the dilaton in the HT scheme at $q=1/6$.

[17] For the same reason we have also a $\frac{1}{2}\tilde{\omega}_m^{(-)\,cd}\tilde{\omega}_{ncd}^{(+)}$ term in the correction above, generated by the scheme-change after the deformation.

4.3 Manifestly covariant form of the correction

The expression (4.16) for the α'-correction is not manifestly covariant but one can show that it is nevertheless covariant. We start by noting that[18]

$$\tilde{\omega}'^{(\pm)ab}_m = -\tilde{e}^{[a|k|}\tilde{e}^{b]l}\nabla_k(\tilde{G}\pm\tilde{B})_{ml} + \frac{1}{2}\tilde{e}^{[a|k|}\tilde{e}^{b]l}\nabla_m(\tilde{G}\pm\tilde{B})_{kl} - \nabla_m[(G-B)\Theta]^{[a|k|}([1+(G-B)\Theta]^{-1})_k{}^{b]} \tag{4.25}$$

where $\tilde{\omega}'^{(\pm)} = \tilde{\omega}^{(\pm)} - \omega$. With a bit of algebra one finds

$$\begin{aligned}\tilde{\omega}'^{(+)}_{mcd} =& \frac{1}{2}\nabla_m B_{cd} - \frac{1}{2}[(G-B)\nabla_m\Theta(G+B)]_{cd} + [1+(G-B)\Theta]_{[c|k|}(\nabla^k B(1+\Theta(G-B))^{-1})_{d]m}\\ &+[1+(G-B)\Theta]_{[c|k|}((G-B)\nabla^k\Theta(1+(G-B)\Theta)^{-1}(G-B))_{d]m}\\ =& \frac{1}{2}H_{mcd} - X^{(+)}_{kcd}[(G-B)(1+\Theta(G-B))^{-1}]^k{}_m\,,\end{aligned} \tag{4.26}$$

where we have defined[19]

$$X^{(\pm)}_{kcd} = \frac{1}{2}\nabla_k\Theta_{cd} - \nabla_{[c}\Theta_{d]k} \pm \frac{1}{2}H_{cdl}\Theta^l{}_k \tag{4.27}$$

and we used the YB equation in the last term of the first expression and also the isometry of B in the next to last term. A similar calculation gives

$$[\tilde{\Lambda}^T\tilde{\omega}'^{(-)}_m\tilde{\Lambda} + \tilde{\Lambda}^T\nabla_m\tilde{\Lambda}]_{cd} = -\frac{1}{2}H_{mcd} + X^{(-)}_{kcd}[(G+B)(1-\Theta(G+B))^{-1}]^k{}_m\,. \tag{4.28}$$

Using these expressions we find that (4.16) can be written instead as

$$\begin{aligned}\delta(\tilde{G}-\tilde{B})_{mn} =& -\frac{1}{4}\nabla_m\tilde{\Lambda}^{cd}\nabla_n\tilde{\Lambda}_{cd} - B^{\text{cov-WZW}}_{mn} + \frac{1}{2}[\nabla_m\tilde{\Lambda}\tilde{\Lambda}^T]^{cd}\left(X^{(+)}_{kcd}(\tilde{G}-\tilde{B})^k{}_n - \frac{1}{2}H_{ncd}\right)\\ &+\frac{1}{2}[\tilde{\Lambda}^T\nabla_n\tilde{\Lambda}]^{cd}\left(X^{(-)}_{kcd}(\tilde{G}-\tilde{B})_m{}^k - \frac{1}{2}H_{mcd}\right)\\ &+\frac{1}{4}\left(\tilde{\Lambda}^c{}_e\tilde{\Lambda}^d{}_f - \delta^c_e\delta^d_f\right)\Big[(\tilde{G}-\tilde{B})_m{}^k X^{(-)ef}_k H_{ncd} + (\tilde{G}-\tilde{B})^k{}_n X^{(+)}_{kcd}H_m{}^{ef}\\ &\qquad - 2(\tilde{G}-\tilde{B})_m{}^k X^{(-)ef}_k X^{(+)}_{lcd}(\tilde{G}-\tilde{B})^l{}_n - \frac{1}{2}H_m{}^{ef}H_{ncd}\Big]\,,\end{aligned} \tag{4.29}$$

where $\tilde{G}-\tilde{B}$ is given by (3.1) and we have defined

$$B^{\text{cov-WZW}}_{mn} = B^{\text{WZW}}_{mn} - \frac{1}{2}\text{tr}\left(\omega_{[m}\tilde{\Lambda}\nabla_{n]}\tilde{\Lambda}^T\right) + \frac{1}{2}\text{tr}\left(\omega_{[m}\tilde{\Lambda}^T\partial_{n]}\tilde{\Lambda}\right)\,. \tag{4.30}$$

The correction to the dilaton is still given by (4.17). All terms except $B^{\text{cov-WZW}}$ are now manifestly covariant. For the latter the identity

$$\begin{aligned}\text{tr}\left([\tilde{\Lambda}^T\nabla\tilde{\Lambda}]^3\right) - \text{tr}\left([\tilde{\Lambda}^T d\tilde{\Lambda}]^3\right) =& -\frac{3}{2}d\text{tr}\left(\omega[d\tilde{\Lambda}\tilde{\Lambda}^T + \tilde{\Lambda}^T d\tilde{\Lambda}]\right) - \frac{3}{2}\nabla\text{tr}\left(\omega[\nabla\tilde{\Lambda}\tilde{\Lambda}^T + \tilde{\Lambda}^T\nabla\tilde{\Lambda}]\right)\\ &+3\text{tr}\left(R[\nabla\tilde{\Lambda}\tilde{\Lambda}^T + \tilde{\Lambda}^T\nabla\tilde{\Lambda}]\right)\,,\end{aligned} \tag{4.31}$$

[18]Here and in the following the covariant derivative is the one for the undeformed metric G. Moreover, unless written explicitly otherwise, one should use the undeformed vielbein to go from curved to flat indices.

[19]When the vielbeins are invariant under the isometries, (4.24) gives $\omega^{(\pm)}_{lcd}\Theta^l{}_k = X^{(\pm)}_{kcd}$.

where $R = d\omega + \omega \wedge \omega$ is the curvature 2-form, implies

$$dB^{\text{cov-WZW}} = -\frac{1}{12}\text{tr}\left([\tilde{\Lambda}^T\nabla\tilde{\Lambda}]^3\right) + \frac{1}{4}\text{tr}\left(R[\nabla\tilde{\Lambda}\tilde{\Lambda}^T + \tilde{\Lambda}^T\nabla\tilde{\Lambda}]\right). \tag{4.32}$$

Therefore also the transformation of B is covariant (up to B-field gauge transformations).

The manifestly covariant form of the correction given by (4.29) is actually more useful than the original form (4.16). The reason is that our derivation has assumed that the vielbeins are invariant under the isometries used to construct Θ, and therefore (4.16) is valid only in this case. Being covariant, (4.29) is valid also when the vielbeins are not invariant under the isometries, as long as there exists a gauge in which they are invariant. In fact, even though it is not guaranteed by our construction, these expressions can be valid more generally, i.e. even in cases where it is not possible to find a gauge in which the vielbeins are invariant. We mention one such example below.

4.4 Tests on examples

We have tested the formulas (4.16), (4.17) for α'-corrections to YB deformations on a number of examples, to check that they generate backgrounds solving the α'-corrected supergravity equations. First we worked out deformations of a Bianchi II background first considered in [21]. We tested our results both on the abelian deformations $\Theta = k_1 \wedge k_4$ and $\Theta = k_2 \wedge k_3$, and on the non-abelian deformation $\Theta = k_1 \wedge k_4 + k_2 \wedge k_3$. We refer to [21] for the α'-correction of the undeformed background and for the definition of the Killing vectors k_i, whose non-trivial commutation relations are just $[k_1, k_2] = k_3$. On this Bianchi II example we find that B^{WZW} is trivial even when considering the non-abelian deformation.

We worked out also deformations of the pure NSNS $\text{AdS}_3 \times S^3$ background.[20] Its YB deformations were classified in [16]. We worked out various abelian deformations corresponding to TsT transformations on the sphere, on AdS, or mixing the two spaces. We worked out also the non-abelian deformation generated by $\Theta = (k_0 + \bar{k}_0) \wedge k_s + k_+ \wedge \bar{k}_-$. Here k_s is a Killing vector on the sphere and we refer to [16] for the definitions we use for the AdS Killing vectors. In this case we cannot immediately apply (4.16) because it is not possible to find a vielbein for the AdS_3 metric that is invariant under all the isometries entering Θ. We can anyway obtain α'-corrections for this non-abelian deformation if we use the covariant formula (4.29). Alternatively, we can interpret this particular deformation as a non-commuting sequence of TsT transformations. Doing so, we can first work out the α'-corrected abelian deformation generated by $\Theta = k_+ \wedge \bar{k}_-$, and after doing that we can work out the abelian deformation $\Theta = (k_0 + \bar{k}_0) \wedge k_s$.[21]

[20]The α' corrections of the undeformed background are simply obtained by multiplying metric and B-field by $1 + 2\alpha'$ on the AdS part and by $1 - 2\alpha'$ on the sphere part.

[21]After doing the first abelian deformation, and before applying the second one, one has to carefully choose the vielbein such that it is invariant under the $k_0 + \bar{k}_0$ isometry. At this stage it is not necessary anymore to impose the invariance under $k_+, \bar{k}_-$, which is what saves the day in this approach.

5 T-duality and TsT transformations

Abelian T-duality transformations are another class of $O(d,d)$ transformations and we can follow exactly the reasoning in section 4 to obtain their α'-corrections. When we remain in the non-covariant scheme that comes from DFT, the corrections to the dualized metric and B-field will be given again by the formula (a hat on the field is used to denote the T-dualization)

$$\delta(\widehat{G} - \widehat{B})_{mn} = -\frac{1}{2}\hat{\omega}^{(-)}_{mcd}(\hat{\Lambda}\partial_n\hat{\Lambda}^T)^{cd} + \frac{1}{4}\partial_m\hat{\Lambda}^{cd}\partial_n\hat{\Lambda}_{cd} - B^{\rm WZW}_{mn}\,, \tag{5.1}$$

where now the Lorentz matrix is

$$\hat{\Lambda}_a{}^b = \delta_a{}^b - 2G^{-1}_{yy}e_{ya}e_y{}^b\,. \tag{5.2}$$

We are assuming that we are dualising along the coordinate y and expressions for the corrections in other schemes will be obtained by implementing the relevant field redefinitions, see appendix A.

In [42] α'-corrections to the T-duality rules from the DFT formulation were also discussed. There however instead of writing the generic form of the corrections in terms of the finite form of the Lorentz transformation as above, it was noted that $\hat{\Lambda}$ reduces to a constant[22] when choosing a specific gauge for the vielbein[23]

$$e_\mu{}^\alpha = \mathsf{e}_\mu{}^\alpha\,, \qquad e_y{}^\alpha = 0\,, \qquad e_\mu{}^\iota = e^\sigma V_\mu\,, \qquad e_y{}^\iota = e^\sigma\,. \tag{5.3}$$

Here we are rewriting the fields in terms of fields of a dimensional reduction

$$\begin{aligned} ds^2 &= G_{mn}dx^m dx^n = g_{\mu\nu}dx^\mu dx^\nu + e^{2\sigma}(dy+V)^2, \\ B &= \frac{1}{2}B_{mn}dx^m\wedge dx^n = \frac{1}{2}b_{\mu\nu}dx^\mu\wedge dx^\nu + \frac{1}{2}W\wedge V + W\wedge dy\,, \\ \Phi &= \phi + \frac{1}{2}\sigma. \end{aligned} \tag{5.4}$$

Since $\hat{\Lambda}$ is constant the anomalous Lorentz transformation is trivial in this gauge, and also the α'-corrections to T-duality will be trivial.[24] (Note that while it is possible to avoid corrections for a single T-duality it is not possible in general for more than one T-duality, as shown in [43].) In [42] this observation was used to obtain the α'-corrections to the T-duality rules in the scheme of Bergshoeff and de Roo (BR) [44, 45]. We use this result as a

[22]It is $\hat{\Lambda} = \mathrm{diag}(-1,1,\dots,1)$ where the dualized coordinate is placed first.

[23]For curved indices we take $m = y,\mu$ and similarly we also have flat indices $a = \iota,\alpha$. We denote by $\mathsf{e}_\mu{}^\alpha$ the vielbein for the reduced metric $g_{\mu\nu}$ appearing below.

[24]Importantly, this statement is gauge dependent, in accordance with the fact that the scheme under discussion is not Lorentz-covariant. Covariant schemes such as HT or MT will not have this type of gauge ambiguity.

JHEP07(2020)103

starting point to write below the T-duality rules to 2 loops in a family of different schemes.

$$
\begin{aligned}
\hat{\sigma} &= -\sigma + \left(a_1 - \frac{a_4}{2} + 2a_5 + 2\gamma_+\right)(\mathsf{D}\sigma)^2 - \frac{1}{8}(a_1 + 4a_2 - a_5 - 2\gamma_+) \\
&\quad \times \left(e^{2\sigma} V^{\lambda\rho} V_{\lambda\rho} + e^{-2\sigma} W^{\lambda\rho} W_{\lambda\rho}\right) - \frac{1}{2}(\gamma_- - a_6)\, V^{\lambda\rho} W_{\lambda\rho}\,, \qquad (5.5)\\
\hat{V}_\mu &= W_\mu + \frac{1}{2}(\gamma_+ - b_3 + a_5)\, W_\beta{}^\alpha \mathsf{w}_{\mu\alpha}{}^\beta + \frac{e^{2\sigma}}{4}(-4a_2 + 2b_1 + b_3 + \gamma_+)\, h_{\mu\lambda\rho} V^{\lambda\rho} + \\
&\quad + \frac{1}{4}(6a_1 - a_4 + 4a_5 + 4b_1 - 2b_2 + 4b_3 + 4\gamma_+)\, W_{\mu p} \mathsf{D}^p \sigma + \frac{1}{2}(a_4 - 2b_2)\, W_{\mu\rho} \mathsf{D}^\rho \phi - \\
&\quad - \frac{1}{2}(a_1 + 2b_1)\, \mathsf{D}^\rho W_{\mu\rho} - \frac{1}{2}(\gamma_- - a_6)\left(e^{2\sigma} V_\beta{}^\alpha \mathsf{w}_{\mu\alpha}{}^\beta + \frac{1}{2} h_{\mu\lambda\rho} W^{\lambda\rho} - 2e^{2\sigma} V_{\mu\rho} \mathsf{D}^\rho \sigma\right), \qquad (5.6)\\
\hat{W}_\mu &= V_\mu - \frac{1}{2}(\gamma_+ - b_3 + a_5)\, V_\beta{}^\alpha \mathsf{w}_{\mu\alpha}{}^\beta - \frac{e^{-2\sigma}}{4}(-4a_2 + 2b_1 + b_3 + \gamma_+)\, h_{\mu\lambda\rho} W^{\lambda\rho} + \\
&\quad + \frac{1}{4}(6a_1 - a_4 + 4a_5 + 4b_1 - 2b_2 + 4b_3 + 4\gamma_+)\, V_{\mu\rho} \mathsf{D}^\rho \sigma - \frac{1}{2}(a_4 - 2b_2)\, V_{\mu\rho} \mathsf{D}^\rho \phi + \\
&\quad + \frac{1}{2}(a_1 + 2b_1)\, \mathsf{D}^\rho V_{\mu\rho} + \frac{1}{2}(\gamma_- - a_6)\left(e^{-2\sigma} W_\beta{}^\alpha \mathsf{w}_{\mu\alpha}{}^\beta + \frac{1}{2} h_{\mu\lambda\rho} V^{\lambda\rho} + 2e^{-2\sigma} W_{\mu\rho} \mathsf{D}^\rho \sigma\right), \\
&\qquad (5.7)\\
\hat{\phi} &= \phi - \frac{1}{16}(a_1 - 4a_2 - a_5 + 4c_1 + 48c_2)\left(e^{2\sigma} V_{\lambda\rho} V^{\lambda\rho} - e^{-2\sigma} W_{\lambda\rho} W^{\lambda\rho}\right) + \\
&\quad + \frac{1}{2}(a_1 - 8c_1 + 2c_4)\, \mathsf{D}^2 \sigma - \frac{1}{2}(a_4 - 4c_3 - 4c_4)\, \mathsf{D}_\rho \sigma \mathsf{D}^\rho \phi\,, \qquad (5.8)\\
\hat{g}_{\mu\nu} &= g_{\mu\nu} - \frac{1}{2}(a_1 + 4a_2 + a_5)\left(e^{2\sigma} V_{\mu\rho} V_\nu{}^\rho - e^{-2\sigma} W_{\mu\rho} W_\nu{}^\rho\right) + \\
&\quad + (-2a_1 + a_4)\, \mathsf{D}_\mu \mathsf{D}_\nu \sigma + 2a_3 \mathsf{D}_{(\mu} \sigma \mathsf{D}_{\nu)} \phi\,, \qquad (5.9)\\
\hat{b}_{\mu\nu} &= b_{\mu\nu} - \frac{1}{2}(\gamma_+ - b_3 + a_5)\left(V_\beta{}^\alpha \mathsf{w}_{[\mu\alpha}{}^\beta W_{\nu]} - W_\beta{}^\alpha \mathsf{w}_{[\mu\alpha}{}^\beta V_{\nu]}\right) + \\
&\quad + \frac{1}{2}(a_1 + 2b_1)\left(\mathsf{D}^\rho W_{\rho[\mu} V_{\nu]} - \mathsf{D}^\rho V_{\rho[\mu} W_{\nu]}\right) + \\
&\quad + \frac{1}{4}(4a_2 - 2b_1 - b_3 - \gamma_+)\left(e^{2\sigma} V_{[\mu} h_{\nu]\lambda\rho} V^{\lambda\rho} - e^{-2\sigma} W_{[\mu} h_{\nu]\lambda\rho} W^{\lambda\rho}\right) + (2b_1 + b_2)\, h_{\mu\nu\rho} \mathsf{D}^\rho \sigma + \\
&\quad + \frac{1}{4}(-6a_1 + a_4 - 4a_5 - 4b_1 + 2b_2 - 4b_3 - 4\gamma_+)\left(V_{[\mu} W_{\nu]\rho} \mathsf{D}^\rho \sigma + W_{[\mu} V_{\nu]\rho} \mathsf{D}^\rho \sigma\right) - \\
&\quad - \frac{1}{2}(a_4 - 2b_2)\left(V_{[\mu} W_{\nu]\rho} \mathsf{D}^\rho \phi - W_{[\mu} V_{\nu]\rho} \mathsf{D}^\rho \phi\right) - 2b_3 V^\rho{}_{[\mu} W_{\nu]\rho} + \\
&\quad + \frac{1}{2}(\gamma_- - a_6)\Big(e^{-2\sigma} W_\beta{}^\alpha \mathsf{w}_{[\mu\alpha}{}^\beta W_{\nu]} - e^{2\sigma} V_\beta{}^\alpha \mathsf{w}_{[\mu\alpha}{}^\beta V_{\nu]} - \frac{1}{2} W^{\lambda\rho} h_{\lambda\rho[\mu} V_{\nu]} + \\
&\qquad\qquad + \frac{1}{2} V^{\lambda\rho} h_{\lambda\rho[\mu} W_{\nu]} - 2e^{-2\sigma} W_{[\mu} W_{\nu]\rho} \mathsf{D}^\rho \sigma - 2e^{2\sigma} V_{[\mu} V_{\nu]\rho} \mathsf{D}^\rho \sigma\Big)\,. \qquad (5.10)
\end{aligned}
$$

Setting $\alpha' \to 0$ they reduce to the Buscher rules that in terms of these fields read simply as $\sigma \to -\sigma$ and $V \leftrightarrow W$. Here D denotes the covariant derivative with respect to the reduced metric $g_{\mu\nu}$, and $\mathsf{w}_{\mu\alpha}{}^\beta$ is the reduced spin-connection. We have also defined $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, $W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$ and $h_{\mu\nu\rho} = 3(\partial_{[\mu} b_{\nu\rho]} - \frac{1}{2} W_{[\mu\nu} V_{\rho]} - \frac{1}{2} V_{[\mu\nu} W_{\rho]}) = H_{\mu\nu\rho} - 3W_{[\mu\nu} V_{\rho]}$. Apart from the order-$\alpha'$ parameters $\gamma_\pm$ needed to interpolate between the bosonic and the heterotic strings (see appendix A), the T-duality rules depend on coefficients a_i, b_i, c_i (that are also of order α') so that they are valid for any scheme related to the one of BR by these field redefinitions

$$\begin{aligned} G_{mn} &= G^{(\mathrm{BR})}_{mn} - a_1 R_{mn} - a_2 H^2_{mn} - a_3 \nabla_m \Phi \nabla_n \Phi - a_4 \nabla_m \nabla_n \Phi - a_5 \omega_{mb}{}^a \omega_{na}{}^b - a_6 \omega_{(m}{}^{ab} H_{n)ab}\,, \\ B_{mn} &= B^{(\mathrm{BR})}_{mn} - b_1 \nabla^p H_{mnp} - b_2 H_{mnp} \nabla^p \Phi - b_3 \omega_{[m}{}^{ab} H_{n]ab}\,, \\ \Phi &= \Phi^{(\mathrm{BR})} - c_1 R - c_2 H^2 - c_3 \nabla_p \Phi \nabla^p \Phi - c_4 \nabla^2 \Phi\,. \end{aligned} \tag{5.11}$$

By turning on these coefficients we can cover all schemes typically considered in the literature, see appendix A for the field redefinitions relating them.[25]

As expected, it is possible to tune the coefficients in order to set to zero all corrections to the T-duality transformations. For generic $\gamma_\pm$ it is enough to set

$$\begin{aligned} &a_2 = -\frac{a_1}{4} + \frac{\gamma_+}{4}\,, \quad a_3 = 0\,, \quad a_4 = 2a_1\,, \quad a_5 = -\gamma_+\,, \quad a_6 = \gamma_-\,, \qquad b_1 = -\frac{a_1}{2}\,, \\ &b_2 = a_1\,, \qquad\quad b_3 = 0\,, \quad c_2 = -\frac{a_1}{24} - \frac{c_1}{12}\,, \qquad\quad c_3 = a_1 - 4c_1\,, \quad c_4 = 4c_1 - \frac{a_1}{2} \end{aligned} \tag{5.12}$$

and T-duality reduces to the Buscher rules even to 2 loops. We will denote the fields in this (gauge-fixed) scheme by G', B', Φ'. When specifying to the bosonic string ($\gamma_+ = \alpha'/2, \gamma_- = 0$), they are related to the HT scheme by[26]

$$\begin{aligned} G'_{mn} &= G^{(\mathrm{HT})}_{mn} - \frac{1}{2}\alpha' \omega^{(-)ab}_{(m} \omega^{(+)}_{n)ab} = G^{(\mathrm{HT})}_{mn} + \alpha' \left(-\frac{1}{2} \omega_{mab} \omega^{ab}_n + \frac{1}{8} H^2_{mn} \right)\,, \\ B'_{mn} &= B^{(\mathrm{HT})}_{mn} + \frac{1}{2}\alpha' \omega^{(-)ab}_{[m} \omega^{(+)}_{n]ab} = B^{(\mathrm{HT})}_{mn} - \frac{1}{2}\alpha' H_{ab[m} \omega^{ab}_{n]}\,, \\ \Phi' &= \Phi^{(\mathrm{HT})} + \alpha' \frac{1+3q}{24} H^2\,. \end{aligned} \tag{5.13}$$

This matches with the field redefinitions that we would write for $\bar{G}, \bar{B}, \bar{\Phi}$ as expected. The difference is that here we are also imposing the specific gauge (5.3) and for that reason we denote the fields differently.

The rules above can be compared to the ones first derived by Kaloper and Meissner in [26] for the bosonic string ($\gamma_+ = \alpha'/2$, $\gamma_- = 0$). The scheme used is obtained setting the coefficients to

$$a_1 = \alpha' \quad a_2 = -\frac{\alpha'}{4}\,, \quad b_1 = -\frac{\alpha'}{2}\,, \quad b_3 = \frac{\alpha'}{2}\,, \quad c_1 = \frac{\alpha'}{8} \quad c_2 = -\frac{5\alpha'}{96} \quad c_3 = -\frac{\alpha'}{2}\,, \tag{5.14}$$

[25] Writing the rules for generic a_i, b_i, c_i coefficients as above, or in other words translating them into new schemes starting from a given one, is straightforward although it requires work to compute all tensors in the dimensional reduction. After that is done we can start from scheme A where $\hat{\sigma}^{(A)} = -\sigma^{(A)} + \alpha'\xi$, for some ξ. To obtain the rules in scheme B related as $\sigma^{(B)} = \sigma^{(A)} + \alpha' s$ for some s, we just have to compute $\hat{\sigma}^{(B)} = \hat{\sigma}^{(A)} + \alpha'\hat{s} = -\sigma^{(A)} + \alpha'(\xi + \hat{s}) = \sigma^{(B)} + \alpha'(\xi + \hat{s} + s)$. Notice that the fields themselves may have some explicit α'-dependence. In this example the field σ is odd under Buscher rules, and then the shift in the corrections $\hat{s} + s$ is even. Fields even under Buscher receive corrections that are odd.

[26] Here we are further setting $a_1 = c_1 = 0$. Turning on a_1, c_1 would introduce terms that vanish by means of 1-loop equations.

and the rest of them equal to zero. To match results, one has to take into account the possibility of transforming the reduced fields by doing diffeomorphisms and gauge transformations. Under such symmetries, the T-dual reduced fields transform as:

$$\hat{V} \to \hat{V} + \alpha' \left(\mathcal{L}_\xi W + dv\right) , \tag{5.15}$$

$$\hat{W} \to \hat{W} + \alpha' \left(\mathcal{L}_\xi V + dw\right) , \tag{5.16}$$

$$\hat{b} \to \hat{b} + \alpha' \left(\mathcal{L}_\xi b + d\beta + \frac{1}{2} V \wedge dv + \frac{1}{2} W \wedge dw\right) , \tag{5.17}$$

and the remaining fields transform normally under diffemorphisms. We are restricting to transformations which are first order in α', both for diffeomorphims and gauge transformations. The dw and $d\beta$ terms come from gauge transformations of the B field with parameter $\beta_\mu dx^\mu + w\, dy$, while v appears when including diffeomorphisms of the form $y \to y + \alpha' v$. Choosing the following set of parameters

$$\xi^\mu = \mathsf{D}^\mu \sigma\,, \quad w = -V_\nu \mathsf{D}^\nu \sigma\,, \quad v = -W_\nu \mathsf{D}^\nu \sigma\,, \quad \beta_\mu = \left(b_{\mu\nu} - \frac{1}{2} V_\mu W_\nu - \frac{1}{2} W_\mu V_\nu\right) \mathsf{D}^\nu \sigma\,, \tag{5.18}$$

we obtain the following set of rules

$$\hat{\sigma} = -\sigma + \frac{\alpha'}{2}\left[\frac{e^{2\sigma}}{4} V_{\lambda\rho} V^{\lambda\rho} + \frac{e^{-2\sigma}}{4} W_{\lambda\rho} W^{\lambda\rho} + 2\left(\mathsf{D}\sigma\right)^2\right] , \tag{5.19}$$

$$\hat{V}_\mu = W_\mu + \frac{\alpha'}{2}\left[\frac{e^{2\sigma}}{2} h_{\mu\lambda\rho} V^{\lambda\rho} + 2 W_{\mu\rho} \mathsf{D}^\rho \sigma\right] , \tag{5.20}$$

$$\hat{W}_\mu = V_\mu - \frac{\alpha'}{2}\left[\frac{e^{-2\sigma}}{2} h_{\mu\lambda\rho} W^{\lambda\rho} - 2 V_{\mu\rho} \mathsf{D}^\rho \sigma\right] , \tag{5.21}$$

$$\begin{aligned} \hat{b}_{\mu\nu} = b_{\mu\nu} + \alpha' \Big[& V_{[\mu}{}^\rho W_{\nu]\rho} - \left(V_{[\mu} W_{\nu]\rho} + W_{[\mu} V_{\nu]\rho}\right) \mathsf{D}^\rho \sigma \\ & - \frac{e^{2\sigma}}{4} V_{[\mu} h_{\nu]\lambda\rho} V^{\lambda\rho} + \frac{e^{-2\sigma}}{4} W_{[\mu} h_{\nu]\lambda\rho} W^{\lambda\rho} \Big] , \end{aligned} \tag{5.22}$$

and both $g_{\mu\nu}$ and ϕ remain invariant. These match with the rules given by Kaloper and Meissner in [26] up to the sign of the α' correction of the b field.[27]

Diffeomorphisms and gauge transformations of the reduced fields can also be used to simplify the rules and to obtain some nice expressions for the T-duality rules without the need of the dimensional reduction. We do this in a Lorentz-covariant scheme, the HT scheme for the bosonic string introduced in (5.13) where we fix $q = -1/3$. Using the same parameters for the transformations presented in the previous paragraph, it is possible to

[27] The fact that this is a typo in [26] is confirmed by the fact that there (4.9) and (4.11) are not compatible. For the field H of [26] (here h) which is even under T-duality at leading order in α', the correction to the T-duality transformation should rather be -2 the expression in (4.9). For odd fields the same contribution would be instead multiplied by $+2$. This easily follows from the first calculation they do to remove by a field redefinition the part of the action that is odd under T-duality, which is later reinterpreted as a correction to the T-duality transformation. Since the expressions in [46] agree with those in [26] we disagree also with that paper.

obtain the following rules for the T-duality transformation[28]

$$\hat{M}_{yy} = \frac{1}{M_{yy}}\,, \quad \hat{M}_{y\mu} = \frac{M_{y\mu}}{M_{yy}}\,, \quad \hat{M}_{\mu y} = -\frac{M_{\mu y}}{M_{yy}}\,, \tag{5.23}$$

$$\hat{M}_{\mu\nu} = M_{\mu\nu} - \frac{M_{\mu y}M_{y\nu}}{M_{yy}} - \alpha' \left[\frac{1}{M_{yy}} R^{(-)}_{\mu y\nu y} - \frac{1}{\hat{M}_{yy}} \hat{R}^{(-)}_{\mu y\nu y}\right]\,, \tag{5.24}$$

$$\hat{\Phi} = \Phi - \frac{1}{2}\log M_{yy} - \frac{\alpha'}{8}\left[R^{(-)} - \hat{R}^{(-)}\right]\,. \tag{5.25}$$

In these expressions

$$R^{(-)}_{mna}{}^{b} = 2\partial_{[m}\omega^{(-)b}_{n]a} + 2\omega^{(-)c}_{[ma}\omega^{(-)b}_{n]c} \tag{5.26}$$

is the Riemann tensor constructed from the torsionful connection $\omega^{(-)}$, $R^{(-)}$ the corresponding Ricci scalar and $M_{mn} = G_{mn} - B_{mn}$. Note also that the dual appears explicitly in the α' corrections but, to the order needed, it can be calculated using just the standard Buscher rules.

5.1 Corrections to TsT transformations

The fact that there exists a (gauge-fixed) scheme — for the sake of the discussion we will call it the "Buscher scheme" — such that T-duality is given just by the Buscher rules is useful. Here we use it to obtain an expression for α'-corrections to TsT transformations that does not necessarily use all the knowledge of DFT. TsT transformations are a special case of YB deformations, and we will show that the result agrees with that in section 4.

In order to do the TsT transformation we assume that there are two U(1) isometries with corresponding coordinates y_1 and y_2, and to avoid burdening the notation we will continue labelling by x^μ the rest of the coordinates.[29] We will do a T-duality $y_1 \to T(y_1)$ followed by a shift $y_2 \to y_2 - \eta T(y_1)$ and by another T-duality $T(y_1) \to y_1$. TsT transformations are special cases of YB if we take $\Theta = \eta k_1 \wedge k_2$, where $k_i = \partial_{y_i}$ are Killing vectors. Each step will be performed in the scheme that is most convenient. Therefore, when doing T-duality we will prefer to move to the Buscher scheme, while when doing the shift we will prefer to go to a covariant scheme. We will show that the α'-corrections to TsT transformations can be understood as arising from these shifts coming from the scheme changes. Because these scheme-changing shifts arise at intermediate steps, we will have to look at how they are further modified by the remaining steps in the TsT transformation.

Suppose we start from the HT scheme. In order to do the first T-duality on y_1 we find convenient to first go to the Buscher scheme. This is achieved by implementing the redefinitions (5.13) after taking care of choosing the vielbein as in (5.3). This effectively shifts the fields at order α' as $\delta_1(G_{mn}-B_{mn}) = -\frac{1}{2}\omega^{(-)}_{mab}\omega^{(+)ab}_{n}$. We can immediately account for this contribution in the final result: because we will have to do a TsT transformation including this contribution δ_1 (and we only care about the order α') we are essentially shifting the original metric and B-field as $G - B \to G - B + \delta_1(G - B)$ appearing in the

[28] Here 1-loop equations of motion were used to simplify the form of the corrections.

[29] The reader should be careful, since when doing T-duality along y_1 the coordinate y_2 should be treated on the same footing as x^μ when using the T-duality rules (5.5).

map (3.1). After expanding to first order in α' we obtain the first contribution to the α' correction of the final result

$$-\frac{1}{2}[(1+(G-B)\Theta)^{-1}]_m{}^p\omega^{(-)}_{pab}\omega^{(+)ab}_q[(1+\Theta(G-B))^{-1}]^q{}_n\,. \tag{5.27}$$

While in the Buscher scheme we can easily do the first T-duality on y_1 because we just need to use the Buscher rules. Notice that under Buscher the gauge choice (5.3) is preserved.

To perform the shift it is more convenient to go back to the HT scheme, which is covariant. That means that we will have to use (5.13) again, although now it will be done using the data of the T-dual background $\delta_2(\hat{G}_{mn}-\hat{B}_{mn}) = +\frac{1}{2}\hat{\omega}^{(-)}_{mab}\hat{\omega}^{(+)ab}_n$. A hat is used to denote that the first T-dualization has already been done. Notice that under the first T-duality and shift, the vielbein $e_m{}^a$ (in matrix form) changes as

$$\begin{pmatrix} e^\sigma & 0 & 0 \\ e^\sigma V_{y_2} & \mathsf{e}_{y_2}{}^2 & \mathsf{e}_{y_2}{}^\alpha \\ e^\sigma V_\mu & \mathsf{e}_\mu{}^2 & \mathsf{e}_\mu{}^\alpha \end{pmatrix} \xrightarrow{T} \begin{pmatrix} e^{-\sigma} & 0 & 0 \\ e^{-\sigma} W_{y_2} & \mathsf{e}_{y_2}{}^2 & \mathsf{e}_{y_2}{}^\alpha \\ e^{-\sigma} W_\mu & \mathsf{e}_\mu{}^2 & \mathsf{e}_\mu{}^\alpha \end{pmatrix} \xrightarrow{s} \begin{pmatrix} e^{-\sigma}(1-\eta W_{y_2}) & -\eta\mathsf{e}_{y_2}{}^2 & -\eta\mathsf{e}_{y_2}{}^\alpha \\ e^{-\sigma} W_{y_2} & \mathsf{e}_{y_2}{}^2 & \mathsf{e}_{y_2}{}^\alpha \\ e^{-\sigma} W_\mu & \mathsf{e}_\mu{}^2 & \mathsf{e}_\mu{}^\alpha \end{pmatrix}. \tag{5.28}$$

The shift is spoiling the choice (5.3) for the vielbein, and that is an important point because we will want to restore this gauge before going back to the Buscher scheme and implementing the last T-duality. To achieve it we implement the Lorentz transformation $e_m{}^a \to e_m{}^b L_b{}^a$ where

$$L_b{}^a = \begin{pmatrix} \frac{1-\eta W_{y_2}}{D^{1/2}} & \frac{\eta e^\sigma\sqrt{g_{y_2y_2}}}{D^{1/2}} & 0 \\ -\frac{\eta e^\sigma\sqrt{g_{y_2y_2}}}{D^{1/2}} & \frac{1-\eta W_{y_2}}{D^{1/2}} & 0 \\ 0 & 0 & \delta_b{}^a \end{pmatrix}, \quad \text{where } D = 1-\eta W_{y_2}(2-\eta W_{y_2})+\eta^2 e^{2\sigma} g_{y_2y_2}. \tag{5.29}$$

At this point one wants to go to the Buscher scheme, in order to perform the last T-duality, which will produce a new correction $\delta_3(\hat{\hat{G}}_{mn}-\hat{\hat{B}}_{mn}) = -\frac{1}{2}\hat{\hat{\omega}}^{(-)}_{mab}\hat{\hat{\omega}}^{(+)ab}_n$. Now a double hat is used to denote that a T-duality and a shift (followed by the compensating Lorentz transformation) have been implemented. The contribution δ_2 (on which we implement the effect of the shift) and δ_3 can be considered together. In fact all expressions from covariant terms cancel out and we are left with

$$\frac{1}{2}\left(-\hat{\hat{\omega}}^{(-)}_{mab}(L^{-1}\partial_n L)^{ab} - \hat{\hat{\omega}}^{(+)}_{nab}(L^{-1}\partial_m L)^{ab} + (L^{-1}\partial_m L)_{ab}(L^{-1}\partial_n L)^{ab}\right). \tag{5.30}$$

In order to account for the effect of the last T-duality on the above expression one uses: the fact that in the first two terms only $(mn)\neq(y_iy_j)$ contribute, that in the summation of a,b only $1,2$ contribute, the fact that under T-duality

$$\hat{\omega}^{(\pm)}_{\iota\iota\beta} = -\omega^{(\pm)}_{\iota\iota\beta}, \qquad \hat{\omega}^{(\pm)}_{\alpha\iota\beta} = \pm\omega^{(\pm)}_{\alpha\iota\beta}, \qquad \hat{\omega}^{(\pm)}_{\iota\alpha\beta} = \mp\omega^{(\pm)}_{\iota\alpha\beta}, \qquad \hat{\omega}^{(\pm)}_{\alpha\beta\gamma} = \omega^{(\pm)}_{\alpha\beta\gamma}, \tag{5.31}$$

and finally that the last term in (5.30) vanishes if m or n are y_i, so that it actually remains the same after T-duality. After taking everything into account the result after the T-duality is simply

$$\frac{1}{2}\left(\tilde{\omega}^{(-)}_{mab}(L^{-1}\partial_n L)^{ab} - \tilde{\omega}^{(+)}_{nab}(L^{-1}\partial_m L)^{ab} + (L^{-1}\partial_m L)_{ab}(L^{-1}\partial_n L)^{ab}\right). \tag{5.32}$$

A tilde denotes the quantities of the TsT-transformed background.

JHEP07(2020)103

After the last T-duality has been performed, we go back from Buscher to the HT scheme using (5.13) obtaining the final contribution to the α' corrections which is $\delta_4(\tilde{G}_{mn}-\tilde{B}_{mn}) = +\frac{1}{2}\tilde{\omega}^{(-)}_{mab}\tilde{\omega}^{(+)ab}_{n}$.

Collecting together all contributions we obtain the α' correction to the TsT deformed background in the HT scheme

$$\begin{aligned}\delta(\tilde{G}-\tilde{B})_{mn} = \frac{1}{2}\left(\tilde{\omega}^{(-)}_{mab} - (L^{-1}\partial_m L)_{ab}\right)\left(\tilde{\omega}^{(+)ab}_{n} + (L^{-1}\partial_n L)^{ab}\right) + (L^{-1}\partial_m L)_{ab}(L^{-1}\partial_n L)^{ab} \\ - \frac{1}{2}[(1+(G-B)\Theta)^{-1}]_m{}^p\omega^{(-)}_{pab}\omega^{(+)ab}_{q}[(1+\Theta(G-B))^{-1}]^q{}_n. \qquad (5.33)\end{aligned}$$

Because of the steps of TsT, the vielbein used to construct the above spin-connection of the deformed model is defined as $\tilde{e}_a{}^m = L_a{}^b e_b{}^n(1-(G+B)\Theta)_n{}^m$, where the undeformed vielbein must respect (5.3), and one can check that the Lorentz transformation used here is related to the one in (3.6) simply as $L^2 = \tilde{\Lambda}$. To compare to the result (4.16) we need to use the same deformed vielbein used there, meaning that we should rather take $\tilde{e}_a{}^m = (L^2)_a{}^b e_b{}^n(1-(G+B)\Theta)_n{}^m$. After taking into account this extra Lorentz transformation we match with (4.16) in the case of TsT if we remember that $B^{\rm WZW}$ can be taken to be zero, and if we use that we for TsT we can write $L^{-1}dL = dLL^{-1}$ because here L is essentially a 2×2 anti-symmetric matrix and it commutes with itself.

With a similar reasoning we can obtain the α'-corrections to the dilaton of the TsT-transformed background. The simplification in this case is that the dilaton is insensitive to the shift, because by assumption it is isometric and the field redefinitions for the dilaton between the schemes of Buscher and HT are covariant. In HT scheme at generic q we get

$$\widetilde{\Phi} = \Phi + \frac{1}{2}\log\frac{\widetilde{G}_{y_1y_1}}{G_{y_1y_1}} + \alpha'\left[\frac{1+3q}{24}(H^2-\widetilde{H}^2) - \frac{1}{2}\left(\frac{\delta_1 G_{y_1y_1}}{G_{y_1y_1}} + \frac{\delta_4 G_{y_1y_1}}{\widetilde{G}_{y_1y_1}}\right)\right]. \qquad (5.34)$$

At $q = 1/6$ on finds

$$e^{-2\widetilde{\Phi}}\sqrt{-\det\widetilde{G}} = e^{-2\Phi}\sqrt{-\det G}\,, \qquad (5.35)$$

which is in agreement with (4.17), since there the result was written when setting $q = 0$, and one therefore has the extra H^2-terms.

6 Concluding comments

In this paper we have demonstrated that it is possible to extend the YB deformation as a solution-generating technique in string theory at least to first order in the α'-expansion. The explicit expression that we found for the corrections allowed us to test successfully our results on explicit examples. We expect our formula to be useful when addressing specific questions on the α'-corrected YB-deformed backgrounds. For example, it would be interesting to see whether the singularities that are sometimes introduced by the deformation procedure are in fact cured by α'-corrections. Another point is the computation of physical observables on the deformed backgrounds — such as entropy calculations in black hole solutions,[30] see e.g. [42, 47–49] — for which the explicit corrections are needed. It would be also interesting to investigate the relation to (quantum) integrability when considering YB-deformations of integrable 2-dimensional σ-models.

We have seen that the α'-correction to YB deformations comes from a compensating Lorentz transformation under which the $O(d,d)$ covariant metric and B-field transform anomalously. It is natural to expect that the same should be true also for T-duality. In fact abelian and non-abelian T-dualities are used to construct the YB deformation and they can also be obtained as a limit (sending the deformation parameter to infinity) of YB deformations. In fact we have already argued that for abelian T-duality the correction is given by precisely the same mechanism. It is therefore very natural to expect the first α'-correction to non-abelian T-duality[31] (on a unimodular algebra) to be given by the same expression, with the Lorentz transformation required for NATD substituted for $\tilde{\Lambda}$ in (4.16) and (4.17).

As in previous works on YB deformations and NATD (see e.g. [5–7]) here it was assumed that the undeformed B-field and vielbein are isometric, i.e. that they have vanishing Lie derivative with respect to the Killing vectors entering Θ. The covariant form of the corrections we have found seems to be valid more generally but it would be interesting to analyze more systematically how to relax these assumptions.

In [16] YB deformations of strings on $\mathrm{AdS}_3 \times S^3$ were studied, and their relation to marginal deformations of WZW models was analyzed. The results of the current paper show that marginal deformations of current algebras include (at least to 2 loops and probably to all loops) also cases which do not solve the "strong version" of the marginality condition of Chaudhuri and Schwartz [52], see [16] for more details. These additional possibilities arise when considering algebras that are not compact. Let us also comment that the deformation generated by the unimodular non-abelian R_9 of [16] must be marginal to all loops, since it can be simply understood as a non-commuting sequence of TsT transformations.

We expect that generalizations of our discussion to a construction in the spirit of the $\mathcal{E}$-model of Klimčik [53–55] will lead to an understanding of the form of α'-corrections for the η-deformation [2, 3], the λ-deformation [56, 57], and to Poisson-Lie T-duality [58].

Another important question we hope to return to is if the structure of the correction found here persists beyond first order in α' or whether novel corrections are required at order α'^2.

Acknowledgments

We thank D. Marqués for discussions and A. Tseytlin for comments on the manuscript. The work of RB is supported by the fellowship of "la Caixa Foundation" (ID 100010434) with code LCF/BQ/PI19/11690019, by FPA2017-84436-P, and by Xunta de Galicia (ED431C 2017/07). AVL is supported by the Spanish MECD fellowship FPU16/06675. RB and

[30] While in this paper we have considered only the case of the bosonic string, it is easy to generalize our results to generic values of the parameters a, b interpolating between the bosonic and the heterotic string.

[31] Using very different arguments NATD has been argued to preserve Weyl invariance at least to 2 loops, and probably to all orders in [50, 51].

AVL are also supported by the European Regional Development Fund (ERDF/FEDER program), by the "María de Maeztu" Units of Excellence program MDM-2016-0692 and by the Spanish Research State Agency. The work of LW is supported by the grant "Integrable Deformations" (GA20-04800S) from the Czech Science Foundation (GACR).

A Field redefinitions between different schemes

In this appendix we collect the field redefinitions needed to relate — to first order in the α' expansion — the schemes we use in this paper and others relevant in the literature. These are the schemes of Hull and Townsend (HT) [59], Metsaev and Tseytlin (MT) [60], Kaloper and Meissner (KP) [26, 61], Bergshoeff and de Roo (BR) [44, 45]. From [33] we read

$$\begin{aligned} G^{(\mathrm{BR})}_{mn} &= G^{(\mathrm{MT})}_{mn} - \frac{1}{2}\gamma_+ H^2_{mn}, \\ B^{(\mathrm{BR})}_{mn} &= B^{(\mathrm{MT})}_{mn} - \gamma_+ \left(\nabla^p H_{mnp} - 2H_{mnp}\nabla^p\Phi + H_{[m}{}^{ab}\omega_{n]ab}\right) \\ &\simeq B^{(\mathrm{MT})}_{mn} - \gamma_+ H_{[m}{}^{ab}\omega_{n]ab}, \\ \Phi^{(\mathrm{BR})} &= \Phi^{(\mathrm{MT})} - \frac{1}{8}\gamma_+ H^2. \end{aligned} \tag{A.1}$$

The symbol $\simeq$ is used when the expressions are simplified by means of the 1-loop equations of motion. We relate the parameters $\gamma_\pm = \mp(a \pm b)/4$ to a, b used in [33]. The bosonic string is obtained at $\gamma_+ = \alpha'/2, \gamma_- = 0$ and the heterotic string at $\gamma_\pm = \pm\alpha'/4$. In the following we will specify to the case of the bosonic string. To relate HT and MT schemes we use

$$\begin{aligned} G^{(\mathrm{HT})}_{mn} &= G^{(\mathrm{MT})}_{mn} - \frac{1}{2}\alpha' H^2_{mn}, \\ B^{(\mathrm{HT})}_{mn} &= B^{(\mathrm{MT})}_{mn}, \\ \Phi^{(\mathrm{HT})} &= \Phi^{(\mathrm{MT})} + \frac{1}{8}\alpha'\left(-1 + \frac{1}{6}(1-6q)\right)H^2. \end{aligned} \tag{A.2}$$

The parameter q appears in [59], and we normally set $q = 0$ in the rest of the paper as in [21]. Notice that the sign of the correction to the metric differs from what one would read in [59]. We have checked that this is the correct sign in order to have the correct α'-corrections for T-duality and YB deformations. From [61] we read that

$$\begin{aligned} G^{(\mathrm{MT})}_{mn} &= G^{(\mathrm{KM})}_{mn} + \alpha' R_{mn}, \\ B^{(\mathrm{MT})}_{mn} &= B^{(\mathrm{KM})}_{mn} - \alpha' H_{mnp}\nabla^P\Phi, \\ \Phi^{(\mathrm{MT})} &= \Phi^{(\mathrm{KM})} + \alpha'\left(\frac{1}{8}R - \frac{1}{2}(\partial\Phi)^2 + \frac{1}{96}H^2\right). \end{aligned} \tag{A.3}$$

The fields of the non-covariant scheme that follows from the DFT formulation are denoted simply with a bar $\bar{G}, \bar{B}, \bar{\Phi}$. They are related to the fields in the HT scheme as

$$\begin{aligned} \bar{G}_{mn} &= G^{(\mathrm{HT})}_{mn} - \frac{1}{2}\alpha'\omega^{(-)ab}_{(m}\omega^{(+)}_{n)ab} = G^{(\mathrm{HT})}_{mn} + \alpha'\left(-\frac{1}{2}\omega_{mab}\omega^{ab}_n + \frac{1}{8}H^2_{mn}\right), \\ \bar{B}_{mn} &= B^{(\mathrm{HT})}_{mn} + \frac{1}{2}\alpha'\omega^{(-)ab}_{[m}\omega^{(+)}_{n]ab} = B^{(\mathrm{HT})}_{mn} - \frac{1}{2}\alpha' H_{ab[m}\omega^{ab}_{n]}. \end{aligned} \tag{A.4}$$

References

[1] C. Klimčík, *Yang-Baxter σ-models and dS/AdS T duality*, JHEP **12** (2002) 051 [hep-th/0210095] [INSPIRE].

[2] C. Klimčík, *On integrability of the Yang-Baxter σ-model*, J. Math. Phys. **50** (2009) 043508 [arXiv:0802.3518] [INSPIRE].

[3] F. Delduc, M. Magro and B. Vicedo, *An integrable deformation of the $AdS_5 \times S^5$ superstring action*, Phys. Rev. Lett. **112** (2014) 051601 [arXiv:1309.5850] [INSPIRE].

[4] I. Kawaguchi, T. Matsumoto and K. Yoshida, *Jordanian deformations of the $AdS_5 \times S^5$ superstring*, JHEP **04** (2014) 153 [arXiv:1401.4855] [INSPIRE].

[5] B. Hoare and A.A. Tseytlin, *Homogeneous Yang-Baxter deformations as non-abelian duals of the AdS_5 σ-model*, J. Phys. A **49** (2016) 494001 [arXiv:1609.02550] [INSPIRE].

[6] R. Borsato and L. Wulff, *Integrable Deformations of T-Dual σ Models*, Phys. Rev. Lett. **117** (2016) 251602 [arXiv:1609.09834] [INSPIRE].

[7] R. Borsato and L. Wulff, *Non-abelian T-duality and Yang-Baxter deformations of Green-Schwarz strings*, JHEP **08** (2018) 027 [arXiv:1806.04083] [INSPIRE].

[8] D. Osten and S.J. van Tongeren, *Abelian Yang-Baxter deformations and TsT transformations*, Nucl. Phys. B **915** (2017) 184 [arXiv:1608.08504] [INSPIRE].

[9] E. Alvarez, L. Álvarez-Gaumé and Y. Lozano, *On nonAbelian duality*, Nucl. Phys. B **424** (1994) 155 [hep-th/9403155] [INSPIRE].

[10] S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer and G. Veneziano, *Remarks on nonAbelian duality*, Nucl. Phys. B **435** (1995) 147 [hep-th/9409011] [INSPIRE].

[11] G. Arutyunov, S. Frolov, B. Hoare, R. Roiban and A.A. Tseytlin, *Scale invariance of the η-deformed $AdS_5 \times S^5$ superstring, T-duality and modified type-II equations*, Nucl. Phys. B **903** (2016) 262 [arXiv:1511.05795] [INSPIRE].

[12] L. Wulff and A.A. Tseytlin, *κ-symmetry of superstring σ-model and generalized* 10d *supergravity equations*, JHEP **06** (2016) 174 [arXiv:1605.04884] [INSPIRE].

[13] R. Borsato and L. Wulff, *Target space supergeometry of η and λ-deformed strings*, JHEP **10** (2016) 045 [arXiv:1608.03570] [INSPIRE].

[14] J.-I. Sakamoto and Y. Sakatani, *Local β-deformations and Yang-Baxter σ-model*, JHEP **06** (2018) 147 [arXiv:1803.05903] [INSPIRE].

[15] L. Wulff, *Trivial solutions of generalized supergravity vs non-abelian T-duality anomaly*, Phys. Lett. B **781** (2018) 417 [arXiv:1803.07391] [INSPIRE].

[16] R. Borsato and L. Wulff, *Marginal deformations of WZW models and the classical Yang-Baxter equation*, J. Phys. A **52** (2019) 225401 [arXiv:1812.07287] [INSPIRE].

[17] J.-i. Sakamoto, Y. Sakatani and K. Yoshida, *Homogeneous Yang-Baxter deformations as generalized diffeomorphisms*, J. Phys. A **50** (2017) 415401 [arXiv:1705.07116] [INSPIRE].

[18] D. Lüst and D. Osten, *Generalised fluxes, Yang-Baxter deformations and the O(d,d) structure of non-abelian T-duality*, JHEP **05** (2018) 165 [arXiv:1803.03971] [INSPIRE].

[19] I. Bakhmatov and E.T. Musaev, *Classical Yang-Baxter equation from β-supergravity*, JHEP **01** (2019) 140 [arXiv:1811.09056] [INSPIRE].

[20] W.H. Baron, J.J. Fernandez-Melgarejo, D. Marques and C. Núñez, *The Odd story of α'-corrections*, JHEP **04** (2017) 078 [arXiv:1702.05489] [INSPIRE].

[21] R. Borsato and L. Wulff, *Two-loop conformal invariance for Yang-Baxter deformed strings*, JHEP **03** (2020) 126 [arXiv:1910.02011] [INSPIRE].

[22] A.A. Tseytlin, *Duality and dilaton*, Mod. Phys. Lett. A **6** (1991) 1721 [INSPIRE].

[23] J. Panvel, *Higher order conformal invariance of string backgrounds obtained by O(d,d) transformations*, Phys. Lett. B **284** (1992) 50 [hep-th/9204024] [INSPIRE].

[24] E. Bergshoeff, B. Janssen and T. Ortín, *Solution generating transformations and the string effective action*, Class. Quant. Grav. **13** (1996) 321 [hep-th/9506156] [INSPIRE].

[25] P.E. Haagensen and K. Olsen, *T duality and two loop renormalization flows*, Nucl. Phys. B **504** (1997) 326 [hep-th/9704157] [INSPIRE].

[26] N. Kaloper and K.A. Meissner, *Duality beyond the first loop*, Phys. Rev. D **56** (1997) 7940 [hep-th/9705193] [INSPIRE].

[27] I. Jack and S. Parsons, *$O(d,d)$ invariance at two loops and three loops*, Phys. Rev. D **62** (2000) 026003 [hep-th/9911064] [INSPIRE].

[28] S. Parsons, *T duality and conformal invariance at two loops*, Phys. Rev. D **61** (2000) 086002 [hep-th/9912105] [INSPIRE].

[29] O. Hohm and B. Zwiebach, *On the Riemann Tensor in Double Field Theory*, JHEP **05** (2012) 126 [arXiv:1112.5296] [INSPIRE].

[30] O. Hohm, W. Siegel and B. Zwiebach, *Doubled α'-geometry*, JHEP **02** (2014) 065 [arXiv:1306.2970] [INSPIRE].

[31] O. Hohm and B. Zwiebach, *Double field theory at order α'*, JHEP **11** (2014) 075 [arXiv:1407.3803] [INSPIRE].

[32] O. Hohm and B. Zwiebach, *Green-Schwarz mechanism and α'-deformed Courant brackets*, JHEP **01** (2015) 012 [arXiv:1407.0708] [INSPIRE].

[33] D. Marques and C.A. Núñez, *T-duality and α'-corrections*, JHEP **10** (2015) 084 [arXiv:1507.00652] [INSPIRE].

[34] W. Siegel, *Superspace duality in low-energy superstrings*, Phys. Rev. D **48** (1993) 2826 [hep-th/9305073] [INSPIRE].

[35] W. Siegel, *Two vierbein formalism for string inspired axionic gravity*, Phys. Rev. D **47** (1993) 5453 [hep-th/9302036] [INSPIRE].

[36] O. Hohm and S.K. Kwak, *Frame-like Geometry of Double Field Theory*, J. Phys. A **44** (2011) 085404 [arXiv:1011.4101] [INSPIRE].

[37] G. Aldazabal, D. Marques and C. Núñez, *Double Field Theory: A Pedagogical Review*, Class. Quant. Grav. **30** (2013) 163001 [arXiv:1305.1907] [INSPIRE].

[38] O. Hohm, D. Lüst and B. Zwiebach, *The Spacetime of Double Field Theory: Review, Remarks and Outlook*, Fortsch. Phys. **61** (2013) 926 [arXiv:1309.2977] [INSPIRE].

JHEP07(2020)103

[39] D.S. Berman and D.C. Thompson, *Duality Symmetric String and M-theory*, Phys. Rept. **566** (2014) 1 [arXiv:1306.2643] [INSPIRE].

[40] T. Araujo, I. Bakhmatov, E.Ó. Colgáin, J. Sakamoto, M.M. Sheikh-Jabbari and K. Yoshida, *Yang-Baxter σ-models, conformal twists and noncommutative Yang-Mills theory*, Phys. Rev. D **95** (2017) 105006 [arXiv:1702.02861] [INSPIRE].

[41] A. Çatal Özer and S. Tunalı, *Yang-Baxter Deformation as an $O(d,d)$ Transformation*, Class. Quant. Grav. **37** (2020) 075003 [arXiv:1906.09053] [INSPIRE].

[42] J.D. Edelstein, K. Sfetsos, J. Sierra-Garcia and A. Vilar López, *T-duality equivalences beyond string theory*, JHEP **05** (2019) 082 [arXiv:1903.05554] [INSPIRE].

[43] C. Eloy, O. Hohm and H. Samtleben, *Green-Schwarz Mechanism for String Dualities*, Phys. Rev. Lett. **124** (2020) 091601 [arXiv:1912.01700] [INSPIRE].

[44] E. Bergshoeff and M. de Roo, *Supersymmetric Chern-Simons Terms in Ten-dimensions*, Phys. Lett. B **218** (1989) 210 [INSPIRE].

[45] E.A. Bergshoeff and M. de Roo, *The Quartic Effective Action of the Heterotic String and Supersymmetry*, Nucl. Phys. B **328** (1989) 439 [INSPIRE].

[46] M.R. Garousi, *Four-derivative couplings via the T-duality invariance constraint*, Phys. Rev. D **99** (2019) 126005 [arXiv:1904.11282] [INSPIRE].

[47] A.S. Arvanitakis and C.D.A. Blair, *Black hole thermodynamics, stringy dualities and double field theory*, Class. Quant. Grav. **34** (2017) 055001 [arXiv:1608.04734] [INSPIRE].

[48] P.A. Cano, P. Meessen, T. Ortín and P.F. Ramírez, *α'-corrected black holes in String Theory*, JHEP **05** (2018) 110 [arXiv:1803.01919] [INSPIRE].

[49] Z. Elgood and T. Ortín, *T duality and Wald entropy formula in the Heterotic Superstring effective action at first order in α'*, arXiv:2005.11272 [INSPIRE].

[50] B. Hoare, N. Levine and A.A. Tseytlin, *Integrable 2d σ-models: quantum corrections to geometry from RG flow*, Nucl. Phys. B **949** (2019) 114798 [arXiv:1907.04737] [INSPIRE].

[51] B. Hoare, N. Levine and A.A. Tseytlin, *Integrable σ-models and 2-loop RG flow*, JHEP **12** (2019) 146 [arXiv:1910.00397] [INSPIRE].

[52] S. Chaudhuri and J.A. Schwartz, *A Criterion for Integrably Marginal Operators*, Phys. Lett. B **219** (1989) 291 [INSPIRE].

[53] C. Klimčík and P. Ševera, *Poisson-Lie T duality and loop groups of Drinfeld doubles*, Phys. Lett. B **372** (1996) 65 [hep-th/9512040] [INSPIRE].

[54] A. Stern, *Hamiltonian approach to Poisson Lie T-duality*, Phys. Lett. B **450** (1999) 141 [hep-th/9811256] [INSPIRE].

[55] C. Klimčík, *η and λ deformations as $\mathcal{E}$-models*, Nucl. Phys. B **900** (2015) 259 [arXiv:1508.05832] [INSPIRE].

[56] K. Sfetsos, *Integrable interpolations: From exact CFTs to non-Abelian T-duals*, Nucl. Phys. B **880** (2014) 225 [arXiv:1312.4560] [INSPIRE].

[57] T.J. Hollowood, J. Miramontes and D.M. Schmidtt, *An Integrable Deformation of the $AdS_5 \times S^5$ Superstring*, J. Phys. A **47** (2014) 495402 [arXiv:1409.1538] [INSPIRE].

[58] C. Klimčík and P. Ševera, *Dual nonAbelian duality and the Drinfeld double*, Phys. Lett. B **351** (1995) 455 [hep-th/9502122] [INSPIRE].

[59] C.M. Hull and P.K. Townsend, *String Effective Actions From σ Model Conformal Anomalies*, *Nucl. Phys. B* **301** (1988) 197 [INSPIRE].

[60] R.R. Metsaev and A.A. Tseytlin, *Order alpha-prime (Two Loop) Equivalence of the String Equations of Motion and the σ-model Weyl Invariance Conditions: Dependence on the Dilaton and the Antisymmetric Tensor*, *Nucl. Phys. B* **293** (1987) 385 [INSPIRE].

[61] K.A. Meissner, *Symmetries of higher order string gravity actions*, *Phys. Lett. B* **392** (1997) 298 [hep-th/9610131] [INSPIRE].

Published for SISSA by Springer

Received: *December 10, 2019*
Revised: *February 12, 2020*
Accepted: *March 6, 2020*
Published: *March 23, 2020*

Two-loop conformal invariance for Yang-Baxter deformed strings

Riccardo Borsato[a] and Linus Wulff[b]

[a] *Instituto Galego de Física de Altas Enerxías (IGFAE), Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain*

[b] *Department of Theoretical Physics and Astrophysics, Masaryk University, 611 37 Brno, Czech Republic*

E-mail: riccardo.borsato@usc.es, wulff@physics.muni.cz

Abstract: The so-called homogeneous Yang-Baxter (YB) deformations can be considered a non-abelian generalization of T-duality-shift-T-duality (TsT) transformations. TsT transformations are known to preserve conformal symmetry to all orders in α'. Here we argue that (unimodular) YB deformations of a bosonic string also preserve conformal symmetry, at least to two-loop order. We do this by showing that, starting from a background with no NSNS-flux, the deformed background solves the α'-corrected supergravity equations to second order in the deformation parameter. At the same time we determine the required α'-corrections of the deformed background, which take a relatively simple form. In examples that can be constructed using, possibly non-commuting sequences of, TsT transformations we show how to obtain the first α'-correction to all orders in the deformation parameter by making use of the α'-corrected T-duality rules. We demonstrate this on the specific example of YB deformations of a Bianchi type II background.

Keywords: Bosonic Strings, Conformal Field Models in String Theory, String Duality

ArXiv ePrint: 1910.02011

https://doi.org/10.1007/JHEP03(2020)126

JHEP03(2020)126

Contents

1 Introduction and summary of results

Yang-Baxter (YB) deformations were first introduced by Klimčik in [1]. It was later understood that they have the remarkable property of preserving integrability [2]. If one starts from an integrable sigma model and performs a YB deformation the resulting model is also integrable. This made people interested in applying them in string theory, which was done for the $\mathrm{AdS}_5 \times S^5$ superstring in [3, 4]. The YB deformation is based on an R-matrix for which there are two basic possibilities — R can solve either the classical Yang-Baxter equation (CYBE) or the modified classical Yang-Baxter equation (mCYBE). The former case is often referred to as *homogeneous* YB deformations and is the case we consider here. It was shown in [5] that these models typically have a Weyl-anomaly[1] unless the R-matrix is unimodular, i.e. its contraction with the structure constants of the isometry algebra of the original model vanishes $R^{IJ} f_{IJ}{}^{K} = 0$. This is similar to the anomaly encountered in non-abelian T-duality (NATD) [8] on a non-unimodular group [9–11]. Indeed it was argued in [12] that homogeneous YB deformations should have a realization in terms of NATD and this was then proven in [13] (see also [14]). While the original YB deformations were defined only for sigma models of the symmetric space type, the realization of the homogeneous

[1] This manifests itself, in the superstring case, as a target space solving the generalized supergravity equations [6, 7] rather than the standard ones.

models using NATD meant that they could be defined for a general string sigma model with isometries. This was carried out for the Green-Schwarz superstring in [15] and rules for writing the supergravity background directly in terms of the R-matrix were derived.[2]

The simplest class of such YB deformations is when R is defined on an abelian subalgebra of the isometry algebra. In this case the deformation is equivalent to a T-duality-shift-T-duality (TsT) transformation [19]. These are also known as $O(d,d)$-transformations [20, 21] and they have been argued to map a consistent string background to another consistent string background, i.e. there should exist corrections to the background fields such that the corrected background solves the α'-corrected supergravity equations to all orders in α' [22–27].[3] Here we want to ask what happens for YB deformations in general at the quantum level.[4] Unimodular YB deformations are known to give a conformal theory at one loop, i.e. the background solves the (super)gravity equations. Here we will analyze the two-loop equations in the bosonic string case. For simplicity we will restrict to deformations of backgrounds with vanishing NSNS-flux. We will show, to second order in the deformation parameter, that the deformed background can be corrected so that it solves the 2-loop equations. Furthermore the correction to the background fields can be cast in a relatively simple form. Using the knowledge of the full corrections in special cases derived using T-duality (see below), we write an expression to all orders in the deformation parameter, which works in some simple cases but not in general.

Since the homogeneous YB deformations can be constructed using NATD, our results indicate that also NATD should preserve conformality at two loops, and possibly all orders in α'. A convincing argument for the preservation of conformality for NATD would follow from a generic analysis to all orders in the deformation parameter η, since NATD is recovered in a $\eta \to \infty$ limit. Another piece of evidence for this comes from the recent analysis of renormalizability of deformed sigma models with two-dimensional target space in [30], and very recently [31] (see also [32]). Some of the deformations considered have a limit where they reduce to NATD and it was found that the models behave nicely beyond lowest order in α' suggesting that things should work out to all orders in α'.

For YB deformations of TsT-type we can also exploit another method to obtain explicit α'-corrections and to promote those backgrounds to two-loop solutions. We can in fact use the known α'-corrections to the T-duality rules when doing the chain of T-duality-shift-T-duality. This strategy will automatically bring in the needed α'-dependence into the deformed background, and will make sure that the deformed background is a solution to the two-loop equations. The interplay between T-duality and higher α'-corrections was studied in various works [26, 33–37]. In this paper we will use the α'-corrections for the T-duality rules of [34], to obtain explicit α'-corrections for YB deformed models. This strategy allows us to start from any background with isometries (it is not necessary to set the NSNS-flux to zero), and to keep the dependence on the deformation parameter exact.

[2]These rules were first guessed, at the supergravity level and restricted to the case of vanishing NSNS flux, in [16] (see also [17] and [18]).

[3]Note however that the form of the α'-corrections are only known in special cases and to low loop order, e.g. [26].

[4]Homogeneous YB deformations also have an $O(d,d)$ interpretation as so called β-shifts [28, 29].

Certain YB deformations, while they cannot be understood as simple TsT transformations, can still be obtained as a non-commuting sequence of TsT's [5]. The non-commutativity is related to the fact that certain isometries needed to perform one TsT transformation may be broken by the application of another TsT. Therefore, in certain cases a sequence of TsT transformations can be implemented only in one precise order. Non-commuting sequences of TsT transformations are nice examples to study, because we can obtain explicit results by applying what is known about abelian T-duality and TsT, and at the same time be able to say something about NATD and more general YB deformations.

In the remaining part of the introduction, we will summarize the main results obtained when expanding the two-loop equations to second order in the deformation parameter.

1.1 First α'-correction to deformed backgrounds

The (homogeneous) Yang-Baxter deformation of a bosonic string background G, B, Φ is given by [15–18]

$$\tilde{G} - \tilde{B} = (G - B)(1 + \eta\Theta(G - B))^{-1}, \qquad \tilde{\Phi} = \Phi - \frac{1}{2}\ln\det\left(1 + \eta\Theta(G - B)\right). \tag{1.1}$$

Here η is the deformation parameter and Θ is constructed by taking an anti-symmetric R-matrix solving the classical Yang-Baxter equation (CYBE), $R^{[I|L|}R^{J|M|}f_{LM}{}^{K]} = 0$, on a subalgebra of the isometry algebra of the original background (with structure constants $f_{IJ}{}^K$) and contracting with the corresponding Killing vectors

$$\Theta^{ij} = k_I{}^i R^{IJ} k_J{}^j \equiv k^i \times k^j, \qquad \nabla_{(i}k_{Ij)} = 0, \tag{1.2}$$

where we simplify the notation by introducing the anti-symmetric product '$\times$'. Assuming that G, B, Φ define a one-loop conformal bosonic string sigma model, the same is true of $\tilde{G}, \tilde{B}, \tilde{\Phi}$ if R is unimodular, i.e. $R^{IJ}f_{IJ}{}^K = 0$ [5].[5]

Here we want to ask what happens at two loops, i.e. the next order in α'. We will work in an expansion in the deformation parameter up to order η^2. To simplify the calculations we will assume that the starting background has $B = 0$ which gives the deformed background

$$\tilde{G}_{ij} = G_{ij} + \eta^2(\Theta^2)_{ij} + \mathcal{O}(\eta^4), \quad \tilde{B}_{ij} = \eta\Theta_{ij} + \mathcal{O}(\eta^3), \quad \tilde{\Phi} = \Phi - \frac{1}{4}\eta^2\Theta_{ij}\Theta^{ij} + \mathcal{O}(\eta^4). \tag{1.3}$$

We find that to this order in η the first α'-correction (i.e. two-loop correction) to the background is given by (in the scheme of Hull and Townsend [40])

$$\begin{aligned}
\delta\tilde{G}_{ij} &= \delta G_{ij} + 2\eta^2(\delta G\Theta^2)_{(ij)} + \eta^2(\Theta\delta G\Theta)_{ij} - 2\eta^2\Theta_{k(i}R_{j)}{}^{klm}\Theta_{lm} + \eta^2\Theta^{mn}\nabla_i\nabla_j\Theta_{mn}\,, \\
\delta\tilde{B}_{ij} &= 2\eta(\delta G\Theta)_{[ij]} - \eta R_{ijkl}\Theta^{kl}\,, \\
\delta\tilde{\Phi} &= \delta\Phi - \frac{1}{2}\eta^2(\delta G\Theta)_{mn}\Theta^{mn} + \frac{1}{16}\eta^2\nabla^k\Theta^{mn}\nabla_k\Theta_{mn} - \frac{3}{8}\eta^2\nabla^k\Theta^{mn}\nabla_m\Theta_{nk} \\
&\quad + \frac{1}{4}\eta^2\nabla_i\Phi\,\nabla^i(\Theta^{mn}\Theta_{mn})\,.
\end{aligned} \tag{1.4}$$

[5]The unimodularity condition is sufficient but not necessary in general. Relaxing it one finds at order η, assuming $B = 0$, the necessary condition $dK = 0$ where $K^n = \nabla_m\Theta^{mn}$. This is equivalent to $\nabla_m k_I^n f_{JK}{}^I R^{JK} = 0$ which is in general weaker than the unimodularity condition $k_I^n f_{JK}{}^I R^{JK} = 0$. The reason for this is that sometimes the anomalous terms generated by a non-unimodular R can be removed by a field redefinition [38] (see also [39]). Here we will take R to be unimodular for simplicity.

JHEP03(2020)126

Here δG, $\delta\Phi$ denote the α' corrections to the undeformed background with $B = \delta B = 0$. Note that the terms involving δG just come from correcting the undeformed metric in (1.3), while the terms involving the Riemann tensor in $\delta\tilde{G}$ and $\delta\tilde{B}$ are obtained simply by replacing $\Theta_{ij} \to \Theta_{ij} - \alpha' R_{ijkl}\Theta^{kl}$ in (1.3). The correction to the dilaton does not look nice in this scheme but by changing the scheme one can arrange it so that

$$e^{-2\tilde{\Phi}}\sqrt{\det\tilde{G}} = e^{-2\Phi}\sqrt{\det G}\,, \tag{1.5}$$

so that the correction to the dilaton just comes from the correction to the determinant of the metric. This is achieved by the scheme change[6]

$$\Phi \to \Phi + \alpha'\left(-\frac{1}{2}\nabla^2\Phi + (\nabla\Phi)^2 - \frac{1}{16}H_{klm}H^{klm}\right). \tag{1.6}$$

With a little help from the corresponding expressions derived to all orders in η for a particular background in (5.30) and (5.31) one can write a completion of (1.4) to all orders in the deformation. First of all it is natural to expect that one should correct the undeformed metric and take $\Theta_{ij} \to \Theta_{ij} - \alpha' R_{ijkl}\Theta^{kl}$ in the expressions in (1.1). On top of this we need to extend the last term in the transformation of the metric and looking at the example in (5.30) and (5.31) suggests the following form for the corrections to all orders in η

$$\begin{aligned}\tilde{G}_{ij} - \tilde{B}_{ij} = {} & \left[G(1+\eta[\Theta - \alpha' R\cdot\Theta])^{-1}\right]_{ij} - \frac{1}{2}\alpha'\partial_i \ln\det(1+\eta\Theta)\partial_j \ln\det(1+\eta\Theta) \\ & + \frac{1}{2}\alpha'\eta\Big(\left[G(1+\eta\Theta)^{-1}\right]_{ik}\nabla^k\nabla_j\Theta^{mn} + \left[G(1-\eta\Theta)^{-1}\right]_{jk}\nabla^k\nabla_i\Theta^{mn}\Big)\left[G(1+\eta\Theta)^{-1}\right]_{nm}\end{aligned} \tag{1.7}$$

with the transformation of the dilaton read off from (1.5) (in the HT scheme after the shift (1.6)). Here indices are raised and lowered with the undeformed metric *including* its α'-corrections. We have also defined the contraction of Θ with the Riemann tensor $(R\cdot\Theta)_{ij} = R_{ijkl}\Theta^{kl}$. Note that this expression can be thought of as an α'-corrected open-closed string map, such as appears for example in the work of Seiberg and Witten on non-commutative gauge theories [41]. While this result works for the rank 2 examples in section 4 it unfortunately does not work in general.

2 Two-loop conformal invariance conditions

The conditions for two-loop conformal invariance of the bosonic string sigma model were worked out in [42–44]. Following Hull and Townsend (HT) the conditions in their scheme are [40][7]

$$F^G_{ij} = F^G_{0,ij} + \alpha' F^G_{1,ij} = 0\,, \quad F^B_{ij} = F^B_{0,ij} + \alpha' F^B_{1,ij} = 0\,, \quad F^\Phi_{ij} = F^\Phi_{0,ij} + \alpha' F^\Phi_{1,ij} = 0\,, \tag{2.1}$$

[6]On-shell this is equivalent to turning on the q parameter in the scheme of Hull and Townsend [40].
[7]To go from their conventions to ours one sends $\Phi \to 2\Phi$ and $H \to \frac{1}{2}H$.

where the one-loop conditions are

$$F^G_{0,ij} = R_{ij} - \frac{1}{4}H_{ikl}H_j{}^{kl} + 2\nabla_i\nabla_j\Phi\,,$$
$$F^B_{0,ij} = \nabla^k H_{ijk} - 2\nabla^k\Phi H_{ijk}\,, \tag{2.2}$$
$$F^\Phi_{0,ij} = 2\nabla^2\Phi - 4\nabla_i\Phi\nabla^i\Phi + \frac{1}{6}H_{ijk}H^{ijk}$$

and the two-loop corrections are

$$F^G_{1,ij} = \frac{1}{2}R_{iklm}R_j{}^{klm} + \frac{1}{4}R_{iklj}H^{kmn}H^l{}_{mn} + \frac{1}{4}R_{klm(i}H_{j)}{}^{mn}H^{kl}{}_n + \frac{1}{24}\nabla_i H_{klm}\nabla_j H^{klm}$$
$$- \frac{1}{8}\nabla^k H^{lm}{}_i\nabla_k H_{lmj} + \frac{1}{16}H_{ikp}H_{jlq}H^{klm}H^{pq}{}_m + \frac{1}{16}H_{ikp}H_{jl}{}^pH^{kmn}H^l{}_{mn}\,, \tag{2.3}$$
$$F^B_{1,ij} = \nabla^k H^{lm}{}_{[i}R_{j]klm} - \frac{1}{4}\nabla_k H_{lij}H^{kmn}H^l{}_{mn} + \frac{1}{2}\nabla^k H^{lm}{}_{[i}H_{j]mn}H_{kl}{}^n\,, \tag{2.4}$$
$$F^\Phi_{1,ij} = -\frac{1}{4}R_{ijkl}R^{ijkl} + \frac{1}{12}(\nabla_i H_{jkl})(\nabla^i H^{jkl}) + \frac{1}{8}H^{ij}{}_m H^{klm}R_{ijkl} + \frac{1}{4}R_{ij}(H^2)^{ij}$$
$$- \frac{5}{96}H_{ijk}H^i{}_{lm}H^{jl}{}_nH^{kmn} - \frac{3}{32}H^2_{ij}(H^2)^{ij}, \tag{2.5}$$

where $H^2_{ij} = H_{ikl}H_j{}^{kl}$. Here we have set to zero the parameter q of [40].

3 Expansion in the deformation parameter

In this section we expand the conditions for two-loop conformal invariance in powers of the deformation parameter η, and we find the explicit α' corrections for the background such that the conditions hold to the quadratic order in η. Here will not need to impose the equation for the dilaton. It is known that when the equations for G and B are satisfied the dilaton equation is satisfied up to a constant [40]. Since we assume the undeformed background to solve all the two-loop equations and since there is no way to introduce a constant at higher orders in η,[8] the dilaton equation will not add anything.

3.1 First order in the deformation parameter

At order η^1 we see, by looking at (1.3), that the metric is not deformed while[9]

$$H^{(1)}_{ijk} = 3\nabla_{[i}\Theta_{jk]}\,. \tag{3.1}$$

Using this in (2.4) we find

$$F^{B(1)}_{1,ij} = \nabla^k H^{(1)lm}{}_{[i}R_{j]klm} = \nabla^k(H^{(1)}_{lm[i}R_{j]k}{}^{lm}) + 2H^{(1)}_{lm[i}\nabla^l R_{j]}{}^m$$
$$= \frac{3}{2}\nabla^k\nabla_{[i}(R_{jk]lm}\Theta^{lm}) - \frac{1}{2}\nabla^k[R_{ijlm}\nabla_k\Theta^{lm}]$$
$$+ 2\nabla_k(R_{[i}{}^{klm}\nabla_{|l|}\Theta_{j]m}) - 2\nabla^k\Phi\, H^{(1)}_{lm[i}R_{j]k}{}^{lm}\,, \tag{3.2}$$

[8] The parameter η is always accompanied by Θ and it is not possible to construct a constant from a general Θ.

[9] We indicate the order in η with a superscript in parenthesis. Since it is clear that this refers to the deformed background we drop the tilde.

where we have used the lowest order equations (2.2). Using the two derivative Killing identity (A.2) we have

$$\begin{aligned}\nabla_k(R_i{}^{klm}\nabla_l\Theta_{jm}) &= \nabla_k R_i{}^{klm}\nabla_l\Theta_{jm} + R_i{}^{klm}\nabla_k\nabla_l\Theta_{jm}\\ &= \nabla_k R_i{}^{klm}\nabla_l\Theta_{jm} + 2R_i{}^{klm}\nabla_k\nabla_{(l}\Theta_{j)m} - R_i{}^{klm}\nabla_k\nabla_j\Theta_{lm}\\ &= -\frac{3}{2}\nabla_k(R_i{}^{klm}\nabla_j\Theta_{lm}) + 2R_i{}^{klm}R_{jkln}\Theta_{mn} - R_{im}{}^{kl}R_{jnkl}\Theta^{mn}\\ &\quad + R^{klmn}R_{klmi}\Theta_{jn} + 3R_{iklm}\nabla^k\Phi\,\nabla_j\Theta^{lm} + 2R_i{}^{klm}\nabla_k\Phi\,\nabla_l\Theta_{jm}\,. \end{aligned} \tag{3.3}$$

Using this together with the identity (A.9) we find

$$F^{B(1)}_{1,ij} = 3\nabla^k\nabla_{[i}(R_{jk]lm}\Theta^{lm}) - 6\nabla^k\Phi\,\nabla_{[i}(R_{jk]lm}\Theta^{lm}) + 2R^{klmn}R_{klm[i}\Theta_{j]n}\,. \tag{3.4}$$

Taking into account the α'-corrections to the classical background, $\alpha'\delta G$ and $\alpha'\delta\Phi$, and the B-field at order η^1, $\alpha'(\delta\tilde{B})^{(1)}$, we have

$$\begin{aligned}\alpha'^{-1}F^B_{ij} =& 3\nabla^k\nabla_{[i}(\delta\tilde{B})^{(1)}_{jk]} - 6\nabla^k\Phi\,\nabla_{[i}(\delta\tilde{B})^{(1)}_{jk]} + 3\nabla^k\nabla_{[i}(R_{jk]lm}\Theta^{lm}) - 6\nabla^k\Phi\,\nabla_{[i}(R_{jk]lm}\Theta^{lm})\\ &+ 3\delta(\nabla^k)\nabla_{[i}\Theta_{jk]} - 6\delta(\nabla^k\Phi)\,\nabla_{[i}\Theta_{jk]} + 2R^{klmn}R_{klm[i}\Theta_{j]n}\,. \end{aligned} \tag{3.5}$$

In the case where the metric and dilaton do not receive corrections, $\delta G = \delta\Phi = 0$, the terms in the second line vanish, and the terms in the first line also vanish provided we take

$$(\delta\tilde{B})^{(1)}_{ij} = -R_{ijkl}\Theta^{kl}\,. \tag{3.6}$$

In the general case the assumption that the corrected original background solves the two-loop equations implies that

$$\begin{aligned}R^{klm}{}_nR_{klmi} = -2\delta(R_{in} + 2\nabla_i\nabla_n\Phi) =& -\nabla^k\nabla_i\delta G_{kn} - \nabla^k\nabla_n\delta G_{ki} + G^{kl}\nabla_i\nabla_n\delta G_{kl}\\ &+ \nabla^2\delta G_{in} + 2\nabla^k\Phi(\nabla_i\delta G_{kn} + \nabla_n\delta G_{ki} - \nabla_k\delta G_{in}) - 4\nabla_i\nabla_n\delta\Phi\,, \end{aligned} \tag{3.7}$$

where we used the expressions for the variation of the Ricci tensor and Christoffel symbols (3.10) and (3.13).

Using this it is not hard to see, noting that $\delta\Phi$ must respect the isometries, that the $\delta\Phi$-terms cancel without any further correction to B. With a little bit more work one can show, using the fact that $\mathcal{L}_k\delta G_{ij} = 0$, i.e. that the correction to the undeformed metric does not break any isometries, that all terms cancel if one takes

$$(\delta\tilde{B})^{(1)}_{ij} = 2(\delta G\Theta)_{[ij]} - R_{ijkl}\Theta^{kl}\,. \tag{3.8}$$

The first term is simply the correction induced by the correction to the undeformed metric, i.e. $\delta(B^{(1)})_{ij} = \delta\Theta_{ij}$, which comes from the fact that the indices on Θ_{ij} were lowered with the metric (note that the Killing vectors k^m_I, with an upper index, are not corrected by assumption). Thus we have proven that a two-loop Weyl invariant sigma-model remains two-loop Weyl invariant under a YB deformation, at least to first order in the deformation parameter. We now consider what happens at second order.

3.2 Second order in the deformation parameter

It is easy to see that at order η^2 the B-field equation, $F^{B(2)}_{1,ij} = 0$, is trivially satisfied. For the metric equation we find

$$F^{G(2)}_{1,ij} = R^{(2)klm}_{(i} R_{j)klm} - \frac{1}{2} R_{(i}{}^{klm} R_{j)nlm} (\Theta^2)_k{}^n - R_{(i}{}^{klm} R_{j)kl}{}^n (\Theta^2)_{mn} \\ + \frac{1}{4} R_{kijl} H^{(1)kmn} H^{(1)l}{}_{mn} + \frac{1}{4} R_{klm(i} H^{(1)mn}_{j)} H^{(1)kl}{}_n + \frac{1}{24} \nabla_i H^{(1)}_{klm} \nabla_j H^{(1)klm} \\ - \frac{1}{8} \nabla^k H^{(1)lm}{}_i \nabla_k H^{(1)}_{lmj} . \tag{3.9}$$

Note that we choose to define all tensors to have lower indices, e.g. R_{ijkl}, and then raise indices with the *undeformed* metric G_{ij}.

The last two terms do not involve the Riemann tensor and the calculations can be simplified somewhat if we remove them by shifting the metric and dilaton. Under a shift of the metric we have

$$\delta(\nabla_i \nabla_j \Phi) = -\delta\Gamma^k_{ij} \nabla_k \Phi = -\frac{1}{2} \nabla^k \Phi (\nabla_j \delta G_{ki} + \nabla_i \delta G_{kj} - \nabla_k \delta G_{ij}) \tag{3.10}$$

and

$$\delta R_{ijkl} = \nabla_k (\delta\Gamma_{ilj} - \Gamma^m_{lj} \delta G_{im}) + \frac{1}{2} R^m{}_{jkl} \delta G_{im} - (k \leftrightarrow l) , \tag{3.11}$$

so that in particular

$$R^{(2)}_{ijkl} = \nabla_k (\Gamma^{(2)}_{[ij]l} + \Gamma^m_{l[i} (\Theta^2)_{j]m}) - \frac{1}{2} (\Theta^2)^m{}_{[i} R_{j]mkl} - (k \leftrightarrow l) \\ = -\nabla_k \nabla_{[i} (\Theta^2)_{j]l} + \nabla_l \nabla_{[i} (\Theta^2)_{j]k} - (\Theta^2)^m{}_{[i} R_{j]mkl} . \tag{3.12}$$

The variation of the Ricci tensor becomes (symmetrization in ij understood)

$$\delta R_{ij} = \delta G^{kl} R_{ikjl} + G^{kl} \delta R_{ikjl} \\ = \delta G_{kl} R^k{}_{ij}{}^l + R^k{}_j \delta G_{ik} + \nabla_j [G^{kl} \delta\Gamma_{ikl} - G^{kl} \Gamma^m_{kl} \delta G_{im}] - \nabla^k [\delta\Gamma_{ijk} - \Gamma^l_{jk} \delta G_{il}] \\ = \nabla^k \nabla_i \delta G_{kj} - \frac{1}{2} G^{kl} \nabla_i \nabla_j \delta G_{kl} - \frac{1}{2} \nabla^2 \delta G_{ij} . \tag{3.13}$$

From this expression we see that the last two terms in (3.9) can be canceled by shifting the metric and dilaton as

$$G_{ij} \to G_{ij} - \frac{1}{8} \alpha' H_{ikl} H_j{}^{kl} , \qquad \Phi \to \Phi - \frac{1}{32} \alpha' H_{klm} H^{klm} . \tag{3.14}$$

The two-loop contribution then becomes (symmetrization in ij understood)

$$F'^{G(2)}_{1,ij} = R^{(2)klm}_i R_{jklm} - \frac{1}{2} R_i{}^{klm} R_{jnlm} (\Theta^2)_k{}^n - R_i{}^{klm} R_{jkl}{}^n (\Theta^2)_{mn} + \frac{1}{8} R_{kijl} H^{(1)kmn} H^{(1)l}{}_{mn} \\ + \frac{1}{2} R_{klmi} H^{(1)mn}_j H^{(1)kl}{}_n - \frac{1}{8} R^{klmn} H^{(1)}_{ikl} H^{(1)}_{jmn} - \frac{1}{24} H^{(1)klm} \nabla_i \nabla_j H^{(1)}_{klm} . \tag{3.15}$$

Here we have used the Bianchi identity for H and the lowest order equations of motion, which in particular imply

$$\nabla^2 H_{klm} = 3 \nabla^n \nabla_{[k} H_{lm]n} = -3 R_{np[kl} H_{m]}{}^{np} + 6 \nabla^n \Phi \, \nabla_{[k} H_{lm]n} . \tag{3.16}$$

JHEP03(2020)126

Note that terms with two derivatives of $H^{(1)}$ indeed give something involving the Riemann tensor since they involve three derivatives acting on a product of two Killing vectors giving at least two derivatives on one Killing vector.

Expressing all terms in terms of the basis defined in appendix B we have (symmetrization in ij understood)

$$\begin{aligned} R_i^{(2)klm}R_{jklm} &= -\nabla\cdot(f_{12}+f_{20}) - \nabla(2\hat{f}_5-\hat{f}_6)+2g_{32}+g_{34}-g_{35}+h_7-\frac{1}{2}h_8 \\ &\quad +\frac{1}{2}h_{10}+2m_7+2m_9 \qquad (3.17)\\ R_{klmi}H_j^{(1)mn}H^{(1)kl}{}_n &= -g_3+2g_4-2g_6+g_8-2g_{14}+g_{15} \qquad (3.18)\\ R^{klmn}H^{(1)}_{ikl}H^{(1)}_{jmn} &= 4g_{16}+4g_{17}+g_{19}\,, \qquad (3.19)\\ H^{(1)klm}\nabla_i\nabla_jH^{(1)}_{klm} &= 3g_3-6g_4-6g_6+3g_8-18g_{14}+9g_{15}+6g_{28}-6g_{29}-3g_{31}+12g_{32}\\ &\quad -12g_{33}+12g_{34}\,. \qquad (3.20)\end{aligned}$$

While the order η α'-correction to $\tilde{B}$ in (3.8) contributes the terms (for the moment we assume that the undeformed background is not corrected) (symmetrization in ij understood)

$$-\frac{1}{2}(\delta\tilde{H})^{(1)}_{ikl}H_j^{(1)kl} = \frac{3}{2}\nabla_{[i}(R_{kl]mn}\Theta^{mn})H_j^{(1)kl} = g_3-g_8-g_{15}+g_{16}+\frac{1}{2}g_{19}\,. \qquad (3.21)$$

For the two-loop correction we therefore get $\frac{1}{8}$ times

$$\begin{aligned}&-8\nabla\cdot(f_{12}+f_{20})-8\nabla(2\hat{f}_5-\hat{f}_6)+3g_3+10g_4-6g_6-5g_8-2g_{14}-7g_{15}+4g_{16}-4g_{17}\\ &+3g_{19}+4g_{30}+2g_{31}+12g_{32}+4g_{33}+4g_{34}-8g_{35}-8h_8+4h_{10}+16m_7+16m_9 \qquad (3.22)\end{aligned}$$

To this we have to add the terms arising from the α'-corrections to $\tilde{G}$ and $\tilde{\Phi}$. We will ignore the corrections to the undeformed background until the end of the section.

Consider the following possible α'-corrections to the metric at order η^2 (symmetrization in ij understood)

$$\begin{aligned}\delta_1\tilde{G}_{ij} &= \nabla_i\Theta_{mn}\nabla_j\Theta^{mn}\,, \qquad (3.23)\\ \delta_2\tilde{G}_{ij} &= k_i\times\nabla_mk_n\,k_j\times\nabla^mk^n\,, \qquad (3.24)\\ \delta_3\tilde{G}_{ij} &= \nabla_i\Theta^{mn}\nabla_m\Theta_{nj}\,, \qquad (3.25)\\ \delta_4\tilde{G}_{ij} &= R_i{}^{klm}\Theta_{jk}\Theta_{lm}\,. \qquad (3.26)\end{aligned}$$

Note that we could write also the second one in terms of Θ as

$$\delta_2\tilde{G}_{ij} = \frac{1}{2}\nabla_m\Theta_{in}\nabla^m\Theta_j{}^n - \frac{1}{2}\nabla^n\Theta_{im}\nabla^m\Theta_{jn} - \nabla_i\Theta^{mn}\nabla_m\Theta_{nj} + \frac{1}{4}\nabla_i\Theta^{mn}\nabla_j\Theta_{mn}\,, \qquad (3.27)$$

but the above expression is more convenient for the following calculation. Using (3.13) and (3.10) these variations give rise to the terms

$$
\begin{aligned}
\delta_1\tilde{G}: \quad & -\nabla\cdot(2f_3+f_{28})-\nabla(\hat{f}_1+2\hat{f}_6)+g_{31}-4m_5-4m_6+2m_{20}\\
\delta_2\tilde{G}: \quad & \frac{1}{2}\nabla\cdot(f_1+2f_7-f_{14}-2f_{17}+f_{22}+2f_{23})+\nabla(-\hat{f}_1+2\hat{f}_2+2\hat{f}_3-\hat{f}_4+2\hat{f}_5)+g_{28}\\
& -g_{29}-2g_{30}+\frac{1}{2}g_{31}-2m_{12}+m_{13}+\frac{3}{8}\nabla_i\nabla_j(2\nabla^k\Theta^{mn}\nabla_m\Theta_{nk}-3\nabla^k\Theta^{mn}\nabla_k\Theta_{mn})\\
\delta_3\tilde{G}: \quad & -\frac{1}{2}\nabla\cdot(f_1+f_3+f_{10}-f_{11}+f_{22}+f_{28}+f_{30}-f_{31})\\
& +\frac{1}{4}\nabla(\hat{f}_1-2\hat{f}_2-2\hat{f}_3+\hat{f}_4-2\hat{f}_5+2\hat{f}_7-4\hat{f}_8)+g_{30}-m_5-m_6+m_7-m_8-m_{10}\\
& +m_{11}-m_{13}+m_{20}+m_{22}-m_{23}\\
\delta_4\tilde{G}: \quad & \frac{1}{2}\nabla\cdot(f_9+f_{14}-f_{26})-\frac{1}{4}\nabla(3\hat{f}_1+2\hat{f}_2+2\hat{f}_3-3\hat{f}_4-2\hat{f}_5+4\hat{f}_6)+h_9-m_1+m_2\\
& -m_3-m_{15}\,,
\end{aligned}
$$

where we used the identity (B.50) in calculating the last variation.

Taking the following correction to the metric and dilaton

$$
(\delta\tilde{G})^{(2)}_{ij}=\frac{1}{4}(-3\delta_1+2\delta_2+2\delta_3+6\delta_4)\tilde{G}_{ij}\,,\ (\delta\tilde{\Phi})^{(2)}=-\frac{3}{32}(2\nabla^k\Theta^{mn}\nabla_m\Theta_{nk}-3\nabla^k\Theta^{mn}\nabla_k\Theta_{mn}) \tag{3.28}
$$

and using appendix B we are left with $\frac{1}{8}$ times the following order α' terms

$$
\begin{aligned}
& 12g_1+8g_2+g_3-6g_4+4g_5-6g_6-12g_7+3g_8+12g_{10}-9g_{12}\\
& +24g_{13}+12g_{14}-6g_{15}+6g_{16}-6g_{19}-6g_{20}+12g_{21}+8g_{22}-2g_{23}-12g_{24}+16g_{25}\\
& +6h_1+8h_2-16h_3-4h_5+16h_6-4h_8+12h_9+8h_{10}-4h_{11}+4\nabla\hat{f}_7
\end{aligned} \tag{3.29}
$$

Next we use the Yang-Baxter equation which, in terms of Θ, reads

$$
\Theta^{k[l}\nabla_k\Theta^{mn]}=0\,. \tag{3.30}
$$

Hitting this with $R_{ipmn}\nabla^p$ we get the identity

$$
0=R_{ilmn}\nabla^l(\Theta_{kj}\nabla^k\Theta^{mn})+2R_i{}^{lmn}\nabla_l(\Theta_{km}\nabla^k\Theta_{nj})=\nabla\cdot(f_{19}-2f_{11})\,. \tag{3.31}
$$

Adding -4 times the r.h.s. to our expression we are left with $\frac{1}{8}$ times

$$
\begin{aligned}
& 12g_1+8g_2-3g_3-6g_4+12g_5-6g_6-12g_7+3g_8+12g_{10}-9g_{12}\\
& +24g_{13}+12g_{14}-6g_{15}+6g_{16}-6g_{19}-6g_{20}+8g_{21}+8g_{22}-6g_{23}-12g_{24}+24g_{25}\\
& +6h_1+8h_2-16h_3+24h_6+12h_9+8h_{10}-4h_{11}-8(m_4-2m_{10}+2m_{11})+4\nabla\hat{f}_7\,,
\end{aligned} \tag{3.32}
$$

where the m-terms vanish by the Yang-Baxter equation. Using the identities (B.47)–(B.49), (B.55) and (B.56) this reduces to (symmetrization in ij understood)

$$
h_{10}-\frac{1}{2}h_{11}+\frac{1}{2}\nabla\hat{f}_7=R_{klmi}R^{klmn}(\Theta^2)_{nj}-\frac{1}{2}R_{klm}{}^nR^{klmp}\Theta_{in}\Theta_{jp}+\frac{1}{4}\nabla_i\nabla_j(\nabla^l\Theta^{mn}\nabla_l\Theta_{mn})\,. \tag{3.33}
$$

The first two terms vanish if the original background does not suffer α'-corrections, while the last term can be canceled by shifting the dilaton.

To summarize we have found that with the following correction to the metric and dilaton in the HT scheme at order η^2, taking into account also (3.14), (symmetrization in ij understood)

$$(\delta\tilde{G})^{(2)}_{ij} = -\frac{3}{4}\nabla_i\Theta^{mn}\nabla_j\Theta_{mn} - \frac{1}{2}\nabla_m\Theta_{ni}\nabla_j\Theta^{mn} - \frac{3}{2}R_i{}^{klm}\Theta_{lm}\Theta_{kj}\,, \tag{3.34}$$

$$(\delta\tilde{\Phi})^{(2)} = \frac{1}{16}\nabla^k\Theta^{mn}\nabla_k\Theta_{mn} - \frac{3}{8}\nabla^k\Theta^{mn}\nabla_m\Theta_{nk}\,, \tag{3.35}$$

the deformed model is Weyl invariant at two loops provided the undeformed model is. The shift in the metric does not look particularly natural but it can be brought to a nicer form by noting that (symmetrization in ij understood)

$$\begin{aligned}\nabla_m\Theta_{ni}\nabla_j\Theta^{mn} &= \nabla_m k_n \times k_i\nabla_j\Theta^{mn} + \frac{1}{2}\nabla_i\Theta^{mn}\nabla_j\Theta_{mn}\\ &= \nabla_i v_j + R_i{}^{klm}\Theta_{kj}\Theta_{lm} + \frac{1}{2}\nabla_i\Theta^{mn}\nabla_j\Theta_{mn}\,,\end{aligned} \tag{3.36}$$

where $v_j = \nabla_m k_n \times k_j\Theta^{mn}$. The first term represents a diffeomorphism, so it can be dropped (note that the dilaton does not transform, $v^i\nabla_i\Phi = 0$, since it is isometric). It will be convenient to perform a further diffeomorphism generated by $v^i = \frac{1}{2}\Theta^{mn}\nabla^i\Theta_{mn}$ after which we have (symmetrization in ij understood)

$$(\delta\tilde{G})^{(2)}_{ij} = -2R_i{}^{klm}\Theta_{lm}\Theta_{kj} + \Theta^{mn}\nabla_i\nabla_j\Theta_{mn}\,, \tag{3.37}$$

$$(\delta\tilde{\Phi})^{(2)} = \frac{1}{16}\nabla^k\Theta^{mn}\nabla_k\Theta_{mn} - \frac{3}{8}\nabla^k\Theta^{mn}\nabla_m\Theta_{nk} + \frac{1}{4}\nabla_i\Phi\,\nabla^i(\Theta^{mn}\Theta_{mn})\,. \tag{3.38}$$

We will now consider what happens when the undeformed background receives α'-corrections.

Taking into account the lowest order correction to the metric and dilaton as well as the first order correction to $\tilde{B}$ (3.8) we have (symmetrization in ij understood)

$$\delta(R^{(2)}_{ij} - \frac{1}{4}H^{(1)}_{ikl}H^{(1)kl}_j + 2[\nabla_i\nabla_j\Phi]^{(2)}) + R_{klmi}R^{klmn}(\Theta^2)_{nj} - \frac{1}{2}R_{klm}{}^nR^{klmp}\Theta_{in}\Theta_{jp}\,. \tag{3.39}$$

Using (3.7) and the variations in (3.13) and (3.10) this becomes, after a tedious calculation,

$$\begin{aligned}&-3\nabla^k\delta G^n{}_i\nabla^l\Theta_{[nk}\Theta_{j]l} - \delta G_{in}k^k \times [k^l, \nabla_l k_j] \times \nabla_k k^n + 2\delta G_{kn}k^k \times [k^l, \nabla_l k_j] \times \nabla_i k^n\\ &+ \delta G_{kn}\nabla_i k^k \times [k^l, \nabla_l k^n] \times k_j - \delta G_{kn}\nabla^n(k_i \times [k^l, \nabla_l k_j] \times k^k)\\ &- 2\nabla_k\Phi\,\delta G_{in}k^n \times [k^l, \nabla_l k^k] \times k_j + 2\nabla^k\Phi\,\delta G_{kn}k_i \times [k^l, \nabla_l k^n] \times k_j\,.\end{aligned} \tag{3.40}$$

The first term vanishes by the Yang-Baxter equation. Using the fact that $k^l_I\nabla_l k^n_J - k^l_J\nabla_l k^n_I = f_{IJ}{}^K k^n_K$ and the YB equation (i.e. $R^{IJ}R^{KL}f_{JK}{}^M$ antisymmetrized in ILM vanishes) this further reduces to

$$\begin{aligned}-\frac{1}{2}R^{IJ}R^{KL}f_{JK}{}^Mf_{IL}{}^N\delta G_{in}k_{Mj}k^n_N &= R^{MJ}R^{KI}f_{JK}{}^Lf_{IL}{}^N\delta G_{in}k_{Mj}k^n_N\\ &= -\frac{1}{2}R^{MJ}R^{KI}f_{KI}{}^Lf_{JL}{}^N\delta G_{in}k_{Mj}k^n_N = 0\,,\end{aligned} \tag{3.41}$$

where we have used first the YB equation, then the Jacobi identity and finally the unimodularity condition $R^{KI}f_{KI}{}^{L}=0$.

This shows that the only additional corrections that arise are the ones coming from correcting the undeformed metric in $\tilde{G}^{(2)}$ and $\Phi^{(2)}$ so that

$$(\delta\tilde{G})^{(2)}_{ij}=2(\delta G\Theta^2)_{(ij)}+(\Theta\delta G\Theta)_{ij}-2\Theta_{k(i}R_{j)}{}^{klm}\Theta_{lm}+\Theta^{mn}\nabla_i\nabla_j\Theta_{mn}\,, \tag{3.42}$$

$$(\delta\tilde{\Phi})^{(2)}=-\frac{1}{2}(\delta G\Theta)_{mn}\Theta^{mn}+\frac{1}{16}\nabla^k\Theta^{mn}\nabla_k\Theta_{mn}-\frac{3}{8}\nabla^k\Theta^{mn}\nabla_m\Theta_{nk}+\frac{1}{4}\nabla_i\Phi\,\nabla^i(\Theta^{mn}\Theta_{mn})\,. \tag{3.43}$$

This completes the proof that, at least to second order in the deformation and when $B=0$, unimodular YB deformations preserve conformality at two loops.

4 α'-corrections from T-duality rules at two loops

Homogeneous Yang-Baxter deformations are closely related to non-abelian T-duality [12, 13] and it can be shown that the non-abelian T-dual model is in fact recovered in the maximally deformed limit $\eta\to\infty$ [13], see also [14, 15]. The simplest class of Yang-Baxter deformations — the "abelian" one — is related to just abelian T-duality, and is equivalent to doing TsT transformations [45, 46]. In general, a Yang-Baxter deformation generated by $\Theta=k_1\wedge k_2$ where $k_1=\partial_{x^1}$ and $k_2=\partial_{x^2}$ are commuting Killing vectors, is equivalent to doing first a T-duality $x^1\to\tilde{x}^1$, then a shift $x^2\to x^2+\eta\tilde{x}^1$, and then a T-duality back $\tilde{x}^1\to x^1$. Some "non-abelian" deformations are non-commuting sequences of TsT's [5, 47]. The non-abelian nature is related to the fact that the order in which the TsT transformations are performed is important, as certain T-dualities would break the isometries that are needed to perform the other T-dualities in the sequence. In this section we want to exploit the relation to TsT transformations and combine it with the knowledge of the first α'-corrections of the T-duality rules, to obtain two-loop corrections for all Yang-Baxter deformations that are obtainable by TsT transformations, or more generically by a non-commuting sequence of them. This strategy allows us to obtain backgrounds at two loops that are exact in the deformation parameter η. Moreover, these tools can be applied to any starting background with isometries, and it is not needed to restrict to $B=0$ as we assume in most of this paper.

Because at each step all that we are doing is (abelian) T-duality and coordinate transformations, we are bound to preserve conformal invariance on the worldsheet to the very end, and we can check explicitly that the solutions we generate do solve the two-loop equations. This argument can be repeated also to higher orders in the α' expansion, and it is enough to conclude that all Yang-Baxter deformations that are obtainable by a generically non-commuting sequence of TsT transformations, do not break the conformality of the original model to all orders in α'.

At leading order in α' the T-duality rules are given by the Buscher rules [48]. At higher loops these rules get corrected in α'. We will use the α'-corrections to the T-duality rules derived by Kaloper and Meissner in [34]. The rules were obtained by carefully analysing the two-loop effective action of the bosonic string, and identifying the terms that are symmetric or anti-symmetric under the Buscher rules. The α'-corrections of the T-duality rules were then fixed by requiring that they give a symmetry of the full two-loop effective action, compensating for the antisymmetry of those terms.[10]

Already at leading order in α', the T-duality rules are more easily presented in terms of fields of a dimensional reduction, where we reduce along the direction that we want to T-dualize. We follow [34] and we rewrite the metric, Kalb-Ramond field and dilaton of the D-dimensional spacetime in terms of the following $(D-1)$-dimensional fields

$$\begin{aligned} ds^2 &= G_{ij}dx^i dx^j = g_{\mu\nu}dx^\mu dx^\nu + e^{2\sigma}(d\underline{x}+V)^2\,, \\ B &= \frac{1}{2}B_{ij}dx^i\wedge dx^j = \frac{1}{2}b_{\mu\nu}dx^\mu\wedge dx^\nu + \frac{1}{2}W\wedge V + W\wedge d\underline{x}\,, \\ \Phi &= \phi + \frac{1}{2}\sigma\,. \end{aligned} \tag{4.1}$$

Here we are assuming that we have brought the solution in a form such that the isometry we want to dualize is simply implemented by a shift of a coordinate, that we denote by $\underline{x}$. We use Greek indices for the $(D-1)$-dimensional spacetime.[11] We have introduced a $(D-1)$-dimensional metric $g_{\mu\nu}$, and antisymmetric $b_{\mu\nu}$, vectors V_μ and W_μ, and scalars ϕ and σ. Above we also used form notation $V = V_\mu dx^\mu, W = W_\mu dx^\mu$. In components, the relations to identify the fields of the dimensional reduction are

$$\begin{aligned} \sigma &= \frac{1}{2}\log G_{\underline{xx}}\,, \qquad & V_\mu &= \frac{G_{\mu\underline{x}}}{G_{\underline{xx}}}\,, \qquad & g_{\mu\nu} &= G_{\mu\nu} - \frac{G_{\mu\underline{x}}G_{\nu\underline{x}}}{G_{\underline{xx}}}\,, \\ \phi &= \Phi - \frac{1}{4}\log G_{\underline{xx}}\,, \qquad & W_\mu &= B_{\mu\underline{x}}\,, \qquad & b_{\mu\nu} &= B_{\mu\nu} + \frac{G_{\underline{x}[\mu}B_{\nu]\underline{x}}}{G_{\underline{xx}}}\,. \end{aligned} \tag{4.2}$$

It is also useful to notice that $G^{\mu\nu} = g^{\mu\nu}$, $G^{\mu\underline{x}} = -V^\mu$, $G^{\underline{xx}} = e^{-2\sigma} + V^2$. The combination

$$h_{\mu\nu\rho} = 3\left(\partial_{[\mu}b_{\nu\rho]} - \frac{1}{2}W_{[\mu\nu}V_{\rho]} - \frac{1}{2}V_{[\mu\nu}W_{\rho]}\right) = H_{\mu\nu\rho} - 3W_{[\mu\nu}V_{\rho]}\,, \tag{4.3}$$

is gauge invariant. In terms of these new fields the Buscher rules are simply

$$\sigma \to -\sigma, \qquad\qquad V \leftrightarrow W\,. \tag{4.4}$$

All other fields remain unchanged under T-duality at leading order in α'.

In [34] Kaloper and Meissner derived the corrections to the T-duality rules in a particular scheme introduced by Meissner in [49]. We will call it the Kaloper-Meissner (KM) scheme. In order to apply the T-duality rules of KM to our case, we will therefore first need to implement the field redefinitions to go from the scheme of HT to that of KM. We can do so by combining the formulas given in [40] (see their equations (61) and (64)) relating the HT scheme to the Metsaev-Tseytlin (MT) scheme of [43], and those given in [49] (see

[10] In [34] the authors claim that their results can be applied also to the heterotic string, but the action they start with is missing the Chern-Simons terms that are expected there. See [37] for α'-corrected T-duality rules that encompass both the bosonic and the heterotic string.

[11] The discussion of the α'-corrected T-duality rules and their derivation simplifies if written in terms of tangent-space indices, but we will not do so here.

his equations (3.7), (4.1) and (4.7)) to go from MT to KM.[12] The field redefinitions that we will use are[13]

$$
\begin{aligned}
G_{ij}^{(\mathrm{HT})} &= G_{ij}^{(\mathrm{KM})} + \alpha' \left(R_{ij} - \frac{1}{2} H_{ij}^2 \right) , \\
B_{ij}^{(\mathrm{HT})} &= B_{ij}^{(\mathrm{KM})} + \alpha' \left(-H_{ijk} \nabla^k \Phi \right) , \\
\Phi^{(\mathrm{HT})} &= \Phi^{(\mathrm{KM})} + \alpha' \left(-\frac{3}{32} H^2 + \frac{1}{8} R - \frac{1}{2} (\nabla\Phi)^2 \right) .
\end{aligned}
\tag{4.5}
$$

Once we are in the scheme of KM we can use their α'-corrected T-duality rules [34]

$$
\begin{aligned}
\sigma &\to -\sigma + \alpha' \left[(\nabla\sigma)^2 + \frac{1}{8} (e^{2\sigma} Z + e^{-2\sigma} T) \right] \\
V_\mu &\to W_\mu + \alpha' \left[W_{\mu\nu} \nabla^\nu \sigma + \frac{1}{4} h_{\mu\nu\rho} V^{\nu\rho} e^{2\sigma} \right] \\
W_\mu &\to V_\mu + \alpha' \left[V_{\mu\nu} \nabla^\nu \sigma - \frac{1}{4} h_{\mu\nu\rho} W^{\nu\rho} e^{-2\sigma} \right] \\
b_{\mu\nu} &\to b_{\mu\nu} + \alpha' \left[V_{\rho[\mu} W^\rho{}_{\nu]} + \left(W_{[\mu\rho} \nabla^\rho \sigma + \frac{1}{4} e^{2\sigma} h_{[\mu\rho\lambda} V^{\rho\lambda} \right) V_{\nu]} + \left(V_{[\mu\rho} \nabla^\rho \sigma - \frac{1}{4} e^{-2\sigma} h_{[\mu\rho\lambda} W^{\rho\lambda} \right) W_{\nu]} \right]
\end{aligned}
\tag{4.6}
$$

Indices are always raised/lowered using the $(D-1)$-dimensional metric $g_{\mu\nu}$, and the transformations are written using also the following definitions

$$
\begin{aligned}
V_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu , \qquad & Z_{\mu\nu} &= V_{\mu\rho} V_\nu{}^\rho , \qquad & Z &= Z_\mu{}^\mu , \\
W_{\mu\nu} &= \partial_\mu W_\nu - \partial_\nu W_\mu , \qquad & T_{\mu\nu} &= W_{\mu\rho} W_\nu{}^\rho , \qquad & T &= T_\mu{}^\mu .
\end{aligned}
\tag{4.7}
$$

In general, at higher loops, not only σ, V and W will change under T-duality. In fact, at two loops in the scheme of KM also $b_{\mu\nu}$ gets modified.[14] It is important to remark that already before doing T-duality the fields will in general have an explicit α'-dependence. In particular, σ, V and W that transform according to (4.6) may in general depend on α', and this must be taken into account already when implementing the leading order T-duality rules (the Buscher rules).

One could in principle combine the T-duality rules of KM in (4.6) with the field redefinitions in (4.5), to obtain the α'-corrections of the T-duality rules in the scheme of

[12]The field redefinitions given in [49] relate the KM and the MT schemes only on-shell, but this is enough for our purposes, since we just want to make sure that we can generate solutions of the two-loop equations.

[13]These are the redefinitions needed when we set the parameter q of [40] to zero. Different values of q would affect the coefficient of H^2 that appears in the redefinition of the dilaton. Importantly, the coefficient in front of H_{ij}^2 that appears in the redefinition of the metric has the opposite sign compared to what one would expect from formulas in [40] or [49]. We have checked in various examples, some not included in this paper, that we must have the sign that we use here, as this is fixed by requiring that we want to have a solution of the two-loop equations after doing T-duality in the KM scheme and going back to the HT scheme.

[14]In [34] the rules were given in terms of transformations of $h_{\mu\nu\rho}$. Here we preferred to rewrite them as a transformation of $b_{\mu\nu}$. Importantly, the α'-corrections to the T-duality rules of $b_{\mu\nu}$ (or equivalently $h_{\mu\nu\rho}$) differ by an overall sign compared to those given in [34], and our formula corrects the one given there. We thank A. Vilar López for discussions on this point. A future paper will contain also more details on this [50].

JHEP03(2020)126

HT. We will not do so here, as the scheme of KM appears to be the minimal scheme for what concerns the complexity of the corrections to the T-duality rules. In other schemes, all other fields of the dimensional reduction will in general receive α'-corrections. Therefore, to obtain Yang-Baxter deformations in the scheme of HT we will follow this strategy:

1. Start from a solution of the two-loop equations in the HT scheme. In general that implies finding α'-corrections for this initial solution.

2. Go to the scheme of KM using (4.5).

3. Do TsT or sequences of TsT transformations, using the α'-corrected T-duality rules in (4.6).

4. Go back to the scheme of HT using (4.5).

We have worked out examples to test this method and obtain explicit results for α'-corrections of Yang-Baxter deformed models. This also allows us to relate to the results of section 3 that are perturbative in η. We will provide an example in the next section.

5 Examples

In this section we consider two particularly simple examples.

5.1 Solvable pp-wave

We start with the pp-wave background considered in [51]

$$ds^2 = 2dx^+dx^- - \frac{k}{(x^+)^2}x_m^2(dx^+)^2 + dx_m^2\,, \qquad \Phi = mx^+ + \frac{d}{2}k\ln x^+\,, \tag{5.1}$$

where $0 < k < \frac{1}{4}$ is a constant, m is another constant and d is the number of transverse dimensions. This background is known not to receive α'-corrections. This follows from the fact that the only non-zero component of the Riemann tensor is $R_{+m+n} = \delta_{mn}k(x^+)^{-2}$.

Consider the following four Killing vectors

$$\begin{aligned} k_1 &= (x^+)^\nu\partial_1 - \nu(x^+)^{\nu-1}x_1\partial_-\,, \qquad & k_3 &= (2\nu-1)\partial_-\,, \\ k_2 &= (x^+)^{1-\nu}\partial_1 - (1-\nu)(x^+)^{-\nu}x_1\partial_-\,, \qquad & k_4 &= (x^+)^\nu\partial_2 - \nu(x^+)^{\nu-1}x_2\partial_-\,, \end{aligned} \tag{5.2}$$

where we have defined the parameter

$$\nu = \frac{1+\sqrt{1-2k}}{2}\,. \tag{5.3}$$

They form a Heisenberg algebra of isometries with the only non-trivial Lie bracket $[k_1, k_2] = k_3$. From the discussion of R-matrices in [5] we see that we can consider the non-abelian rank 4 deformation

$$\Theta = k_1\wedge k_4 + sk_2\wedge k_3\,, \tag{5.4}$$

where we introduced the parameter s to keep track of the contribution from the second term. We will show below that in this case this deformation is equivalent to the abelian one obtained by setting $s=0$. First we construct the matrix

$$\Theta^{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & -a & 0 & c \\ 0 & -b & -c & 0 \end{pmatrix}, \tag{5.5}$$

where

$$a = \nu(x^+)^{2\nu-1}x_2 - s(2\nu-1)(x^+)^{1-\nu}\,, \qquad b = -\nu(x^+)^{2\nu-1}x_1\,, \qquad c = (x^+)^{2\nu}\,. \tag{5.6}$$

The deformed background takes the form

$$\begin{aligned} \tilde{ds}^2 &= 2dx^+\left(dx^- + \eta^2\frac{ac}{1+\eta^2c^2}dx^2 - \eta^2\frac{bc}{1+\eta^2c^2}dx^1\right) \\ &\quad - \left(\frac{k}{(x^+)^2}x_m^2 + \eta^2\frac{a^2+b^2}{1+\eta^2c^2}\right)(dx^+)^2 + \frac{dx_1^2+dx_2^2}{1+\eta^2c^2} + dx_{m'}^2\,. \end{aligned} \tag{5.7}$$

With the B-field and dilaton given by

$$\tilde{B} = -\frac{\eta}{1+\eta^2c^2}\left[(adx^1+bdx^2)\wedge dx^+ + cdx^2\wedge dx^1\right], \qquad \tilde{\Phi} = \Phi - \frac{1}{2}\ln(1+\eta^2c^2)\,. \tag{5.8}$$

One sees from this that

$$\tilde{H} = 4\eta\nu(x^+)^{2\nu-1}dx^2\wedge dx^1\wedge dx^+\,, \tag{5.9}$$

which is independent of the parameter s. The fact that also Φ is independent of s suggests that it might be possible to remove the s dependence also from the metric. Consider the change of coordinates $x_2 \to x_2 + f$ and $x^- \to x^- + gx_2 + h$ where f, g, h are functions only of x^+. One finds that the choice

$$\begin{aligned} f &= \frac{s}{2}\eta^2(2\nu-1)(x^+)^{\nu+2}\,, \quad g = -\frac{s}{2}\eta^2(2\nu-1)\nu(x^+)^{\nu+1}\,, \\ h &= \frac{s^2}{8}\eta^2(2\nu-1)^2\left[4(3-2\nu)^{-1}(x^+)^{3-2\nu} - \eta^2\nu(x^+)^{3+2\nu}\right], \end{aligned} \tag{5.10}$$

removes the dependence on s completely and reduces the background to the one obtained by the TsT with

$$\Theta = k_1 \wedge k_4\,. \tag{5.11}$$

Explicitly, the metric is

$$\begin{aligned} \tilde{ds}^2 &= 2dx^+\Big(dx^- + \nu\eta^2c^2(1+\eta^2c^2)^{-1}(x_1dx_1 + x_2dx^2)/x^+\Big) \\ &\quad - (x^+)^{-2}\Big(kx_m^2 + \nu^2\eta^2c^2(1+\eta^2c^2)^{-1}(x_1^2+x_2^2)\Big)(dx^+)^2 + \frac{dx_1^2+dx_2^2}{1+\eta^2c^2} + dx_{m'}^2\,. \end{aligned} \tag{5.12}$$

From (1.4) we find the only correction to the deformed background is given by

$$\delta G_{++} = -4\eta^2(2\nu^2-\nu)(x^+)^{4\nu-2}\,, \tag{5.13}$$

which can be canceled by a diffeomorphism $\delta G_{++} = \nabla_+ v_+$. In fact the change of coordinates $x^- \to x^- + \nu\eta^2 c^2 \frac{x_1^2+x_2^2}{2x^+(1+\eta^2c^2)}$, $x_{1,2} \to \sqrt{1+\eta^2c^2}\, x_{1,2}$ brings the deformed metric to the form

$$\tilde{ds}^2 = 2dx^+dx^- + (x^+)^{-2}\left[-kx_m^2 + \eta^2c^2[-3\nu+5\nu^2-k\eta^2c^2]\frac{x_1^2+x_2^2}{1+\eta^2c^2}\right](dx^+)^2 + dx_m^2 \,. \quad (5.14)$$

Therefore this background is exact at two loops, as is easily checked directly, and possibly to all loops.

5.2 Bianchi type II background

Next we consider the Bianchi type II background [52, 53] (the α'-corrections to Bianchi type I were considered in [54])

$$ds^2 = -\cosh(\tau)e^{(a+b+c)\tau}d\tau^2 + \frac{e^{a\tau}}{\cosh(\tau)}(dx - zdy)^2 + \cosh(\tau)e^{(a+b)\tau}dy^2 + \cosh(\tau)e^{(a+c)\tau}dz^2 \,, \quad (5.15)$$

supported by a dilaton linear in τ

$$\Phi = a\tau/2 \,. \quad (5.16)$$

This solves the Einstein equations provided that the parameters a, b, c are related as

$$bc = a^2 + 1 \,. \quad (5.17)$$

The solution has three Killing vectors

$$k_1 = -\partial_z - y\partial_x \,, \quad k_2 = \partial_y \,, \quad k_3 = \partial_x \,, \quad (5.18)$$

which again satisfy a Heisenberg algebra $[k_1, k_2] = k_3$.

From now on we will simplify things by taking $a = 0$ and $b = c = 1$. The two-loop equations are not automatically satisfied, and we need to find α'-corrections for this background. It is convenient to introduce a new coordinate system $\{v, x, y, z\}$ where $v = e^\tau$, since the metric then has a rational dependence on v

$$ds^2 = \frac{2v(dx - zdy)^2}{v^2+1} + \frac{(v^2+1)\left(v\left(dy^2+dz^2\right) - dv^2\right)}{2v} \,. \quad (5.19)$$

We assume that the correction to the metric δG_{ij} respects the isometries of the background. We turn on the diagonal components δG_{ii} and $\delta G_{12} = -z\delta G_{11}$. We also allow for a correction to the dilaton $\delta\Phi$ that, together with δG_{ii}, is allowed to depend only on v. The two-loop equation for the B-field is already satisfied. First it is simpler to solve the two-loop equation for the dilaton, because there only the correction $\delta\Phi$ contributes. One finds a second order differential equation $-3v^6+45v^4-45v^2+3-\left(v^2+1\right)^5(v\delta\Phi''(v)+\delta\Phi'(v)) = 0$ solved by

$$\delta\Phi = \frac{v}{2\left(v^2+1\right)} + \frac{2v}{\left(v^2+1\right)^3} + \frac{1}{2}\arctan v + c_\Phi \log v \,, \quad (5.20)$$

where c_Φ is a constant. Looking at the two-loop equations for the metric, one can find a linear combination of those equations that gives an algebraic constraint imposing $\delta G_{11} = 0$.

To find $\delta G_{00}, \delta G_{22}, \delta G_{33}$, we first identify linear combinations of the equations that give first order differential equations for δG_{00} and δG_{33}, and we solve them obtaining results written in terms of δG_{22}. These are then used to get a third order differential equation for δG_{22} only, that we also solve. The final result is

$$\begin{aligned}\delta G_{00} &= \frac{-2v^8+20v^4+8v^2-2\left(v^2-3\right)\left(v^2+1\right)^3 v\arctan v+6}{\left(v^4-1\right)^2}\\ &\quad+\frac{\left(v^2+1\right)\left(c_{00}\left(v^2-1\right)^2+v^2(c_{22}-2f_{22})+c_{22}-2f_{22}-4c_\Phi\left(v^2-3\right)v^2\log v+8c_\Phi\right)}{v\left(v^2-1\right)^2},\\ \delta G_{22} &= \frac{\left(3v^2-1\right)\left(\left(v^2+1\right)^3\arctan v+v\left(v^4+2v^2+5\right)\right)}{2\left(v^2-1\right)\left(v^2+1\right)^2}\\ &\quad+\frac{\left(v^2+1\right)\left(v^2(d_{22}-c_{22})+3c_{22}-d_{22}+2\log v\left(f_{22}\left(v^2-1\right)+4c_\Phi\right)-4f_{22}+8c_\Phi\right)}{4\left(v^2-1\right)},\\ \delta G_{33} &= \delta G_{22}-\frac{1}{2}\left(v^2+1\right)\left(2c_{00}-2c_{22}+d_{22}+2(f_{22}-6c_\Phi)\log v+2f_{22}\right).\end{aligned} \tag{5.21}$$

For simplicity in what follows we will set all integration constants $c_\Phi = c_{00} = c_{22} = d_{22} = f_{22} = 0$. This background admits a non-abelian deformation with

$$\Theta = \alpha k_1 \wedge k_4 + \beta k_2 \wedge k_3\,, \tag{5.22}$$

where α, β are parameters and we have introduced an additional flat direction w so that we can have a fourth Killing vector $k_4 = \partial_w$. If both α and β are non-zero, they can be reabsorbed by redefining w and the deformation parameter η. For simplicity we set $\alpha = 0$, $\beta = 1$ and analyze the abelian deformation given by

$$\Theta = k_2 \wedge k_3\,. \tag{5.23}$$

The Yang-Baxter deformation to lowest order in α' yields the following deformed background[15]

$$\begin{aligned}ds^2 &= \frac{\left(\left(v^2+1\right)^2+4vz^2\right)dy^2-8vz\,dx\,dy+4v\,dx^2}{2\left(v^2+1\right)(1+\eta^2 v)}-\frac{\left(v^2+1\right)dv^2}{2v}+\frac{1}{2}\left(v^2+1\right)dz^2,\\ B &= \frac{\eta v\,dx\wedge dy}{1+\eta^2 v},\\ \Phi &= -\frac{1}{2}\log\left(1+\eta^2 v\right).\end{aligned} \tag{5.24}$$

We can obtain the first α'-correction exactly in the deformation parameter η if we follow the strategy outlined in section 4. The deformation generated by $\Theta = k_2 \wedge k_3$ is equivalent to doing first a T-duality along x, then shifting $y \to y - \eta\tilde{x}$ where $\tilde{x}$ is the dual coordinate to x, and then T-dualising $\tilde{x}$ back.

We first start from the background given by the metric (5.19) and the α'-corrections (5.21). This background solves the two-loop equations in the HT scheme, and we

[15] We remind that in this paper we use the convention $B = \frac{1}{2}B_{ij}dx^i \wedge dx^j$.

need to apply (4.5) in order to find a solution in the KM scheme. Obviously, since the corrections in (4.5) are multiplied by an explicit power of α', it is enough to use the uncorrected background to derive them, which simplifies the calculation. Because $B = 0$, we can in principle get a non-trivial modification only for the metric from the Ricci tensor, and for the dilaton from the Ricci scalar. But the Bianchi II background is also Ricci-flat, therefore it is the same in the KM scheme and in the HT scheme. The next step is that of identifying the fields of the dimensional reduction as in (4.1). Because we want to do T-duality along x here, we are taking $\underline{x} = x$. This is a straightforward exercise, and instead of writing down all fields of the dimensional reduction, we only write those that can potentially change under the corrected T-duality rules

$$\sigma = \frac{1}{2}\log\left(\frac{2v}{1+v^2}\right)\,, \qquad V = -z dy\,, \qquad W = 0\,, \qquad b = 0\,. \tag{5.25}$$

These particular fields of the dimensional reduction happen not to depend on α' in this particular example. We then implement the α'-corrected T-duality rules of KM as in (4.6) and obtain the fields of the dimensional reduction after T-duality

$$\sigma = -\frac{1}{2}\log\left(\frac{2v}{1+v^2}\right) - \alpha'\frac{\left(v^4-6v^2+1\right)}{2v\left(v^2+1\right)^3}\,, \qquad V = 0\,, \qquad W = -z dy\,, \qquad b = 0\,. \tag{5.26}$$

After T-duality the scalar σ does depend explicitly on α'. The explicit form of the two-loop background after performing this first T-duality along x is

$$\begin{aligned}
ds^2 =& \frac{1}{2}\left(v+v^{-1}-\frac{\alpha'\left(v^4-6v^2+1\right)}{\left(v^3+v\right)^2}\right)d\tilde{x}^2 \\
&+\frac{1}{2}\left(1+v^2+\frac{\alpha'\left(3v^2-1\right)\left(\left(v^2+1\right)^3\arctan v+v\left(v^4+2v^2+5\right)\right)}{\left(v^2-1\right)\left(v^2+1\right)^2}\right)\left(dy^2+dz^2\right) \\
&+\left(-\frac{v^2+1}{2v}-\frac{2\alpha'\left(v^8-10v^4-4v^2+\left(v^2-3\right)\left(v^2+1\right)^3 v\arctan v-3\right)}{\left(v^4-1\right)^2}\right)dv^2\,, \\
B =& z d\tilde{x}\wedge dy\,, \\
\Phi =& -\frac{1}{2}\log\left(\frac{2v}{v^2+1}\right)+\alpha'\left[\frac{\left(2v^6+3v^4+16v^2-1\right)}{4v\left(v^2+1\right)^3}+\frac{1}{2}\arctan v\right].
\end{aligned} \tag{5.27}$$

In the T-dual frame the metric is diagonal (even to two loops) at the cost of having a non-vanishing B-field. We can now do the shift $y \to y - \eta\tilde{x}$, that here will have only the effect of modifying the metric. To perform another T-duality along $\tilde{x}$ we have to first repeat the identification of the fields of the dimensional reduction. We find in particular

$$
\begin{aligned}
\sigma =&\, \frac{1}{2}\log\left(\frac{1}{2}\left(\eta^2\left(v^2+1\right)+v+v^{-1}\right)\right)\\
&+\frac{1}{2}\alpha'\left[-\frac{\left(v^4-6v^2+1\right)}{v\left(v^2+1\right)^3\left(1+\eta^2 v\right)}+\frac{\eta^2 v\left(3v^2-1\right)\left(\left(v^2+1\right)^3\arctan v+v\left(v^4+2v^2+5\right)\right)}{\left(v^2-1\right)\left(v^2+1\right)^3\left(\eta^2 v+1\right)}\right],\\
V =&\, \frac{-\eta\, dy}{(1+\eta^2 v)^2}\left(v\left(1+\eta^2 v\right)+\frac{\alpha'\left(v\left(3v^2-1\right)\left(v^2+1\right)^2\arctan v+3\left(v^6+v^4+v^2\right)-1\right)}{\left(v^2-1\right)\left(v^2+1\right)^2}\right),\\
W =&\, -z dy\,, \qquad\qquad b=0\,.
\end{aligned}
\tag{5.28}
$$

At this point we can use again the T-duality rules of KM (4.6). After doing that we obtain the following background

$$
\begin{aligned}
ds^2 =&\, -\frac{\left(v^2+1\right)dv^2}{2v}+\frac{2(dx-zdy)^2}{\eta^2\left(v^2+1\right)+v+v^{-1}}+\frac{\left(v^2+1\right)dy^2}{2(1+\eta^2 v)}+\frac{1}{2}\left(v^2+1\right)dz^2\\
&+\alpha'\delta G_{00}dv^2-4\alpha'\eta^2 v^2\left(\frac{\delta G_{22}}{\left(v^2+1\right)^2\left(1+\eta^2 v\right)^2}+2v\frac{v^2-1}{\left(v^2+1\right)^4\left(1+\eta^2 v\right)^2}\right)(dx-zdy)^2\\
&+\alpha'\left(\frac{\delta G_{22}}{\left(1+\eta^2 v\right)^2}-\eta^2\frac{v^4-6v^2+1}{2\left(v^2+1\right)^2\left(1+\eta^2 v\right)^2}\right)dy^2+\alpha'\delta G_{22}dz^2\,,\\
\tilde{B} =&\, \frac{\alpha'\eta dv\wedge dz}{\left(v^2+1\right)\left(1+\eta^2 v\right)}\\
&+\eta v dx\wedge dy\left(\frac{1}{1+\eta^2 v}+2\alpha'\frac{\delta G_{22}}{\left(v^2+1\right)\left(1+\eta^2 v\right)^2}+\alpha'\frac{2v(v^2-3)+\eta^2\left(3v^2-1\right)\left(v^2-1\right)}{\left(v^2+1\right)^3\left(1+\eta^2 v\right)^3}\right),\\
\tilde{\Phi} =&\, -\frac{1}{2}\log\left(1+\eta^2 v\right)+\alpha'\delta\Phi-\alpha'\eta^2\frac{4v(v^2+1)^2\delta G_{22}+5v^4-10v^2+1}{4\left(v^2+1\right)^3\left(1+\eta^2 v\right)}\,,
\end{aligned}
\tag{5.29}
$$

where δG_{ij} and $\delta\Phi$ are the corrections to the undeformed background given in (5.20) and (5.21). This is a TsT of the initial Bianchi II that solves the two-loop equations in the KM scheme. To go to the HT scheme we use again (4.5). Because of the deformation, now the dictionary to go to the new scheme is non-trivial, and the background in the HT scheme reads

$$
\begin{aligned}
ds^2 =&\, \tilde{G}_{ij}dx^i dx^j\,,\\
\tilde{B} =&\, \frac{\alpha'\eta dv\wedge dz}{\left(v^2+1\right)\left(1+\eta^2 v\right)}\\
&+\eta v dx\wedge dy\left(\frac{1}{1+\eta^2 v}+2\alpha'\frac{\delta G_{22}}{\left(v^2+1\right)\left(1+\eta^2 v\right)^2}+2\alpha' v\frac{v^2-3}{\left(v^2+1\right)^3\left(1+\eta^2 v\right)^2}\right),\\
\tilde{\Phi} =&\, -\frac{1}{2}\log\left(1+\eta^2 v\right)+\alpha'\delta\Phi-\alpha'\eta^2\frac{v\delta G_{22}}{\left(v^2+1\right)\left(1+\eta^2 v\right)}-\alpha'\eta^2\frac{3v^4-14v^2-1}{4\left(v^2+1\right)^3\left(1+\eta^2 v\right)}\,,
\end{aligned}
\tag{5.30}
$$

where

$$\begin{aligned}
\tilde{G}_{00} &= -\frac{v^2+1}{2v} + \alpha'\delta G_{00} - \alpha'\eta^2 \frac{\eta^2 + 3\eta^2 v^2 + 2v}{2\left(v^2+1\right)\left(1+\eta^2 v\right)^2}\,,\\
\tilde{G}_{11} &= \frac{2v}{\left(v^2+1\right)\left(1+\eta^2 v\right)} - 4\alpha'\eta^2 v^2 \frac{\delta G_{22}}{\left(v^2+1\right)^2\left(1+\eta^2 v\right)^2} - 4v\alpha'\eta^2 v^2 \frac{v^2-3}{\left(v^2+1\right)^4\left(1+\eta^2 v\right)^2}\,,\\
\tilde{G}_{22} &= \frac{v^2+1}{2\left(1+\eta^2 v\right)} - \alpha'\eta^2 v^2 \frac{v^2-3}{\left(v^2+1\right)^2\left(1+\eta^2 v\right)^2} + \alpha' \frac{\delta G_{22}}{\left(1+\eta^2 v\right)^2} + z^2 \tilde{G}_{11}\,,\\
\tilde{G}_{33} &= \frac{1}{2}\left(v^2+1\right) + \alpha'\delta G_{22} - \alpha'\eta^2 \frac{v^2}{\left(v^2+1\right)\left(1+\eta^2 v\right)}\,,\\
\tilde{G}_{12} &= -z\tilde{G}_{11}\,.
\end{aligned} \tag{5.31}$$

Performing the redefinition of the dilaton given in (1.6) this background agrees precisely with that obtained from the all order expression (1.7).

When we want to work out a deformation generated by $\Theta = k_1 \wedge k_4$ following the strategy of section 4, we first need to find a coordinate system in which k_1 acts as a simple shift of a coordinate. We can redefine

$$x = x' + y'z'\,, \qquad y = y'\,, \qquad z = z'\,, \tag{5.32}$$

so that in the new coordinate system $k_1 = -\partial_{z'}$. As should be clear from the discussion at the beginning of this section, the isometry generated by k_1 is not broken by α' corrections, therefore the metric will not depend on z' also at two loops. The deformation generated by $\Theta = k_1 \wedge k_4$ can be obtained by doing T-duality $w \to \tilde{w}$, then the shift $z' \to z' - \eta\tilde{w}$, and then T-duality back $\tilde{w} \to w$. We will omit the explicit results for this particular deformation, since they involve very long expressions, and we have already presented our method in the previous deformation generated by $\Theta = k_2 \wedge k_3$. We have checked that the resulting background again agrees with that obtained by the α'-corrected open-closed string map (1.7).

The interesting point is that we can combine these two TsT transformations. We can first do a TsT involving x and y corresponding to $\Theta = k_2 \wedge k_3$. At the end of this result the background is still invariant under isometries generated by k_1 and k_4, and we can do a second TsT transformation involving z' and w, equivalent to $\Theta = k_1 \wedge k_4$. The composition of the two deformations is equivalent to the deformation given by $\Theta = k_1 \wedge k_4 + k_2 \wedge k_3$, as explained in [15]. The non-abelian nature of the deformation is related to the fact that if we had started from $\Theta = k_1 \wedge k_4$ instead, we would have broken the isometries that we would need to perform the deformation with $\Theta = k_2 \wedge k_3$. As follows from the results of [15], in the maximally deformed limit $\eta \to \infty$ we recover the non-abelian T-dual of the original Bianchi II solution, where the isometries dualized are those corresponding to the Killing vectors k_1, k_2, k_3 forming a Heisenberg algebra, and k_4. By this argument it follows that non-abelian T-dual models related to this class of Yang-Baxter deformations remain conformal on the worldsheet to two loops. Because T-duality remains a symmetry of the string at higher orders in an α'-expansion, we can argue that this is true to all loops. Unfortunately the all order expression (1.7) turns out not to give the correct answer in this case.

6 Conclusions

We have argued that (homogeneous) YB deformed string σ-models that are conformal at one loop remain conformal at two loops,[16] i.e. including the first correction in α'. We showed this to second order in the deformation parameter η for a generic unimodular deformation of a background with vanishing B-field. We also argued that using the α'-corrected T-duality rules of [34] one can verify this to all orders in the deformation parameter for the cases that can be built from TsT transformations, and we explained that this strategy can be used also for the non-abelian YB deformations that are equivalent to a non-commuting sequence of TsT transformations.[17] We exemplified our results in the case of a deformation of a Bianchi type II background.

Our findings suggest that one-loop conformal YB σ-models should in fact remain conformal to first order in α', and likely all orders. Since these models can be thought of as a generalization of non-abelian T-duality [12, 13, 15] (which can be recovered in an appropriate $\eta \to \infty$ limit) our findings suggest that the same should be true for NATD. This was also argued recently from a different perspective in [30, 31], studying renormalizability of a different type of integrable deformation of σ-models.[18] To test this idea one should start from a model which is conformal to all orders in α' and then deform it. A good candidate is therefore the unimodular deformation of $\mathrm{AdS}_3 \times S^3$ constructed in [39].

We saw that the expression (1.7) for the all order in η form of the first α'-correction to YB deformations works in simple cases but fails in general. It is an important problem to fix it so that it holds in general. If a simple solution exists for the corrections, it is also interesting in the special case of TsT transformations, whose corrections have, to our knowledge, not been analyzed before. If, further more, this continues to work to higher orders in α' it could even help in determining the structure of higher α'-corrections to the target space equations of motion. This approach could be said to be an example of using $O(d, d)$ symmetry to determine/constrain higher α'-corrections.

We plan to address some of these questions in the near future.

Acknowledgments

We thank B. Hoare, N. Levine and A. Tseytlin for useful and interesting discussions. LW wishes to thank the participants of the workshop "New frontiers of integrable deformations" in Villa Garbald, Castasegna for interesting discussions. RB is grateful to J. Edelstein, J.A. Sierra-Garcia, and in particular to A. Vilar López for very useful discussions. The work of RB is supported by the fellowship of "la Caixa Foundation" (ID 100010434) with code LCF/BQ/PI19/11690019. He is also supported by the Maria de Maeztu Unit of Excellence MDM-2016-0692, by FPA2017-84436-P, by Xunta de Galicia (ED431C 2017/07), and by FEDER.

[16] Provided, of course, the undeformed background is conformal to two loops.

[17] See e.g. [5, 47].

[18] Early work on α'-corrections in NATD include [55–58].

JHEP03(2020)126

A Killing identities

The Killing vectors satisfy the equations (suppressing the Lie algebra index)

$$\nabla_{(i}k_{j)} = 0 \qquad \nabla_i\nabla_j k_l = R_{ljin}k^n\,. \tag{A.1}$$

Using this and the expression for Θ in (1.2) we can derive the useful two-derivative identity

$$\begin{aligned}2\nabla_k\nabla_{(i}\Theta_{j)l} &= 2\nabla_k k_{(i}\times\nabla_{j)}k_l + 2R_{knl(i}\Theta_{j)}{}^n\\ &= -\nabla_{(i}\nabla_{j)}\Theta_{kl} + 2R_{knl(i}\Theta_{j)}{}^n - R_{k(ij)n}\Theta_l{}^n + R_{l(ij)n}\Theta_k{}^n\,.\end{aligned} \tag{A.2}$$

A special case of this is

$$\nabla^2\Theta_{ij} = -R_{ijkl}\Theta^{kl} + R_{ik}\Theta_j{}^k - R_{jk}\Theta_i{}^k\,. \tag{A.3}$$

In addition we have the unimodularity condition, which in terms of Θ, takes the form

$$\nabla_k\Theta^{kl} = 0\,. \tag{A.4}$$

We also know that the dilaton respects the isometries so that

$$k^i\nabla_i\Phi = 0\,. \tag{A.5}$$

Using these facts we can prove the useful identity

$$\nabla_k(R_{ijlm}\Theta^{lm}) = -\frac{1}{2}R_{iklm}\nabla_j\Theta^{lm} + R_{imkl}\nabla^m\Theta_j{}^l - R_{ilmk}\nabla^m\Theta_j{}^l - (i\leftrightarrow j)\,. \tag{A.6}$$

This follows by noting that

$$\begin{aligned}2R_{ijlm}\nabla_k\Theta^{lm} &= -4\nabla_m\nabla_j k_i\times\nabla_k k^m = -4\nabla_m(\nabla_j k_i\times\nabla_k k^m) + 4R_{kl}\nabla_j k_i\times k^l\\ &= -2\nabla_m\nabla_j(k_i\times\nabla_k k^m) + 2\nabla_m(R_{mkjl}\Theta_i{}^l) + 2R_{kl}\nabla_j k_i\times k^l - (i\leftrightarrow j)\\ &= -\nabla^m\nabla_j\nabla_k\Theta_{im} + \nabla^m\nabla_j\nabla_m\Theta_{ik} + \nabla^m\nabla_j\nabla_i\Theta_{mk} + 2\nabla^m(R_{mkjl}\Theta_i{}^l)\\ &\quad + 2R_{kl}\nabla_j k_i\times k^l - (i\leftrightarrow j)\\ &= \frac{1}{2}R_{ijlm}\nabla_k\Theta^{lm} - \frac{1}{2}\nabla_k R_{ijlm}\Theta^{lm} - \frac{1}{2}R_{iklm}\nabla_j\Theta^{lm} + R_{imkl}\nabla^m\Theta_j{}^l\\ &\quad - R_{ilmk}\nabla^m\Theta_j{}^l - (i\leftrightarrow j)\,,\end{aligned} \tag{A.7}$$

where we have used the fact that

$$\nabla^l\Phi\,\nabla_k\Theta_{lj} + \nabla^l\Phi\,\nabla_l\Theta_{kj} + \nabla^l\Phi\,\nabla_j\Theta_{kl} = 0\,, \tag{A.8}$$

as is easily verified. Acting with ∇^k, and using also $\nabla^k\Phi$ times the above identity, one finds

$$\begin{aligned}4\nabla^k(R_{ijlm}\nabla_k\Theta^{lm}) &= 3\nabla^k\nabla_{[i}(R_{jk]lm}\Theta^{lm}) - 2R_{imkl}R_{jn}{}^{kl}\Theta^{mn} + 4R_i{}^{klm}R_{jkln}\Theta_m{}^n\\ &\quad + 2R_{ijlm}\nabla^k\Phi\,\nabla_k\Theta^{lm} + 4R_{iklm}\nabla^k\Phi\,\nabla_j\Theta^{lm} - (i\leftrightarrow j)\,.\end{aligned} \tag{A.9}$$

JHEP03(2020)126

B Relations needed for second order calculation

For the second order calculations we define the following 'basis' of terms (for readability we write all indices as lower indices)

$$\begin{aligned}
f_1 &= R_{ilmn}\nabla_j\Theta_{mn}\Theta_{kl} & f_{12} &= R_{imkn}\nabla_m\Theta_{lj}\Theta_{ln} & f_{23} &= R_{klmn}\nabla_m\Theta_{in}\Theta_{jl} \\
f_2 &= R_{ilmn}\nabla_j\Theta_{kl}\Theta_{mn} & f_{13} &= R_{imkn}\nabla_l\Theta_{mj}\Theta_{ln} & f_{24} &= R_{klmn}\nabla_l\Theta_{im}\Theta_{jn} \\
f_3 &= R_{ikmn}\nabla_j\Theta_{ml}\Theta_{ln} & f_{14} &= R_{ilmn}\nabla_k\Theta_{mn}\Theta_{lj} & f_{25} &= \nabla_l R_{ikmn}\Theta_{jl}\Theta_{mn} \\
f_4 &= R_{imnk}\nabla_j\Theta_{ml}\Theta_{ln} & f_{15} &= R_{ilmn}\nabla_l\Theta_{mn}\Theta_{kj} & f_{26} &= \nabla_k R_{ilmn}\Theta_{jl}\Theta_{mn} \\
f_5 &= R_{ilmn}\nabla_l\Theta_{mj}\Theta_{nk} & f_{16} &= R_{ilmn}\nabla_l\Theta_{km}\Theta_{nj} & f_{27} &= \nabla_i\nabla_j\Theta_{mn}\nabla_k\Theta_{mn} \\
f_6 &= R_{ilmn}\nabla_m\Theta_{lj}\Theta_{nk} & f_{17} &= R_{ilmn}\nabla_m\Theta_{kn}\Theta_{lj} & f_{28} &= \nabla_i\nabla_k\Theta_{mn}\nabla_j\Theta_{mn} \\
f_7 &= R_{ilmn}\nabla_m\Theta_{nj}\Theta_{lk} & f_{18} &= R_{ikmn}\nabla_m\Theta_{ln}\Theta_{lj} & f_{29} &= \nabla_i\nabla_j\Theta_{mn}\nabla_m\Theta_{nk} \\
f_8 &= R_{ilmn}\nabla_l\Theta_{kj}\Theta_{mn} & f_{19} &= R_{ikmn}\nabla_l\Theta_{mn}\Theta_{lj} & f_{30} &= \nabla_i\nabla_k\Theta_{mn}\nabla_m\Theta_{nj} \\
f_9 &= R_{ilmn}\nabla_k\Theta_{lj}\Theta_{mn} & f_{20} &= R_{imkn}\nabla_m\Theta_{ln}\Theta_{lj} & f_{31} &= \nabla_i\nabla_m\Theta_{nk}\nabla_j\Theta_{mn} \\
f_{10} &= R_{ikmn}\nabla_m\Theta_{lj}\Theta_{ln} & f_{21} &= R_{klmn}\nabla_i\Theta_{jl}\Theta_{mn} & f_{32} &= \nabla_i\nabla_m\Theta_{nk}\nabla_m\Theta_{nj} \\
f_{11} &= R_{ikmn}\nabla_l\Theta_{mj}\Theta_{ln} & f_{22} &= R_{klmn}\nabla_i\Theta_{mn}\Theta_{jl} & f_{33} &= \nabla_i\nabla_m\Theta_{nk}\nabla_n\Theta_{mj}
\end{aligned} \tag{B.1}$$

where we suppress the free indices ijk and assume symmetry in ij throughout. We also define the terms with only one free index

$$\begin{aligned}
\hat{f}_1 &= R_{klmn}\nabla_j\Theta_{kl}\Theta_{mn} & \hat{f}_4 &= R_{jlmn}\nabla_k\Theta_{mn}\Theta_{kl} & \hat{f}_7 &= \nabla_j\nabla_l\Theta_{mn}\nabla_l\Theta_{mn} \\
\hat{f}_2 &= R_{klmn}\nabla_k\Theta_{lj}\Theta_{mn} & \hat{f}_5 &= R_{jlmn}\nabla_m\Theta_{nk}\Theta_{kl} & \hat{f}_8 &= \nabla_j\nabla_l\Theta_{mn}\nabla_m\Theta_{nl} \\
\hat{f}_3 &= R_{klmn}\nabla_m\Theta_{kl}\Theta_{nj} & \hat{f}_6 &= R_{jlmn}\nabla_l\Theta_{km}\Theta_{kn} & &
\end{aligned} \tag{B.2}$$

We will denote for example $\nabla^k f_{1ijk}$ as $\nabla\cdot f_1$, again suppressing the indices, and similarly for example $\nabla_{(i}\hat{f}_{1j)}$ as $\nabla\hat{f}_1$. Using the Killing vector identities, unimodularity and isometry of the dilaton one finds

$$\nabla\cdot f_1 = \frac{1}{2}g_{12} + g_{23} - 2h_6 \tag{B.3}$$

$$\nabla\cdot f_2 = \frac{1}{2}g_{12} + g_{15} - \frac{1}{2}h_1 + 2m_1 \tag{B.4}$$

$$\nabla\cdot f_3 = g_{13} - g_{25} - h_5 - h_7 - 2m_5 - 2m_6 \tag{B.5}$$

$$\nabla\cdot f_4 = g_{14} - g_{24} + g_{25} + \frac{1}{2}h_5 + h_7 - \frac{1}{2}h_8 + 2m_6 \tag{B.6}$$

$$\nabla\cdot f_5 = -\frac{1}{2}g_1 + \frac{1}{2}g_{23} - g_{25} + \frac{1}{2}h_3 - h_5 - h_6 - \frac{1}{2}h_8 \tag{B.7}$$

$$\nabla\cdot f_6 = -\frac{1}{2}g_1 - \frac{1}{2}g_{10} - \frac{1}{2}g_{23} + g_{24} - g_{25} - \frac{1}{2}h_3 - h_5 + \frac{1}{2}h_8 \tag{B.8}$$

$$\nabla\cdot f_7 = -\frac{1}{2}g_{10} - g_{23} + g_{24} - h_3 + h_6 + h_8 \tag{B.9}$$

$$\nabla\cdot f_8 = g_1 + g_3 + \frac{1}{2}h_1 + 2m_2 \tag{B.10}$$

$$\nabla\cdot f_9 = g_1 + g_8 + g_{10} + h_1 + 2m_1 - 2m_2 \tag{B.11}$$

$$\nabla \cdot f_{10} = g_4 + g_7 + g_{24} - 2g_{25} + h_4 - \frac{1}{2}h_5 - h_6 - h_7 + \frac{1}{2}h_8 + 2m_7 - 2m_8 \tag{B.12}$$

$$\nabla \cdot f_{11} = g_5 - \frac{1}{2}g_{23} + g_{25} - h_4 + \frac{1}{2}h_5 + h_6 + \frac{1}{2}h_8 + 2m_{10} - 2m_{11} \tag{B.13}$$

$$\nabla \cdot f_{12} = g_4 + g_{24} - 2g_{25} - \frac{1}{2}h_3 + h_4 - \frac{1}{2}h_5 - h_6 - h_7 + 2m_7 \tag{B.14}$$

$$\nabla \cdot f_{13} = g_6 + \frac{1}{2}g_{23} - g_{24} + g_{25} + \frac{1}{2}h_3 - h_4 + h_5 - h_6 + \frac{1}{2}h_8 + 2m_{10} \tag{B.15}$$

$$\nabla \cdot f_{14} = g_2 + g_8 + h_2 + 2m_3 \tag{B.16}$$

$$\nabla \cdot f_{15} = g_2 - g_{11} + g_{21} \tag{B.17}$$

$$\nabla \cdot f_{16} = -\frac{1}{2}g_2 - g_5 + \frac{1}{2}g_{11} - \frac{1}{4}h_2 + 2m_{16} \tag{B.18}$$

$$\nabla \cdot f_{17} = \frac{1}{2}g_2 + g_6 + \frac{1}{2}h_2 - m_3 \tag{B.19}$$

$$\nabla \cdot f_{18} = g_4 - \frac{3}{2}g_{21} + h_3 + h_4 - h_{10} + 2m_9 + 2m_{17} \tag{B.20}$$

$$\nabla \cdot f_{19} = g_3 + g_{21} - 2h_4 + 2m_4 \tag{B.21}$$

$$\nabla \cdot f_{20} = g_4 + g_7 - \frac{3}{2}g_{21} + \frac{3}{2}h_3 + h_4 - \frac{1}{2}h_{10} + 2m_9 \tag{B.22}$$

$$\nabla \cdot f_{21} = g_9 - \frac{1}{2}g_{20} - \frac{1}{2}h_2 + h_9 + 2m_{14} \tag{B.23}$$

$$\nabla \cdot f_{22} = -g_{16} - g_{22} + 2h_4 - 2m_{13} \tag{B.24}$$

$$\nabla \cdot f_{23} = g_{17} + \frac{3}{2}g_{22} - h_3 - h_4 - h_{11} + 2m_{12} \tag{B.25}$$

$$\nabla \cdot f_{24} = -g_{18} - \frac{1}{2}h_3 + \frac{1}{2}h_{11} + 2m_{18} \tag{B.26}$$

$$\nabla \cdot f_{25} = g_{11} - g_1 - g_2 - \frac{1}{2}h_2 - 2h_4 - 4m_2 + 2m_{19} \tag{B.27}$$

$$\nabla \cdot f_{26} = -g_1 - g_2 - g_{10} - h_2 - 4h_4 - 2m_{15} \tag{B.28}$$

$$\nabla \cdot f_{27} = 2g_5 - 2g_6 + 2g_7 - 2g_{13} - 2g_{14} - g_{20} + 2g_{28} + 2g_{32} + 4m_{21} \tag{B.29}$$

$$\nabla \cdot f_{28} = -g_{12} + 2g_{13} - g_{19} - 2g_{25} + g_{26} + 2m_{20} \tag{B.30}$$

$$\nabla \cdot f_{29} = \frac{1}{2}g_3 - g_4 - g_5 - g_7 + \frac{1}{2}g_8 + g_{13} + \frac{1}{2}g_{15} + \frac{1}{2}g_{20} - g_{29} + g_{30} - g_{33} + g_{34} + 2m_{21} \tag{B.31}$$

$$\nabla \cdot f_{30} = g_4 - g_5 + g_7 - g_{10} - g_{16} - g_{22} + g_{23} - g_{25} + g_{26} - g_{27} + 2m_{22} \tag{B.32}$$

$$\nabla \cdot f_{31} = \frac{1}{2}g_{12} - g_{13} - g_{14} + \frac{1}{2}g_{15} + \frac{1}{2}g_{19} - \frac{1}{2}g_{23} - g_{24} + g_{25} + g_{27} + 2m_{23} \tag{B.33}$$

$$\nabla \cdot f_{32} = -g_7 + \frac{1}{2}g_8 + \frac{1}{2}g_{10} + \frac{1}{2}g_{16} + \frac{1}{2}g_{22} - g_{23} + g_{24} + g_{25} - g_{26} + g_{27} + 2m_{24} \tag{B.34}$$

$$\nabla \cdot f_{33} = \frac{1}{2}g_3 - g_5 + g_6 - \frac{1}{2}g_{10} - \frac{1}{2}g_{16} - \frac{1}{2}g_{22} - \frac{1}{2}g_{23} + 2m_{25} \tag{B.35}$$

and

$$\nabla \hat{f}_1 = g_{12} + g_{19} + g_{20} \tag{B.36}$$

$$\nabla \hat{f}_2 = g_{10} + g_{16} + \frac{1}{2}g_{20} + h_2 \tag{B.37}$$

$$\nabla\hat{f}_3 = -g_9+g_{11}+g_{22} \tag{B.38}$$
$$\nabla\hat{f}_4 = g_{15}+g_{23}-2g_{33} \tag{B.39}$$
$$\nabla\hat{f}_5 = -g_{14}+g_{24}+g_{32}+g_{34} \tag{B.40}$$
$$\nabla\hat{f}_6 = g_{13}+g_{25}+g_{34}+g_{35} \tag{B.41}$$
$$\nabla\hat{f}_7 = -g_4-g_6-3g_{14}+g_{26}+g_{28}-g_{29}-g_{30}+2g_{32}+2g_{34} \tag{B.42}$$
$$\nabla\hat{f}_8 = \frac{1}{2}(g_3-g_4-g_6+g_8-3g_{14}+3g_{15}+2g_{27}+g_{28}-g_{29}+g_{30}-g_{31}+2g_{32}-4g_{33}+2g_{34}) \tag{B.43}$$

where we have defined the $\nabla^2 R\Theta^2$-terms

$$\begin{aligned}
g_1 &= \nabla_k R_{ilmn}\nabla_l\Theta_{kj}\Theta_{mn} & g_{13} &= R_{ilmn}\nabla_j\Theta_{mk}\nabla_l\Theta_{kn} & g_{25} &= R_{ilmn}\nabla_j\nabla_l\Theta_{km}\Theta_{kn}\\
g_2 &= \nabla_k R_{ilmn}\nabla_k\Theta_{mn}\Theta_{lj} & g_{14} &= R_{ilmn}\nabla_j\Theta_{kl}\nabla_m\Theta_{kn} & g_{26} &= \nabla_i\nabla_k\Theta_{mn}\nabla_j\nabla_k\Theta_{mn}\\
g_3 &= R_{ilmn}\nabla_l\Theta_{kj}\nabla_k\Theta_{mn} & g_{15} &= R_{ilmn}\nabla_j\Theta_{kl}\nabla_k\Theta_{mn} & g_{27} &= \nabla_i\nabla_k\Theta_{mn}\nabla_j\nabla_m\Theta_{nk}\\
g_4 &= R_{ilmn}\nabla_l\Theta_{kj}\nabla_m\Theta_{kn} & g_{16} &= R_{klmn}\nabla_i\Theta_{mn}\nabla_k\Theta_{lj} & g_{28} &= R_{kijl}\nabla_m\Theta_{nk}\nabla_m\Theta_{nl}\\
g_5 &= R_{ilmn}\nabla_k\Theta_{mj}\nabla_l\Theta_{kn} & g_{17} &= R_{klmn}\nabla_k\Theta_{li}\nabla_m\Theta_{nj} & g_{29} &= R_{kijl}\nabla_m\Theta_{nk}\nabla_n\Theta_{ml}\\
g_6 &= R_{ilmn}\nabla_k\Theta_{lj}\nabla_m\Theta_{kn} & g_{18} &= R_{klmn}\nabla_k\Theta_{mi}\nabla_l\Theta_{nj} & g_{30} &= R_{kijl}\nabla_k\Theta_{mn}\nabla_m\Theta_{nl}\\
g_7 &= R_{ilmn}\nabla_m\Theta_{kj}\nabla_n\Theta_{kl} & g_{19} &= R_{klmn}\nabla_i\Theta_{kl}\nabla_j\Theta_{mn} & g_{31} &= R_{kijl}\nabla_k\Theta_{mn}\nabla_l\Theta_{mn}\\
g_8 &= R_{ilmn}\nabla_k\Theta_{lj}\nabla_k\Theta_{mn} & g_{20} &= R_{klmn}\nabla_i\nabla_j\Theta_{kl}\Theta_{mn} & g_{32} &= \nabla_m R_{kijl}\nabla_m\Theta_{nk}\Theta_{nl}\\
g_9 &= R_{klmn}\nabla_i\Theta_{jl}\nabla_k\Theta_{mn} & g_{21} &= R_{ilmn}\nabla_k\nabla_l\Theta_{mn}\Theta_{kj} & g_{33} &= \nabla_m R_{kijl}\nabla_n\Theta_{mk}\Theta_{nl}\\
g_{10} &= \nabla_i R_{klmn}\nabla_k\Theta_{lj}\Theta_{mn} & g_{22} &= R_{klmn}\nabla_i\nabla_k\Theta_{mn}\Theta_{lj} & g_{34} &= \nabla_m R_{kijl}\nabla_k\Theta_{mn}\Theta_{nl}\\
g_{11} &= \nabla_i R_{klmn}\nabla_k\Theta_{mn}\Theta_{lj} & g_{23} &= R_{ilmn}\nabla_j\nabla_k\Theta_{mn}\Theta_{kl} & g_{35} &= \nabla_m R_{kijl}\nabla_k\Theta_{ln}\Theta_{mn}\\
g_{12} &= \nabla_i R_{klmn}\nabla_j\Theta_{kl}\Theta_{mn} & g_{24} &= R_{ilmn}\nabla_j\nabla_m\Theta_{nk}\Theta_{kl}
\end{aligned} \tag{B.44}$$

the $R^2\Theta^2$-terms

$$\begin{aligned}
h_1 &= R_{ipkl}R_{jpmn}\Theta_{kl}\Theta_{mn} & h_5 &= R_{ilmn}R_{jlkp}\Theta_{mk}\Theta_{np} & h_9 &= R_{kijp}R_{klmn}\Theta_{mn}\Theta_{pl}\\
h_2 &= R_{ilmn}R_{mnkp}\Theta_{kp}\Theta_{jl} & h_6 &= R_{ikmp}R_{jlnp}\Theta_{kl}\Theta_{mn} & h_{10} &= R_{klmi}R_{klmn}(\Theta^2)_{nj}\\
h_3 &= R_{ilmn}R_{mnkp}\Theta_{kl}\Theta_{jp} & h_7 &= R_{ilmn}R_{jlmk}(\Theta^2)_{nk} & h_{11} &= R_{klmn}R_{klmp}\Theta_{in}\Theta_{jp}\\
h_4 &= R_{ilmn}R_{klmp}\Theta_{np}\Theta_{jk} & h_8 &= R_{ilmn}R_{jkmn}(\Theta^2)_{lk}
\end{aligned} \tag{B.45}$$

and the terms involving the dilaton

$$\begin{aligned}
m_1 &= R_{ilmn}\nabla_k\Phi\,\nabla_j\Theta_{kl}\Theta_{mn} & m_{10} &= R_{ilmn}\nabla_m\Phi\,\nabla_k\Theta_{lj}\Theta_{kn} & m_{19} &= \nabla_k R_{ilmn}\nabla_l\Phi\,\Theta_{kj}\Theta_{mn}\\
m_2 &= R_{ilmn}\nabla_k\Phi\,\nabla_l\Theta_{kj}\Theta_{mn} & m_{11} &= R_{ilmn}\nabla_m\Phi\,\nabla_k\Theta_{nj}\Theta_{kl} & m_{20} &= \nabla_k\Phi\,\nabla_i\nabla_k\Theta_{mn}\nabla_j\Theta_{mn}\\
m_3 &= R_{ilmn}\nabla_k\Phi\,\nabla_k\Theta_{mn}\Theta_{lj} & m_{12} &= R_{klmn}\nabla_k\Phi\,\nabla_m\Theta_{ni}\Theta_{lj} & m_{21} &= \nabla_k\Phi\,\nabla_i\nabla_j\Theta_{mn}\nabla_m\Theta_{nk}\\
m_4 &= R_{ilmn}\nabla_l\Phi\,\nabla_k\Theta_{mn}\Theta_{kj} & m_{13} &= R_{klmn}\nabla_k\Phi\,\nabla_i\Theta_{mn}\Theta_{lj} & m_{22} &= \nabla_k\Phi\,\nabla_i\nabla_k\Theta_{mn}\nabla_m\Theta_{nj}\\
m_5 &= R_{ilmn}\nabla_m\Phi\,\nabla_j\Theta_{nk}\Theta_{kl} & m_{14} &= R_{klmn}\nabla_k\Phi\,\nabla_i\Theta_{jl}\Theta_{mn} & m_{23} &= \nabla_k\Phi\,\nabla_i\nabla_m\Theta_{nk}\nabla_j\Theta_{mn}\\
m_6 &= R_{ilmn}\nabla_m\Phi\,\nabla_j\Theta_{kl}\Theta_{kn} & m_{15} &= \nabla_k R_{ilmn}\nabla_k\Phi\,\Theta_{mn}\Theta_{lj} & m_{24} &= \nabla_k\Phi\,\nabla_i\nabla_m\Theta_{nk}\nabla_m\Theta_{nj}\\
m_7 &= R_{ilmn}\nabla_m\Phi\,\nabla_l\Theta_{kj}\Theta_{kn} & m_{16} &= R_{ilmn}\nabla_k\Phi\,\nabla_l\Theta_{km}\Theta_{nj} & m_{25} &= \nabla_k\Phi\,\nabla_i\nabla_m\Theta_{nk}\nabla_n\Theta_{mj}\\
m_8 &= R_{ilmn}\nabla_m\Phi\,\nabla_n\Theta_{kj}\Theta_{kl} & m_{17} &= R_{ilmn}\nabla_m\Phi\,\nabla_n\Theta_{lk}\Theta_{kj}\\
m_9 &= R_{ilmn}\nabla_m\Phi\,\nabla_l\Theta_{kn}\Theta_{kj} & m_{18} &= R_{klmn}\nabla_k\Phi\,\nabla_l\Theta_{mi}\Theta_{nj}
\end{aligned} \tag{B.46}$$

JHEP03(2020)126

B.1 Additional identities

Contracting (A.6) with Θ and one covariant derivative, or the derivative of the dilaton, in all possible ways gives the identities

$$0 = g_{12} - 4g_{13} - 2g_{14} + g_{15} + g_{19}\,, \tag{B.47}$$

$$0 = 2g_1 + g_3 - 2g_4 - 4g_7 + 2g_8 + 2g_{10} + g_{16} + 2g_{17} - 4g_{18}\,, \tag{B.48}$$

$$0 = 2g_1 + 2g_3 - 4g_5 + 2g_6 + g_8 - g_{16} + 2g_{17} - 4g_{18}\,, \tag{B.49}$$

$$0 = \hat{f}_1 + \hat{f}_4 + 2\hat{f}_5 - 4\hat{f}_6 + \nabla_i R_{klmn}\Theta_{mn}\Theta_{kl}\,, \tag{B.50}$$

$$0 = 2m_4 + 2m_{12} - m_{13} + 4m_{16} + 4m_{18} + 2m_{19}\,, \tag{B.51}$$

$$0 = 2m_3 + m_4 - 2m_9 + 2m_{12} + m_{13} + 2m_{15} + 2m_{17} + 4m_{18}\,, \tag{B.52}$$

$$0 = 2f_{14} + 2f_{18} + f_{19} - 4f_{20} - f_{22} + 2f_{23} + 4f_{24} - 2f_{26}\,, \tag{B.53}$$

$$0 = f_{14} + 4f_{16} + 2f_{17} + 2f_{19} + f_{22} + 2f_{23} + 4f_{24} - 2f_{25}\,. \tag{B.54}$$

The last two imply, using the previous ones, that $m_{19} = m_2$ and

$$0 = g_2 + g_{21} + g_{22} + h_2 - 2h_3 \tag{B.55}$$

In addition we can derive the following identity

$$\begin{aligned}
2h_5 &= 2R_{iklp}R_{jkmn}\Theta^{pm}\Theta_{nl} = -2\nabla_l\nabla_k k_i \times \nabla_n\nabla_k k_j\Theta_{nl} \\
&= -2\nabla_l(R_{jknm}\nabla_k k_i \times k_m\Theta_{nl}) + R_{lnkm}\nabla_k k_i \times \nabla_m k_j\Theta_{nl} + R_{lnjm}\nabla_k k_i \times \nabla_k k_m\Theta_{nl} \\
&= \nabla_l(R_{jknm}\nabla_k\Theta_{mi}\Theta_{nl}) + \nabla_l(R_{jknm}\nabla_m\Theta_{ki}\Theta_{nl}) + \nabla_l(R_{jknm}\nabla_i\Theta_{km}\Theta_{nl}) \\
&\quad - \frac{1}{2}R_{klmn}\Theta_{kl}\nabla_i\nabla_j\Theta_{mn} - R_{lnkm}R_{mijp}\Theta_{kp}\Theta_{nl} - \frac{1}{2}R_{klpi}R_{jpmn}\Theta_{kl}\Theta_{mn} \\
&= -\frac{1}{2}\nabla\cdot f_1 - \nabla\cdot f_5 - \nabla\cdot f_6 - \frac{1}{2}g_{20} + \frac{1}{2}h_1 + h_9\,.
\end{aligned} \tag{B.56}$$

References

[1] C. Klimčík, *Yang-Baxter σ-models and dS/AdS T duality*, *JHEP* **12** (2002) 051 [hep-th/0210095] [INSPIRE].

[2] C. Klimčík, *On integrability of the Yang-Baxter σ-model*, *J. Math. Phys.* **50** (2009) 043508 [arXiv:0802.3518] [INSPIRE].

[3] F. Delduc, M. Magro and B. Vicedo, *An integrable deformation of the $AdS_5 \times S^5$ superstring action*, *Phys. Rev. Lett.* **112** (2014) 051601 [arXiv:1309.5850] [INSPIRE].

[4] I. Kawaguchi, T. Matsumoto and K. Yoshida, *Jordanian deformations of the $AdS_5 \times S^5$ superstring*, *JHEP* **04** (2014) 153 [arXiv:1401.4855] [INSPIRE].

[5] R. Borsato and L. Wulff, *Target space supergeometry of η and λ-deformed strings*, *JHEP* **10** (2016) 045 [arXiv:1608.03570] [INSPIRE].

JHEP03(2020)126

[6] G. Arutyunov, S. Frolov, B. Hoare, R. Roiban and A.A. Tseytlin, *Scale invariance of the η-deformed $AdS_5 \times S^5$ superstring, T-duality and modified type-II equations*, Nucl. Phys. **B 903** (2016) 262 [arXiv:1511.05795] [INSPIRE].

[7] L. Wulff and A.A. Tseytlin, *κ-symmetry of superstring σ-model and generalized* 10d *supergravity equations*, JHEP **06** (2016) 174 [arXiv:1605.04884] [INSPIRE].

[8] X.C. de la Ossa and F. Quevedo, *Duality symmetries from nonAbelian isometries in string theory*, Nucl. Phys. **B 403** (1993) 377 [hep-th/9210021] [INSPIRE].

[9] M. Gasperini, R. Ricci and G. Veneziano, *A Problem with nonAbelian duality?*, Phys. Lett. **B 319** (1993) 438 [hep-th/9308112] [INSPIRE].

[10] E. Alvarez, L. Álvarez-Gaumé and Y. Lozano, *On nonAbelian duality*, Nucl. Phys. **B 424** (1994) 155 [hep-th/9403155] [INSPIRE].

[11] S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer and G. Veneziano, *Remarks on nonAbelian duality*, Nucl. Phys. **B 435** (1995) 147 [hep-th/9409011] [INSPIRE].

[12] B. Hoare and A.A. Tseytlin, *Homogeneous Yang-Baxter deformations as non-abelian duals of the AdS_5 σ-model*, J. Phys. **A 49** (2016) 494001 [arXiv:1609.02550] [INSPIRE].

[13] R. Borsato and L. Wulff, *Integrable Deformations of T-Dual σ Models*, Phys. Rev. Lett. **117** (2016) 251602 [arXiv:1609.09834] [INSPIRE].

[14] R. Borsato and L. Wulff, *On non-abelian T-duality and deformations of supercoset string σ-models*, JHEP **10** (2017) 024 [arXiv:1706.10169] [INSPIRE].

[15] R. Borsato and L. Wulff, *Non-abelian T-duality and Yang-Baxter deformations of Green-Schwarz strings*, JHEP **08** (2018) 027 [arXiv:1806.04083] [INSPIRE].

[16] T. Araujo, I. Bakhmatov, E. Ó Colgáin, J. Sakamoto, M.M. Sheikh-Jabbari and K. Yoshida, *Yang-Baxter σ-models, conformal twists and noncommutative Yang-Mills theory*, Phys. Rev. **D 95** (2017) 105006 [arXiv:1702.02861] [INSPIRE].

[17] I. Bakhmatov, Ö. Kelekci, E. Ó Colgáin and M.M. Sheikh-Jabbari, *Classical Yang-Baxter Equation from Supergravity*, Phys. Rev. **D 98** (2018) 021901 [arXiv:1710.06784] [INSPIRE].

[18] J.-i. Sakamoto, Y. Sakatani and K. Yoshida, *Homogeneous Yang-Baxter deformations as generalized diffeomorphisms*, J. Phys. **A 50** (2017) 415401 [arXiv:1705.07116] [INSPIRE].

[19] D. Osten and S.J. van Tongeren, *Abelian Yang-Baxter deformations and TsT transformations*, Nucl. Phys. **B 915** (2017) 184 [arXiv:1608.08504] [INSPIRE].

[20] K.A. Meissner and G. Veneziano, *Symmetries of cosmological superstring vacua*, Phys. Lett. **B 267** (1991) 33 [INSPIRE].

[21] K.A. Meissner and G. Veneziano, *Manifestly $O(d,d)$ invariant approach to space-time dependent string vacua*, Mod. Phys. Lett. **A 6** (1991) 3397 [hep-th/9110004] [INSPIRE].

[22] A. Sen, *$O(d) \otimes O(d)$ symmetry of the space of cosmological solutions in string theory, scale factor duality and two-dimensional black holes*, Phys. Lett. **B 271** (1991) 295 [INSPIRE].

[23] M. Gasperini, J. Maharana and G. Veneziano, *From trivial to nontrivial conformal string backgrounds via $O(d,d)$ transformations*, Phys. Lett. **B 272** (1991) 277 [INSPIRE].

[24] S.F. Hassan and A. Sen, *Marginal deformations of WZNW and coset models from $O(d,d)$ transformation*, Nucl. Phys. **B 405** (1993) 143 [hep-th/9210121] [INSPIRE].

[25] E. Kiritsis, *Exact duality symmetries in CFT and string theory*, in *International Workshop on String Theory, Quantum Gravity and the Unification of Fundamental Interactions*, Rome, Italy, 21–26 September 1992, pp. 347–376 (1992) [hep-th/9211081] [INSPIRE].

[26] J. Panvel, *Higher order conformal invariance of string backgrounds obtained by $O(d,d)$ transformations*, *Phys. Lett.* **B 284** (1992) 50 [hep-th/9204024] [INSPIRE].

[27] E. Kiritsis, *Exact duality symmetries in CFT and string theory*, *Nucl. Phys.* **B 405** (1993) 109 [hep-th/9302033] [INSPIRE].

[28] D. Lüst and D. Osten, *Generalised fluxes, Yang-Baxter deformations and the $O(d,d)$ structure of non-abelian T-duality*, *JHEP* **05** (2018) 165 [arXiv:1803.03971] [INSPIRE].

[29] J.-I. Sakamoto and Y. Sakatani, *Local β-deformations and Yang-Baxter σ-model*, *JHEP* **06** (2018) 147 [arXiv:1803.05903] [INSPIRE].

[30] B. Hoare, N. Levine and A.A. Tseytlin, *Integrable 2d σ-models: quantum corrections to geometry from RG flow*, *Nucl. Phys.* **B 949** (2019) 114798 [arXiv:1907.04737] [INSPIRE].

[31] B. Hoare, N. Levine and A.A. Tseytlin, *Integrable σ-models and 2-loop RG flow*, *JHEP* **12** (2019) 146 [arXiv:1910.00397] [INSPIRE].

[32] A. Eghbali, *Exact conformal field theories from mutually T-dualizable σ-models*, *Phys. Rev.* **D 99** (2019) 026001 [arXiv:1812.07664] [INSPIRE].

[33] P.E. Haagensen and K. Olsen, *T duality and two loop renormalization flows*, *Nucl. Phys.* **B 504** (1997) 326 [hep-th/9704157] [INSPIRE].

[34] N. Kaloper and K.A. Meissner, *Duality beyond the first loop*, *Phys. Rev.* **D 56** (1997) 7940 [hep-th/9705193] [INSPIRE].

[35] I. Jack and S. Parsons, *$O(d,d)$ invariance at two loops and three loops*, *Phys. Rev.* **D 62** (2000) 026003 [hep-th/9911064] [INSPIRE].

[36] S. Parsons, *T duality and conformal invariance at two loops*, *Phys. Rev.* **D 61** (2000) 086002 [hep-th/9912105] [INSPIRE].

[37] J.D. Edelstein, K. Sfetsos, J.A. Sierra-Garcia and A. Vilar López, *T-duality equivalences beyond string theory*, *JHEP* **05** (2019) 082 [arXiv:1903.05554] [INSPIRE].

[38] L. Wulff, *Trivial solutions of generalized supergravity vs non-abelian T-duality anomaly*, *Phys. Lett.* **B 781** (2018) 417 [arXiv:1803.07391] [INSPIRE].

[39] R. Borsato and L. Wulff, *Marginal deformations of WZW models and the classical Yang-Baxter equation*, *J. Phys.* **A 52** (2019) 225401 [arXiv:1812.07287] [INSPIRE].

[40] C.M. Hull and P.K. Townsend, *String Effective Actions From σ Model Conformal Anomalies*, *Nucl. Phys.* **B 301** (1988) 197 [INSPIRE].

[41] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142] [INSPIRE].

[42] C.M. Hull and P.K. Townsend, *The Two Loop β-function for σ Models With Torsion*, *Phys. Lett.* **B 191** (1987) 115 [INSPIRE].

[43] R.R. Metsaev and A.A. Tseytlin, *Order alpha-prime (Two Loop) Equivalence of the String Equations of Motion and the σ-model Weyl Invariance Conditions: Dependence on the Dilaton and the Antisymmetric Tensor*, *Nucl. Phys.* **B 293** (1987) 385 [INSPIRE].

JHEP03(2020)126

[44] D. Zanon, *Two Loop β-functions and Low-energy String Effective Action for the Two-dimensional Bosonic Nonlinear σ Model With a Wess-Zumino-Witten Term*, Phys. Lett. **B 191** (1987) 363 [INSPIRE].

[45] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, JHEP **05** (2005) 069 [hep-th/0503201] [INSPIRE].

[46] L.F. Alday, G. Arutyunov and S. Frolov, *Green-Schwarz strings in TsT-transformed backgrounds*, JHEP **06** (2006) 018 [hep-th/0512253] [INSPIRE].

[47] S.J. van Tongeren, *Almost abelian twists and AdS/CFT*, Phys. Lett. **B 765** (2017) 344 [arXiv:1610.05677] [INSPIRE].

[48] T.H. Buscher, *A Symmetry of the String Background Field Equations*, Phys. Lett. **B 194** (1987) 59 [INSPIRE].

[49] K.A. Meissner, *Symmetries of higher order string gravity actions*, Phys. Lett. **B 392** (1997) 298 [hep-th/9610131] [INSPIRE].

[50] R. Borsato, A.V. López and L. Wulff, *The first α'-correction to homogeneous Yang-Baxter deformations using $O(d,d)$*, arXiv:2003.05867 [INSPIRE].

[51] G. Papadopoulos, J.G. Russo and A.A. Tseytlin, *Solvable model of strings in a time dependent plane wave background*, Class. Quant. Grav. **20** (2003) 969 [hep-th/0211289] [INSPIRE].

[52] M. Gasperini and R. Ricci, *Homogeneous conformal string backgrounds*, Class. Quant. Grav. **12** (1995) 677 [hep-th/9501055] [INSPIRE].

[53] N.A. Batakis and A.A. Kehagias, *Anisotropic space-times in homogeneous string cosmology*, Nucl. Phys. **B 449** (1995) 248 [hep-th/9502007] [INSPIRE].

[54] F. Naderi, A. Rezaei-Aghdam and F. Darabi, *Noncritical anisotropic Bianchi type-I string cosmology with α' corrections*, Phys. Rev. **D 98** (2018) 026009 [arXiv:1712.03581] [INSPIRE].

[55] A. Subbotin and I.V. Tyutin, *On the equivalence of dual theories*, Int. J. Mod. Phys. **A 11** (1996) 1315 [*Erratum ibid.* **A 11** (1996) 2231] [hep-th/9506132] [INSPIRE].

[56] J. Balog, P. Forgacs, Z. Horvath and L. Palla, *Perturbative quantum (in)equivalence of dual σ-models in two-dimensions*, Nucl. Phys. Proc. Suppl. **49** (1996) 16 [hep-th/9601091] [INSPIRE].

[57] L.K. Balazs, J. Balog, P. Forgacs, N. Mohammedi, L. Palla and J. Schnittger, *Quantum equivalence of σ-models related by nonAbelian duality transformations*, Phys. Rev. **D 57** (1998) 3585 [hep-th/9704137] [INSPIRE].

[58] G. Bonneau and P.-Y. Casteill, *Dualized σ-models at the two loop order*, Nucl. Phys. **B 607** (2001) 293 [hep-th/0103260] [INSPIRE].

Journal of Physics A: Mathematical and Theoretical

PAPER

Marginal deformations of WZW models and the classical Yang–Baxter equation

To cite this article: Riccardo Borsato and Linus Wulff 2019 *J. Phys. A: Math. Theor.* **52** 225401

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **52** (2019) 225401 (30pp) https://doi.org/10.1088/1751-8121/ab1b9c

Marginal deformations of WZW models and the classical Yang–Baxter equation

Riccardo Borsato[1,2] and **Linus Wulff**[3]

[1] Instituto Galego de Física de Altas Enerxías (IGFAE), Universidade de Santiago de Compostela, Santiago de Compostela, Spain
[2] Nordita, Stockholm University and KTH Royal Institute of Technology, Stockholm, Sweden
[3] Department of Theoretical Physics and Astrophysics, Masaryk University, 611 37 Brno, Czech Republic

E-mail: riccardo.borsato@usc.es and wulff@physics.muni.cz

Received 15 January 2019, revised 12 April 2019
Accepted for publication 23 April 2019
Published 7 May 2019

Abstract

We show how so-called Yang–Baxter (YB) deformations of sigma models, based on an R-matrix solving the classical Yang–Baxter equation (CYBE), give rise to marginal current–current deformations when applied to the Wess–Zumino–Witten (WZW) model. For non-compact groups these marginal deformations are more general than the ones usually considered, since they can involve a non-Abelian current subalgebra. We classify such deformations of the $\mathrm{AdS}_3 \times \mathrm{S}^3$ string.

Keywords: Wess–Zumino–Witten model, marginal deformations, string on AdS(3), classical Yang–Baxter equation

1. Introduction

Conformal field theories (CFTs) in two dimensions are of interest for various areas of physics, from condensed matter physics to string theory. In string theory they naturally arise on the worldsheet of the string. In the context of holographic duality, certain two-dimensional CFTs are also known to be *dual to* string theories on three-dimensional anti de Sitter spacetimes [1–3]. An important instance of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality is obtained by studying string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times T^4$. In the case of pure NSNS flux the string is described by a Wess–Zumino–Witten (WZW) model on the group $F = SL(2,\mathbb{R}) \times SU(2)$, see [4–6] and references there. We will be interested mainly in this setup. The CFT description of the worldsheet theory allows to make precise statements about the AdS/CFT duality in this case. A recent example is the duality between the symmetric product orbifold CFT and the string WZW model at level $k = 1$ [7].

Generally speaking it is interesting to understand the conformal manifold of a CFT, i.e. the space of marginal deformations generated by adding a local perturbation to the Lagrangian.

When applied to the WZW model under study, the marginal deformations correspond to deformations of the supergravity background. They give us (at least in principle) a way to go beyond the usual AdS_3/CFT_2 duality and extend it to cases in which e.g. the supersymmetry or the conformal symmetry of the dual CFT_2 are broken. Local marginal deformations of WZW models were studied by Chaudhuri and Schwartz (CS) [8]. They found that a necessary condition for local operators constructed out of the chiral and antichiral currents as

$$O(\sigma,\bar{\sigma}) = c^{ab} J_a(\sigma) \bar{J}_b(\bar{\sigma}), \tag{1.1}$$

with c^{ab} some constant coefficients, to give a marginal deformation is that c^{ab} satisfy

$$C \cdot C + \bar{C} \cdot \bar{C} = 0, \tag{1.2}$$

where we have defined $C^{abc} \equiv c^{da} c^{eb} f_{de}{}^{c}$, and $\bar{C}^{abc} \equiv c^{ad} c^{be} f_{de}{}^{c}$, and the product is obtained using the Killing metric K_{ab}, e.g. $C \cdot C \equiv C^{abc} C^{def} K_{ad} K_{be} K_{cf}$. Equation (1.2) is quartic in c^{ab}, and it involves also the structure constants of the algebra of F, the Lie group of the WZW model. We will call it the *weak* CS condition. CS were interested in the case of CFTs where the group F is compact. In that case (1.2) reduces to

$$C^{abc} = 0, \qquad \text{and} \qquad \bar{C}^{abc} = 0, \tag{1.3}$$

which is an equation quadratic in c^{ab} that we will call the *strong* CS condition. It imposes, in fact, a stronger constraint, since it is only in the case of compact groups that it is equivalent to (1.2). CS also showed that solutions of the strong condition correspond to Abelian subalgebras of $\mathrm{Lie}(F)$, since when (1.3) holds it is always possible to identify linear combinations of $J_a, \bar{J}_a$ such that their OPEs do not have the term involving the structure constants—the so-called 'no simple-pole condition', see (2.16). In that case the correlation functions of O are the same as for an O constructed out of free bosons, which in turn implies that the deformation can be completed to all orders in conformal perturbation theory in the deformation parameter. For deformations satisfying only the weak CS condition on the other hand there is no guarantee that they remain marginal beyond lowest order in the deformation parameter. In the literature on marginal deformations of WZW models, see e.g. [9–18], we did not find examples that satisfy the weak CS condition but not the strong one. Here we will construct such examples by involving sufficient components of the chiral and antichiral currents of $SL(2,\mathbb{R})$. In this sense our results identify new directions to explore the conformal manifold.

In [10, 11, 19] it was argued that $O(d,d)$ transformations provides the correct language to obtain the exact (in the deformation parameter) version of the CFTs deformed by the Abelian current–current operators. Indeed, such transformations do not break the isometries involved in the deformation and one can show that the derivative of the action with respect to the deformation parameter is given by

$$\frac{\mathrm{d}S}{\mathrm{d}\eta} = -\frac{T}{2} \int \mathrm{d}^2\sigma \, J^\eta \bar{J}^\eta, \tag{1.4}$$

where J^η and $\bar{J}^\eta$ are (anti)chiral currents of the *deformed* theory corresponding to the isometries involved in the deformation. This makes it clear that the infinitesimal deformation can be integrated to a finite one. The relevant so-called β-shifts of $O(d,d)$, corresponding to a simple shift of the B-field of the dual model obtained by performing T-duality on two $U(1)$ isometries, are also known as TsT (T-duality, shift, T-duality) transformations [20–23]. A TsT transformation exploits an Abelian $U(1)^2$ global symmetry of the sigma model to construct a deformation parameterised by a continuous deformation parameter. Since the deformation is constructed by exploiting T-duality, on-shell the deformed model is equivalent to the undeformed one, and the deformation can be equivalently understood as a twist in the boundary conditions in the compact direction of the worldsheet.

TsT transformations are known to belong to a larger class of deformations of sigma models, usually called Yang–Baxter (YB) deformations. They first appeared in the context of integrable models, since the deformations do not break the classical integrability of the original model [24–26]. YB deformations are particularly interesting in the context of the *AdS*/*CFT* correspondence, since they can be used to generate string backgrounds deforming the standard ones appearing in the $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ dualities. Particularly important cases are those for which integrability techniques may be applied. In the case of $\mathrm{AdS}_5/\mathrm{CFT}_4$ it was proposed that the deformations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background should correspond to non-commutative deformations of $\mathcal{N} = 4$ super Yang–Mills [27–30]. The name YB comes from the fact that the deformation is controlled by an object R which is an element of $\mathfrak{g} \wedge \mathfrak{g}$ (where $\mathfrak{g}$ is the algebra of isometries of the starting background) and solves the classical Yang–Baxter equation[4] (CYBE) on $\mathfrak{g}$. The simplest solutions to the CYBE are the so-called Abelian R-matrices, e.g. $R = T_1 \wedge T_2$ with $T_i \in \mathfrak{g}$ and $[T_1, T_2] = 0$. In this case the CYBE is trivially satisfied because the relevant structure constants vanish. Abelian YB deformations were shown to be equivalent to TsT transformations in [31]. On compact algebras, the CYBE only admits Abelian solutions. On non-compact algebras, instead, more interesting non-Abelian solutions (i.e. R-matrices constructed out of generators of a non-Abelian subalgebra) are possible.

Originally, in the construction of the YB-deformed sigma models, the CYBE was necessary in order to preserve the classical integrability. Later it was understood that YB deformations may be obtained from non-Abelian T-duality (NATD) [32, 33]. That interpretation revealed a consistent generalisation of what is known about TsT transformations, since it became clear that YB deformations correspond to a shift of the B-field of the dual (undeformed) sigma model; the deformed model is then obtained by applying NATD on the subalgebra of $\mathfrak{g}$ where R is non-degenerate. After restricting the domain, R may be inverted and its inverse R^{-1} is a Lie algebra 2-cocycle. The shift of the dual B-field is given by this 2-cocycle. In other words, it is possible to go beyond the construction related to integrable models, and understand the CYBE as being a constraint necessary to shift the dual B-field without modifying its field strength $H = \mathrm{d}B$. Consistently with this interpretation, in [34, 35] it was proposed to identify YB deformations with the β-shifts of a larger group extending the known $O(d, d)$ group of Abelian T-duality, that in [34] was dubbed 'non-Abelian T-duality group'. The logic of NATD/β-shifts may be used to construct YB deformations of generic sigma models [34–36], beyond those for which YB deformations were first introduced, the Principal Chiral Model and (super)cosets[5]. Here we will use the transformation rules of [36], which were derived from the NATD construction and have the advantage of being applicable to a generic background with isometries (even when the initial $G - B$ is not invertible, as is the case for the background metric and Kalb–Ramond field that we have to consider in this paper).

Because of their realization via NATD, YB models will be Weyl invariant at least to one loop in σ–model perturbation theory (and exactly in the deformation parameter), provided that the Lie algebra on which the R-matrix is non-trivial is unimodular (i.e. the trace of its structure constants vanish). In this case the deformed background solves the standard supergravity equations of motion. When the algebra is not unimodular there is a potential Weyl anomaly [38, 39]. In that case the resulting background solves instead a generalization of the standard supergravity equations [40, 41] controlled by a Killing vector field K. Even in these

[4] There is also a version of these models where R solves instead the *modified* CYBE [24, 25]. The story in that case is quite different and we will not consider it here.

[5] Bakhmatov *et al* [37] put forward a first proposal for a set of transformation rules to go beyond these cases.

non-unimodular cases, it can happen that there is actually no Weyl anomaly. This is reflected in the fact that the generalized supergravity equations can have 'trivial' solutions, i.e. solutions with $K \neq 0$ but where nevertheless the other fields solve the standard supergravity equations [42]. In [42] it was shown that this can happen if K is null. In appendix E we show that this condition can be weakened and K does not have to be null if the one-form X appearing in the generalized supergravity equations takes the form $\mathrm{X} = \mathrm{d}\phi + \tilde{\mathrm{K}}$ with ϕ the dilaton and $\tilde{K}$ another Killing vector. We will see that YB deformations of the $\mathrm{AdS}_3 \times \mathrm{S}^3$ WZW model give rise also to 'trivial' solutions, both ones with $K^2 = 0$ and with $K^2 \neq 0$. We will also find some examples with a genuine anomaly, corresponding to K not being null (and $\tilde{K}$ not defining a Killing vector).

As we will discuss in more detail in section 3, at leading order in the deformation parameter YB deformations correspond to current–current deformations. We are therefore led to study YB deformations of strings on backgrounds containing an AdS_3 subspace, expecting to find marginal deformations of the corresponding WZW model. Particularly interesting for us are the deformations that do not solve the strong version of the CS condition, but only the weak one. We will construct explicitly such examples. Such possibilities are allowed because we exploit also the non-compact part of the current algebra to generate the deformations.

The paper is organised as follows. In section 2 we review some aspects of the $SL(2,\mathbb{R}) \times SU(2)$ WZW model and of marginal current–current deformations that are important for our discussion. In section 3 we review the transformation rules of YB deformations and explain in which cases we can understand them as compositions of simpler YB transformations. We will also explain the connection to the marginal current–current deformations. Using the classification of R-matrices in appendix C, we later study deformations of AdS_3 and $\mathrm{AdS}_3 \times \mathrm{S}^3$. We give our conclusions in section 4. Appendix A collects some details on the field redefinition used in section 3, and appendix B discusses the on-shell equivalence of the YB models to the undeformed ones. In appendix D we consider the case of the $\mathfrak{sl}_3$ algebra, which is separate from the rest of the paper. In appendix E we extend the triviality condition of [42].

2. Wess–Zumino–Witten model and marginal current–current deformations

In this section we review certain aspects of WZW models and their marginal current–current deformations. Although the discussion can be made general, for concreteness we will take the example of the $SL(2,\mathbb{R}) \times SU(2)$ WZW model, since it is important for string theory applications and it already contains all the salient features.

2.1. The $\mathrm{AdS}_3 \times \mathrm{S}^3$ sigma model

We start with a sigma model describing the propagation of a string in $\mathrm{AdS}_3 \times \mathrm{S}^3$, that can be viewed (after adding four free bosons) as the bosonic sector of the superstring . The sigma model action is[6]

$$S = \frac{k}{2\pi} \int \mathrm{d}^2\sigma \, \left(\frac{-\partial x^- \bar{\partial} x^+ + \partial z \bar{\partial} z}{z^2} + \frac{1}{4}\partial\phi_i \bar{\partial}\phi_i + \frac{1}{2}\partial\phi_2\bar{\partial}\phi_1 \sin\phi_3 \right). \quad (2.1)$$

[6] We work with a Lorentzian worldsheet and we introduce worldsheet coordinates $\sigma^{\pm} = \sigma^0 \pm \sigma^1$, so that $\eta^{+-} = \eta^{-+} = -2$, $\epsilon^{+-} = -\epsilon^{-+} = -2$ and $\mathrm{d}^2\sigma = \frac{1}{2}\mathrm{d}\sigma^+\mathrm{d}\sigma^-$. We also use the standard notation $\sigma, \bar{\sigma}$ in place of σ^+, σ^-, as well as $\partial = \partial_\sigma, \bar{\partial} = \partial_{\bar{\sigma}}$.

Here we are considering the pure NSNS background, and k will be the level of the WZW model. The string tension is $T = k/\pi$, and the metric and B-field appearing in the sigma model action follow from $S = T\int \mathrm{d}^2\sigma\ L = \frac{T}{2}\int \mathrm{d}^2\sigma\ \partial x^m(G_{mn} - B_{mn})\bar{\partial}x^n$ and are[7]

$$
\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}s^2_{\mathrm{AdS}_3} + \mathrm{d}s^2_{\mathrm{S}^3} = \frac{-\mathrm{d}x^+\mathrm{d}x^- + \mathrm{d}z^2}{z^2} + \frac{1}{4}\left[\mathrm{d}\phi_3^2 + \cos^2\phi_3\mathrm{d}\phi_1^2 + (\mathrm{d}\phi_2 + \sin\phi_3\mathrm{d}\phi_1)^2\right],\\
B &= \frac{\mathrm{d}x^+\wedge\mathrm{d}x^-}{2z^2} - \frac{1}{4}\sin\phi_3\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2.
\end{aligned}
\tag{2.2}
$$

AdS_3 is parameterised by the boundary coordinates $x^\pm$ and the radial coordinate z, while the angles ϕ_i parameterise the sphere. The AdS_3 metric admits the following Killing vectors

$$
\begin{aligned}
&k_0^m\partial_m = x^+\partial_{x^+} + \frac{1}{2}z\partial_z, \qquad k_+^m\partial_m = \partial_{x^+}, \qquad k_-^m\partial_m = -(x^+)^2\partial_{x^+} - z^2\partial_{x^-} - x^+z\partial_z,\\
&\bar{k}_0^m\partial_m = -x^-\partial_{x^-} - \frac{1}{2}z\partial_z, \qquad \bar{k}_-^m\partial_m = \partial_{x^-}, \qquad \bar{k}_+^m\partial_m = -(x^-)^2\partial_{x^-} - z^2\partial_{x^+} - x^-z\partial_z.
\end{aligned}
\tag{2.3}
$$

They satisfy $[k_a^m\partial_m, k_b^n\partial_n] = -f_{ab}{}^c k_c^p\partial_p$ (and similarly for $\bar{k}_a$), where $f_{ab}{}^c$ are the structure constants of the algebra of $SL(2,\mathbb{R})$

$$
[S_0, S_\pm] = \pm S_\pm, \qquad [S_+, S_-] = 2S_0. \tag{2.4}
$$

In these formulas and in the following we use a bar to distinguish the right copy of the algebra from the left copy[8]. For the sphere we have two copies of $SU(2)$, whose algebra is generated by T_a $(a = 1, 2, 3)$ with commutation relations $[T_a, T_b] = -\epsilon_{abc}T_c$. We will not write explicitly all Killing vectors of S^3 since we will not need them. For our purposes it will be enough to use the two commuting Killing vectors

$$
k_1 = -\partial_{\phi_1} \qquad \text{and} \qquad \bar{k}_2 = \partial_{\phi_2}. \tag{2.5}
$$

The sigma-model action is invariant under the transformations generated by the above Killing vectors, although in certain cases the B-field is not invariant but changes by a total derivative. Therefore in general the corresponding Noether currents are given by

$$
\mathcal{J}_{A,\pm} = k_A^m(G_{mn} \pm B_{mn})\partial_\pm x^n + j_{A,\pm}, \tag{2.6}
$$

where $j_{A,\pm}$ is defined by looking at the variation of the Lagrangian $\delta_A L = \varepsilon\partial_i j_A^i$ under the infinitesimal global transformation. Because of our choice of gauge, in the AdS_3 part only j_-^i and $\bar{j}_+^i$ are non-zero, and we also have $j_1^i = \bar{j}_2^i = 0$. In the following we will ignore the transformations generated by $S_-, \bar{S}_+$, since for our discussion it will be enough to focus on the (maximal solvable) subalgebra generated by

$$
S_0, \quad S_+, \quad \bar{S}_0, \quad \bar{S}_-, \quad T_1, \quad \bar{T}_2. \tag{2.7}
$$

All Noether currents that we will need to consider will therefore have $j_{A,\pm} = 0$. Let us anticipate that these Noether currents are not always equal to the chiral (resp. antichiral) currents of the WZW description, which we shall denote by J (resp. $\bar{J}$) and write explicitly in the next subsection. They agree up to 'improvement terms' that do not spoil the current conservation, of the type $\epsilon^{ij}\partial_j c$ for some c. Restricting to the generators in (2.7), for AdS_3 we have

[7] In our conventions $B = \frac{1}{2}B_{nm}\mathrm{d}x^m\wedge\mathrm{d}x^n$.

[8] We will interchangeably place the bar on an object or on its index, in other words $\bar{k}_a$ or $k_{\bar{a}}$ have the same meaning. For readability sometimes we will prefer the former.

$$\mathcal{J}_{0,+} = J_0 - \frac{1}{2}\partial \log z, \qquad \mathcal{J}_{0,-} = +\frac{1}{2}\bar{\partial} \log z, \qquad \mathcal{J}_{+,+} = J_+, \qquad \mathcal{J}_{+,-} = 0,$$
$$\bar{\mathcal{J}}_{0,-} = \bar{J}_0 + \frac{1}{2}\bar{\partial} \log z, \qquad \bar{\mathcal{J}}_{0,+} = -\frac{1}{2}\partial \log z, \qquad \bar{\mathcal{J}}_{-,-} = \bar{J}_-, \qquad \bar{\mathcal{J}}_{-,+} = 0, \tag{2.8}$$

while for S^3

$$\mathcal{J}_{1,+} = J_1 + \frac{1}{4}\partial\phi_1, \qquad \mathcal{J}_{1,-} = -\frac{1}{4}\bar{\partial}\phi_1,$$
$$\bar{\mathcal{J}}_{2,-} = \bar{J}_2 - \frac{1}{4}\bar{\partial}\phi_2, \qquad \bar{\mathcal{J}}_{2,+} = \frac{1}{4}\partial\phi_2. \tag{2.9}$$

This fact will later play an important role in our discussion.

2.2. The $SL(2,\mathbb{R}) \times SU(2)$ WZW model

The action of the WZW model is $S_{\rm WZW} = S_1 + k\Gamma$ where k is the level and

$$S_1[g] = \frac{k}{4\pi}\int_{\partial\mathcal{B}} d^2\sigma \mathrm{Tr}(\partial^i g^{-1}\partial_i g), \qquad \Gamma[g] = -\frac{1}{6\pi}\int_{\mathcal{B}} d^3\sigma\epsilon^{ijk}\mathrm{Tr}(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g). \tag{2.10}$$

Here g is an element of a group G, depending on coordinates on $\mathcal{B}$, whose boundary $\partial\mathcal{B}$ is the worldsheet of the string. In the following we will take the action[9] $S_{\rm WZW}[g_{\rm a}] - S_{\rm WZW}[g_{\rm s}]$ with

$$g_{\rm a} = e^{x^+ S_+} z^{2S_0} e^{-x^- S_-} \in SL(2,\mathbb{R}), \qquad g_{\rm s} = e^{\phi_1 T_1} e^{\phi_3 T_3} e^{\phi_2 T_2} \in SU(2). \tag{2.11}$$

We realise the generators of the algebra of $SL(2,\mathbb{R})$ in terms of the Pauli matrices as $S_0 = \sigma_3/2$, $S_+ = (\sigma_1 + i\sigma_2)/2$, $S_- = (\sigma_1 - i\sigma_2)/2$, and similarly for $SU(2)$ we take $T_a = \frac{i}{2}\sigma_a$. The Killing form is related to the trace in this representation as $K_{ab} = f_{ac}{}^d f_{bd}{}^c = 4\mathrm{Tr}(S_a S_b)$, and similarly for T_a. We will use the bilinear form induced by the trace, rather than the Killing form, to raise and lower algebra indices.

The equations of motion for the action $S_{\rm WZW}[g]$ imply chirality for the current $J = \partial g g^{-1}$, and equivalently antichirality for the current $\bar{J} = -g^{-1}\bar{\partial} g$, i.e. $\bar{\partial} J = 0$, $\partial\bar{J} = 0$. We decompose the currents as $J = J_a S^a$ for AdS_3 and $J = J_a T^a$ for S^3, and similarly for $\bar{J}$, where $S^0 = 2S_0$, $S^\pm = S_\mp$ and $T^a = -2T_a$. Thanks to these definitions the component J_a of the chiral current corresponds to the action of the generator S_a (or T_a in the case of the sphere) from the left, while the component $\bar{J}_a$ of the antichiral current corresponds to the action of the same generator from the right. The same holds for the corresponding Killing vectors k_a and $\bar{k}_a$. In particular we have $k_a^m \partial_m g_{\rm a} = +S_a g_{\rm a}$, $\bar{k}_a^m \partial_m g_{\rm a} = -g_{\rm a} S_a$ for AdS and $k_1^m \partial_m g_{\rm s} = -T_1 g_{\rm s}$, $\bar{k}_2^m \partial_m g_{\rm s} = +g_{\rm s} T_2$ for the sphere[10]. In our parameterisation the components of the $SL(2)$ currents read

$$J_0 = \frac{z\partial z - x^+\partial x^-}{z^2}, \qquad J_- = \frac{x^+(x^+\partial x^- - 2z\partial z)}{z^2} + \partial x^+, \qquad J_+ = -\frac{\partial x^-}{z^2},$$
$$\bar{J}_0 = -\frac{z\bar{\partial} z - x^-\bar{\partial} x^+}{z^2}, \qquad \bar{J}_+ = \frac{x^-(x^-\bar{\partial} x^+ - 2z\bar{\partial} z)}{z^2} + \bar{\partial} x^-, \qquad \bar{J}_- = -\frac{\bar{\partial} x^+}{z^2}, \tag{2.12}$$

[9] The relative minus sign is needed to get the correct sign in front of the S^3 metric. In the supersymmetric case it is naturally accounted for by the supertrace.
[10] The relative minus sign between AdS_3 and S^3 is again related to the fact that we are not using the supertrace in the action and in order to define the components of the currents.

while for the $SU(2)$ currents we have

$$J_1 = \frac{1}{2}(-\partial\phi_1 - s_3\partial\phi_2), \qquad J_2 = \frac{1}{2}(-c_1c_3\partial\phi_2 - s_1\partial\phi_3), \qquad J_3 = \frac{1}{2}(-c_1\partial\phi_3 + c_3s_1\partial\phi_2),$$
$$\bar{J}_2 = \frac{1}{2}(+\bar{\partial}\phi_2 + s_3\bar{\partial}\phi_1), \qquad \bar{J}_1 = \frac{1}{2}(+c_2c_3\bar{\partial}\phi_1 + s_2\bar{\partial}\phi_3), \qquad \bar{J}_3 = \frac{1}{2}(+c_2\bar{\partial}\phi_3 - c_3s_2\bar{\partial}\phi_1), \tag{2.13}$$

where we use the shorthand notation $s_i = \sin\phi_i$, $c_i = \cos\phi_i$. The (anti)chiral currents appear also when computing the Noether currents from the action $S_{\mathrm{WZW}} = S_1 + k\Gamma$. In fact, invariance of the WZW action under left transformations $g \to (1 + \varepsilon_L + \ldots)g$ implies the conservation of the Noether current $\mathscr{J}^i = \frac{1}{4}(\eta^{ij} - \epsilon^{ij})\partial_j g g^{-1}$ which is related to the chiral current as

$$\mathscr{J} = (\mathscr{J}_+, \mathscr{J}_-) = (J, 0). \tag{2.14}$$

Similarly, from the right transformations $g \to g(1 + \varepsilon_R + \ldots)$ one finds the Noether current $\bar{\mathscr{J}}^i = -\frac{1}{4}(\eta^{ij} + \epsilon^{ij})g^{-1}\partial_j g$ related to the antichiral current as

$$\bar{\mathscr{J}} = (\bar{\mathscr{J}}_+, \bar{\mathscr{J}}_-) = (0, \bar{J}). \tag{2.15}$$

Conservation of the Noether current $\partial_i \mathscr{J}^i = 0$ (respectively $\partial_i \bar{\mathscr{J}}^i = 0$) implies chirality of J (respectively antichirality of $\bar{J}$). As we have already pointed out, in general these Noether currents are not the same as those of the sigma model description, which we denoted by $\mathcal{J}$.

2.3. Marginal deformations

In [8] Chaudhuri and Schwartz considered two-dimensional CFTs with $J_a, \bar{J}_a$ satisfying current algebra relations[11]

$$J_a(\sigma)J^b(\sigma') \sim \frac{i\,\delta_a^b}{2k(\sigma - \sigma')^2} + \frac{i f_{ac}{}^b J^c}{2k(\sigma - \sigma')},$$
$$\bar{J}_a(\bar{\sigma})\bar{J}^b(\bar{\sigma}') \sim \frac{i\,\delta_a^b}{2k(\bar{\sigma} - \bar{\sigma}')^2} + \frac{i f_{ac}{}^b \bar{J}^c}{2k(\bar{\sigma} - \bar{\sigma}')}, \tag{2.16}$$

where we use $\sim$ since we are omitting regular terms and $f_{ab}{}^c$ are structure constants of a Lie algebra $\mathfrak{f}$. The authors of [8] were interested in exploring the space of marginal deformations induced by dimension $(1, 1)$ operators of the type

$$g\mathcal{O}(\sigma, \bar{\sigma}) = g c^{ab} J_a(\sigma)\bar{J}_b(\bar{\sigma}), \tag{2.17}$$

where c^{ab} are constant coefficients. The above operator is 'integrably' or exactly marginal (i.e. can be completed to all orders in conformal perturbation theory in g) if it has no anomalous dimension, and they found that a necessary condition for this to hold is that

$$C^{abc}C^{def}K_{ad}K_{be}K_{cf} + \bar{C}^{abc}\bar{C}^{def}K_{ad}K_{be}K_{cf} = 0, \tag{2.18}$$

where K_{ab} is the Killing form and we have defined

$$C^{abc} \equiv c^{da}c^{eb}f_{de}{}^c, \qquad \bar{C}^{abc} \equiv c^{ad}c^{be}f_{de}{}^c. \tag{2.19}$$

We will call (2.18) the *weak* Chaudhuri–Schwartz (CS) condition. Chaudhuri–Schwartz [8] considered only the case of compact algebras, meaning that the Killing form K_{ab} is negative

[11] Since we are not normalising the currents with an explicit k and we raise/lower indices with the bilinear form induced by the trace, as opposed to the Killing form, certain factors differ from [8].

definite and can be taken to be diagonal. In this case (2.18) becomes a sum of squares of C^{abc} and $\bar{C}^{abc}$, and it holds if and only if

$$C^{abc} = 0, \qquad \text{and} \qquad \bar{C}^{abc} = 0. \tag{2.20}$$

We will call (2.20) the *strong* CS condition, because it is a stronger constraint in the case of non-compact algebras. In [8] it was also shown that the strong condition is equivalent to being able to rewrite

$$\mathcal{O}(\sigma,\bar{\sigma}) = \tilde{c}^{ab}\tilde{J}_a(\sigma)\tilde{\bar{J}}_b(\bar{\sigma}), \tag{2.21}$$

where $\tilde{J}_a(\sigma), \tilde{\bar{J}}_b(\bar{\sigma})$ are linear combinations of the original $J_a(\sigma)$, $\bar{J}_b(\bar{\sigma})$ such that

$$\tilde{J}_a(\sigma)\tilde{J}^b(\sigma') \sim \frac{i\,\delta_a^b}{2k(\sigma-\sigma')^2}, \qquad \tilde{\bar{J}}_a(\bar{\sigma})\tilde{\bar{J}}^b(\bar{\sigma}') \sim \frac{i\,\delta_a^b}{2k(\bar{\sigma}-\bar{\sigma}')^2}, \tag{2.22}$$

i.e. the structure constants for this particular set of currents vanish. The absence of a simple pole in these OPEs means that they are the same as similar ones for free bosons, which in turn means that the β-function for the deformation parameter g vanishes and the deformation is exactly marginal. In other words, deformations corresponding to Abelian subalgebras, which is the only possibility in the compact case, are exactly marginal. When the Lie algebra $\mathfrak{f}$ is non-compact it is possible to find deformations that satisfy only the weak CS condition, as we will see. *A priori* they are not guaranteed to be marginal beyond lowest order, and indeed we will find both examples which are and those which are not.

In fact a *sufficient* condition on c^{ab} such that the weak CS (2.18) holds is that the coefficients c^{ab} identify two solvable subalgebras of $\mathfrak{f}$ (one corresponding to J_a and one to $\bar{J}_b$). This follows directly from Cartan's criterion for a solvable Lie algebra $\mathfrak{h}$

$$\mathfrak{h}\ \text{solvable} \iff \mathrm{Tr}(ab) = 0, \qquad \forall a \in \mathfrak{h}, b \in [\mathfrak{h},\mathfrak{h}]. \tag{2.23}$$

If we are in such a situation then the two terms in (2.18) separately vanish because

$$C^{abc}C^{def}K_{cf} = 0, \qquad \bar{C}^{abc}\bar{C}^{def}K_{cf} = 0. \tag{2.24}$$

In the case of the $SL(2,\mathbb{R}) \times SU(2)$ WZW model, we may for example identify the two solvable subalgebras generated by $\{S_0, S_+, T_1\}$ and $\{\bar{S}_0, \bar{S}_-, \bar{T}_2\}$. Then if we call $Y_a \equiv \{J_0, J_+, J_1\}$ and $\bar{Y}_a = \{\bar{J}_0, \bar{J}_-, \bar{J}_2\}$ the list of the corresponding (anti)chiral currents, an operator

$$\mathcal{O}(\sigma,\bar{\sigma}) = c^{ab}Y_a(\sigma)\bar{Y}_b(\bar{\sigma}), \tag{2.25}$$

will be marginal to lowest order for *generic* coefficients c^{ab}. Notice that generically c^{ab} will not solve the strong CS condition (2.20).

All the solutions to the weak CS condition that we will generate from the CYBE on $\mathfrak{g} = \mathfrak{f}_{\mathrm{L}} \oplus \mathfrak{f}_{\mathrm{R}} = \mathfrak{sl}(2,\mathbb{R})_{\mathrm{L}} \oplus \mathfrak{su}(2)_{\mathrm{L}} \oplus \mathfrak{sl}(2,\mathbb{R})_{\mathrm{R}} \oplus \mathfrak{su}(2)_{\mathrm{R}}$ will be of this type. Indeed to solve the CYBE it is enough to look at the subalgebra generated by $\{S_0, S_+, T_1, \bar{S}_0, \bar{S}_-, \bar{T}_2\}$. When they come from the YB construction, the coefficients c^{ab} will obviously not be generic, and we will relate them to certain components of the R-matrix, see the discussion at the end of section 3.2. The YB construction has the advantage of giving a way to go beyond the infinitesimal deformation driven by $\mathcal{O}(\sigma,\bar{\sigma})$, and gives a sigma-model action that is exact in the deformation parameter.

As we have argued, we expect the CYBE to give solutions to the weak CS condition in more generic situations. In appendix D we discuss a solution of the CYBE that provides coefficients c^{ab} that solve the weak CS condition without identifying solvable subalgebras.

3. Yang–Baxter and current–current deformations

3.1. Yang–Baxter deformations

We now review the transformation rules for the target space fields for YB deformations derived in [36]. Given an initial sigma model with metric and Kalb–Ramond fields G_{mn}, B_{mn}, the background of the YB deformed model is given by

$$\tilde{G} - \tilde{B} = (G - B)[1 + \eta\Theta(G - B)]^{-1}, \tag{3.1}$$

where for simplicity we are suppressing all spacetime indices. Here $\Theta^{mn} = k_A^m R^{AB} k_B^n$ is a tensor constructed out of the Killing vectors k_A^m and of R^{AB}, which is a solution to the CYBE on the Lie algebra $\mathfrak{g}$

$$R^{D[A} R^{|E|B} f_{DE}^{\ \ C]} = 0. \tag{3.2}$$

In our case $\mathfrak{g} = \mathfrak{f}_{\mathrm{L}} \oplus \mathfrak{f}_{\mathrm{R}}$ is the sum of a left and a right copy of $\mathfrak{f} = \mathfrak{sl}(2) \oplus \mathfrak{su}(2)$, and G, B were given in section 2.1. The derivation of [36] assumes that the B-field is invariant under the isometries used in the deformation, i.e. the ones appearing in Θ. This is ensured by picking the form of B in section 2.1 and using only the isometries generated by (2.7), which is enough to generate any Yang–Baxter deformation (see appendix C).

The deformation produces also a shift of the dilaton calculated from the determinant[12]

$$\mathrm{e}^{-2\tilde{\Phi}} = \mathrm{e}^{-2\Phi} \det[1 + \eta\Theta(G - B)], \tag{3.3}$$

where Φ is the dilaton of the original background (in our case $\Phi = 0$). In general YB backgrounds are solutions to the equations of generalised supergravity [40, 41], so that in addition to the usual fields one may have also a vector K computed as[13]

$$K^m = -\frac{\eta}{2} R^{AB} f_{AB}^{\ \ C} k_C^m, \tag{3.4}$$

which is a Killing vector of the YB background $\nabla_{(m} K_{n)} = 0$. For such generalised supergravity solutions the role of (the derivative of) the dilaton is replaced by the vector[14]

$$\mathrm{X}_m = \partial_m \tilde{\Phi} - B_{mn} K^n. \tag{3.5}$$

When K^m vanishes one goes back to a standard supergravity solution. From (3.4) the relation to the unimodularity condition of [43] is manifest. There exist also so-called 'trivial solutions' of generalised supergravity [42], i.e. when K does not vanish but it decouples from the equations. A trivial solution is therefore both a solution of the generalised and the standard supergravity equations. Later we will encounter examples of this type.

Let us comment on the fact that the YB transformations constructed in [36] were derived by assuming a group of left isometries for the sigma model. This is necessary in order to apply the NATD construction and twist the model with the corresponding Killing vectors k_A. The

[12] In the supersymmetric case the determinant is replaced by the superdeterminant.

[13] Equation (3.4) may be obtained from the formula derived in [36] (i.e. $K^m = \eta\Theta^{mn} n_n = \eta k_A^m R^{AB} n_B$ with $n_A = f_{AB}^{\ \ B}$) after using the identity $R^{AB} f_{AB}^{\ \ C} = -2 f_{AB}^{\ \ A} R^{BC}$, which is a consequence of the CYBE. It is also easy to check that (3.4) agrees with $K^m = \eta \nabla_n^{(0)} \Theta^{mn}$ proposed in [30], where $\nabla_n^{(0)}$ is the covariant derivative of the original undeformed background. Indeed, using first the Killing equation for k_A^m and then the anti-symmetry of R, we have $\nabla_m^{(0)} \Theta^{mn} = k_A^m R^{AB} \nabla_m^{(0)} k_B^n = R^{AB} k_A^m \partial_m k_B^n$. Knowing that Killing vectors satisfy $[k_A^m \partial_m, k_B^n \partial_n] = -f_{AB}^{\ \ C} k_C^p \partial_p$ we obtain again (3.4).

[14] This expression applies in a gauge where B is invariant under the isometry generated by K, $\mathcal{L}_K B = 0$. Here we stick with the original notation of [41]. In [36] and [42] X_m was used instead, but there is a risk of confusing it with $X_m = \mathrm{X}_m + K_m$ of [41].

isometries that we will exploit here to deform $AdS_3 \times S^3$, corresponding to the generators in (2.7), belong both to the left and to the right copy of the symmetry group of the WZW model. The reason why we can apply the above rules of YB transformations is that the corresponding sigma model may be constructed as a coset on $SO(2,2)/SO(1,2) \times SO(4)/SO(3)$ with a WZ term. For example, focusing on AdS_3, we may relate the generators of $\mathfrak{sl}(2,\mathbb{R})$ to those of the conformal algebra as

$$\begin{aligned} S_0 &= +\frac{1}{2}(D - J_{01}), & S_+ &= p_+, & S_- &= k_-, \\ \bar{S}_0 &= -\frac{1}{2}(D + J_{01}), & \bar{S}_+ &= k_+, & \bar{S}_- &= p_-, \end{aligned} \qquad (3.6)$$

where e.g. $p_\pm = \frac{1}{2}(p_0 \pm p_1)$. Then we obtain the wanted sigma model action from $S = \frac{k}{2\pi}\int d^2\sigma \mathrm{Tr}[g^{-1}\partial g(P+b)g^{-1}\bar{\partial}g]$ where $g = \exp(x^+ p_+ + x^- p_-)\exp(D\log z)$, P projects on the generators of the coset $p_i - k_i$ and D, and finally $b(p_\pm - k_\pm) = \pm(p_\pm - k_\pm)$ produces the B-field. In this formulation the isometries that we want to exploit, generated by $S_0, S_+, \bar{S}_0, \bar{S}_-$, act from the left as $g \to hg$ and leave also the B-field invariant.

For later convenience, let us say at this point that it is easy to check that when an R-matrix is given by the sum of two R-matrices, the corresponding background can be understood as the composition of two successive YB transformations. This is easily seen using the following identity valid when $\Theta = \Theta_1 + \Theta_2$

$$[1+\eta\Theta_1(G-B)]^{-1}\left[1+\eta\Theta_2(G-B)[1+\eta\Theta_1(G-B)]^{-1}\right]^{-1} = [1+\eta\Theta(G-B)]^{-1}, \qquad (3.7)$$

which holds without assuming any property[15] for Θ_i, neither antisymmetry nor CYBE. Thanks to this formula it is straightforward to argue that the background metric and B-field of a YB deformation generated by $\Theta = \Theta_1 + \Theta_2$ are equivalent to those coming from the composition of two successive deformations, e.g. first one generated by Θ_1 and then one generated by Θ_2 (or vice versa). The same holds for the transformation rule of the dilaton, and of the vector K, which is linear in Θ (or equivalently R). Obviously, the interpretation as YB deformations in the intermediate steps will be possible only if Θ_1, Θ_2 separately solve the CYBE, and if the isometries needed to implement the second deformation are not broken by the first one. In this case we will say that Θ is 'decomposable'. Apart from these subtleties, it will often prove useful to interpret a deformation generated by $\Theta = \Theta_1 + \Theta_2$ as a composition of two transformations.

Later we will encounter examples in which Θ_1 generates the undeformed background *up to* a (η-dependent) field redefinition. In this case one can say that $\Theta = \Theta_1 + \Theta_2$ is equivalent to the YB deformation generated by Θ_2 alone *only if* the field redefinition $x^m \to x^m(x') = x'^m + \eta f^m(x')$ needed to trivialise Θ_1 is compatible with Θ_2. It is easy to convince oneself that the necessary compatibility condition is

$$A^{-1}\Theta_2(x'^m + \eta f^m(x'))A^{-T} = \Theta_2(x'^m), \qquad (3.8)$$

where $A^m{}_n = \frac{\partial x^m}{\partial x'^n}$, and we are writing the explicit dependence of Θ_2 on the coordinates.

3.2. Relation to marginal current–current deformations

Before discussing YB deformations of $AdS_3 \times S^3$, let us make a simple observation: at leading order in the deformation parameter, the YB deformation is of the form $\mathcal{J}\mathcal{J}$, where $\mathcal{J}$ are

[15] Obviously we need to assume invertibility of the above operators.

the Noether currents of the sigma model. This is straightforwardly checked by expanding the sigma model action $S=\frac{T}{2}\int \mathrm{d}^2\sigma\ \partial x^m(\tilde{G}_{mn}-\tilde{B}_{mn})\bar{\partial}x^n$ to lowest order in the deformation parameter[16]

$$S=S_0-\eta\frac{T}{2}\int \mathrm{d}^2\sigma\, R^{AB}\mathcal{J}_{A+}\mathcal{J}_{B-}+\mathcal{O}(\eta^2). \tag{3.9}$$

While this is true for a generic sigma model, this observation is particularly interesting when the original sigma model is related to a WZW model. If the Noether currents $\mathcal{J}_{Ai}$ coincided with the chiral Noether currents of the WZW model $\mathscr{J}_{Ai}=\{\mathscr{J}_{ai},\bar{\mathscr{J}}_{ai}\}$, then we would automatically obtain a current–current deformation of the type $J\bar{J}$. In fact, from (2.14) and (2.15) one immediately finds that[17] $\int \mathrm{d}^2\sigma\ \epsilon^{ij}\,R^{BA}\,\mathscr{J}_{Ai}\mathscr{J}_{Bj}=4\int \mathrm{d}^2\sigma\ R^{a\bar{b}}J_a\bar{J}_{\bar{b}}$. As we have seen, though, in general $\mathcal{J}^i_A=\mathscr{J}^i_A+\epsilon^{ij}\partial_j c_A$, and the discussion is more subtle because (3.9) will contain additional terms together with the wanted $J\bar{J}$ ones. We are about to show that for YB deformations of the $\mathrm{AdS}_3\times\mathrm{S}^3$ sigma model these additional terms can be removed by proper field redefinitions. We can therefore relate YB deformations to the deformations of the type $c^{ab}J_a\bar{J}_b$ considered by CS. From this discussion it is also clear that we should identify the coefficients c^{ab} of CS with an 'off-diagonal' block of the R-matrix. More explicitly, since $R\in\mathfrak{g}\wedge\mathfrak{g}$ we are solving the CYBE on an algebra which is the sum of a left and a right copy $\mathfrak{g}=\mathfrak{f}_{\mathrm{L}}\oplus\mathfrak{f}_{\mathrm{R}}$, the R-matrix can be decomposed as

$$R=\begin{pmatrix}R_{\mathrm{LL}} & R_{\mathrm{LR}}\\ R_{\mathrm{RL}} & R_{\mathrm{RR}}\end{pmatrix} \tag{3.10}$$

with $R^T_{\mathrm{LL}}=-R_{\mathrm{LL}}, R^T_{\mathrm{RR}}=-R_{\mathrm{RR}}$ and $R^T_{\mathrm{LR}}=-R_{\mathrm{RL}}$. The relation to the coefficients of CS is therefore $c^{ab}=R^{ab}_{\mathrm{LR}}$. We can therefore generate solutions to the (weak) CS condition from solutions of the CYBE, and we will find several non-trivial examples in the following. Obviously Abelian R-matrices ($R=a\wedge b, [a,b]=0$) will give solutions of the strong CS condition. When dealing with non-compact algebras the CYBE allows also for solutions that are not of the Abelian type. Some of them will give coefficients c^{ab} that do not solve the strong CS condition. They all solve the weak CS condition as already explained at the end of section 2.3.

It would be very interesting to understand more deeply the relation between the space of solutions of the CYBE (3.2) and the weak CS condition (2.18) in the case of a generic algebra $\mathfrak{g}=\mathfrak{f}_{\mathrm{L}}\oplus\mathfrak{f}_{\mathrm{R}}$. It is interesting to notice that in order to solve the CYBE one may need also components of the diagonal blocks R_{LL} and R_{RR}, while in the weak CS condition these will not enter. In fact, given $R^{a\bar{b}}_{\mathrm{LR}}=c^{a\bar{b}}$ and taking the CYBE on mixed left/right indices (where we use an explicit bar for indices of the right copy of the algebra), one gets for example $c^{d\bar{a}}R^{eb}f_{de}{}^{c}+c^{b\bar{d}}c^{c\bar{e}}f_{\bar{d}\bar{e}}{}^{\bar{a}}+R^{dc}c^{e\bar{a}}f_{de}{}^{b}=0$. Depending on the coefficients $c^{a\bar{b}}$ one may also need non-vanishing left-left R^{ab} components in order to solve this equation[18], but the CS condition is not sensitive to them.

Let us also comment that, differently from what was claimed in [44], the strong CS condition (2.20) is *not* the CYBE, not even when one further imposes the unimodularity condition (and in fact our $\mathbf{R}_9$ in table 3 is a counter example to that claim).

[16] Recall that we are restricting to the case when the isometries used to construct the deformation leave not just the action but also the Lagrangian invariant, so that $j_{Ai}=0$. This was the assumption also in the derivation done in [36].

[17] Notice that the R-matrix may have also non-vanishing components with both indices in the left (or both in the right) copy of the algebra, but these will not contribute in the final expression, and the contributions coupling currents with the same chirality cancel out.

[18] Such an example is given by the fourth R-matrix in table 1.

3.3. Field redefinition

In order to display the current–current structure of the deformed model it is convenient to write the Lagrangian in the form

$$L = L_0 - \frac{1}{2}\eta \mathcal{J}_{A+}[(1+\eta RM)^{-1}R]^{AB}\mathcal{J}_{B-}, \tag{3.11}$$

where L_0 is the undeformed Lagrangian and $M_{AB} = k_A^m k_B^n (G-B)_{mn}$. This follows directly from the form of $\tilde{G}-\tilde{B}$ in (3.1) and the definition of the Noether currents in (2.6) upon recalling that we will pick R so that the last term in $\mathcal{J}$ does not contribute. To compare this to the discussion of current–current deformations of the WZW model we need to perform a field redefinition that replaces the Noether currents in the above expression with the chiral currents[19]. In appendix A we find such a field redefinition for a general deformation specifying to $\mathrm{AdS}_3\times\mathrm{S}^3$ for concreteness. Here we will just say that for all deformations that we consider we find that the Lagrangian can be written in the form

$$L = L_0 - \frac{1}{2}\eta \hat{J}_a[(1+\eta RM')^{-1}R]^{a\bar{a}}\hat{\bar{J}}_{\bar{a}}, \tag{3.12}$$

where M' is a shorthand for M after the field redefinition. $\hat{J}_a, \hat{\bar{J}}_{\bar{a}}$ are modifications of the chiral currents of the undeformed WZW model. Their explicit form is given in (A.25) for deformations of AdS_3, and in section 3.5 for deformations of $\mathrm{AdS}_3\times\mathrm{S}^3$.

At leading order in η the above Lagrangian becomes

$$L = L_0 - \frac{1}{2}\eta J_a R^{a\bar{a}}\bar{J}_{\bar{a}} + \mathcal{O}(\eta^2), \tag{3.13}$$

so that the comparison to the current–current deformations considered by CS is now manifest.

3.4. YB deformations of AdS_3

Let us start by looking at YB deformations that deform only AdS_3. We will start with the simplest ones which are TsT-transformations. They come from the Abelian R-matrices of $\mathfrak{sl}(2,\mathbb{R})_\mathrm{L}\oplus\mathfrak{sl}(2,\mathbb{R})_\mathrm{R}$ which (up to automorphisms) are[20,21]

$$S_+\wedge\bar{S}_-, \qquad S_0\wedge\bar{S}_0, \qquad S_+\wedge\bar{S}_0. \tag{3.14}$$

For the first one we obtain, from (3.1) and (3.3), the supergravity background of a deformation of $\mathrm{AdS}_3\times\mathrm{S}^3\times T^4$

$$\begin{aligned}
\mathrm{d}s^2 &= -\frac{\mathrm{d}x^+\mathrm{d}x^-}{z^2-\eta} + \frac{\mathrm{d}z^2}{z^2} + \mathrm{d}s^2_{\mathrm{S}^3} + \mathrm{d}s^2_{T^4},\\
B &= \frac{\mathrm{d}x^+\wedge\mathrm{d}x^-}{2(z^2-\eta)} - \frac{1}{4}\sin\phi_3\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2, \qquad \mathrm{e}^{-2\tilde{\Phi}} = 1-\frac{\eta}{z^2}.
\end{aligned} \tag{3.15}$$

In this case the isometries involved in the deformation procedure correspond to Noether currents that agree with the (anti)chiral currents, see (2.8). We therefore automatically get that to

[19] It is actually enough to look at the lowest order in η if we are considering infinitesimal deformations. The action written in terms of the Noether currents as in (3.11) was given also in [44] but the rewriting in terms of the chiral currents is missing there.

[20] From now on we will use the components R^{AB} to construct $R = R^{AB}\mathbf{T}_A\wedge\mathbf{T}_B\in\mathfrak{g}\wedge\mathfrak{g}$.

[21] We could consider also $R = S_0\wedge\bar{S}_-$, but it is related to $R = S_+\wedge\bar{S}_0$ if we also exchange the left and right copy of the algebra.

Table 1. Non-Abelian R-matrices of $\mathfrak{sl}(2)_\mathrm{L} \oplus \mathfrak{sl}(2)_\mathrm{R}$ up to $SL(2,\mathbb{R})_\mathrm{L} \times SL(2,\mathbb{R})_\mathrm{R}$ inner automorphisms and swaps of $\mathrm{L} \leftrightarrow \mathrm{R}$. For convenience we also write the marginal operators that they give rise to, and whether they satisfy the strong CS condition.

$R = R^{AB}\mathbf{T}_A \wedge \mathbf{T}_B$	Deformation?	$c^{ab}J_a\bar{J}_b$	Strong CS?
$R_1 = S_0 \wedge S_+$	Trivial deformation	0	Yes
$R_2 = (S_0 \mp \bar{S}_-) \wedge S_+$	TsT	$\pm J_+\bar{J}_-$	Yes
$R_3 = (S_0 - a\bar{S}_0) \wedge S_+$	TsT	$aJ_+\bar{J}_0$	Yes
$R_4 = (S_0 - \bar{S}_0) \wedge (S_+ \pm \bar{S}_-)$	Not SUGRA	$J_+\bar{J}_0 \pm J_0\bar{J}_-$	No
$R_5 = S_0 \wedge S_+ \pm \bar{S}_0 \wedge \bar{S}_- + \lambda S_+ \wedge \bar{S}_-$	TsT	$\lambda J_+\bar{J}_-$	Yes

leading order the deformation of the Lagrangian is given by the marginal operator $\eta J_+\bar{J}_-$. In [45] the above background was argued to be the dual of the 'single-trace' $T\bar{T}$ deformation of the symmetric product orbifold CFT_2. This is also is accordance with the fact this particular YB deformation is just a TsT transformation involving the two boundary coordinates[22]. At finite order in the deformation parameter the Lagrangian is given by (3.12), (A.25) and (A.1)

$$L = L_0 - \frac{1}{2}\frac{\eta}{1 - \frac{\eta}{z^2}} J_+\bar{J}_-. \tag{3.16}$$

The derivative of the action with respect to the deformation parameter is given by

$$\frac{\mathrm{d}S}{\mathrm{d}\eta} = -\frac{T}{2}\int \mathrm{d}^2\sigma\, J^\eta_+ \bar{J}^\eta_-, \tag{3.17}$$

where $J^\eta_+ = (1-\eta z^{-2})^{-1}J_+$ and $\bar{J}^\eta_- = (1-\eta z^{-2})^{-1}\bar{J}_-$ are (anti)chiral currents of the *deformed* model. This background was analysed in [46]. Let us also note that in this case the deformation parameter can be absorbed by a rescaling of the coordinates. There are therefore only three cases: $\eta > 0$, $\eta = 0$ and $\eta < 0$. The first of these is not globally well behaved since the dilaton becomes imaginary when crossing $z = \sqrt{\eta}$. For $\eta < 0$ the solution interpolates between two CFTs: the $SL(2,\mathbb{R})$ WZW model and a linear dilaton background (plus two decoupled bosons). It would be interesting to study further the implications of the existence of this interpolating solution but we will not do so here. See also [45] for a discussion on the different interpretations depending on the sign of the deformation parameter.

For the remaining two Abelian R-matrices we obtain by a similar calculation the Lagrangians[23]

$$\begin{aligned} L &= L_0 - \frac{1}{2}\eta\frac{(4+\eta)z^2}{4z^2 + \eta(4+\eta)x^+x^-}J_0\bar{J}_0, \\ L &= L_0 - \frac{1}{2}\eta\frac{z^2}{z^2+\eta x^-}J_+\bar{J}_0. \end{aligned} \tag{3.18}$$

For all the Abelian examples one finds, as already mentioned, that

$$\frac{\mathrm{d}S}{\mathrm{d}\eta} = -\frac{T}{2}\int \mathrm{d}^2\sigma\, R^{a\bar{a}} J^\eta_a \bar{J}^\eta_{\bar{a}}, \tag{3.19}$$

[22] We can have a TsT interpretation because we can implement the sequence T-duality, shift, T-duality in terms of the coordinates x^0, x^1, instead of the null coordinates $x^\pm$.
[23] The $J_0\bar{J}_0$-deformation was considered also in [47].

where $J^\eta_a, \bar{J}^\eta_{\bar{a}}$ are (anti)chiral currents of the deformed model. For the last two Abelian examples they are

$$R = S_0 \wedge \bar{S}_0: \qquad J^\eta_0 = \alpha J_0, \qquad \bar{J}^\eta_0 = \alpha \bar{J}_0, \qquad \alpha \equiv \frac{4z^2\sqrt{1+\frac{\eta}{2}}}{4z^2 + \eta(4+\eta)x^+x^-},$$
$$R = S_+ \wedge \bar{S}_0: \qquad J^\eta_+ = \alpha J_+, \qquad \bar{J}^\eta_0 = \alpha \bar{J}_0, \qquad \alpha \equiv \frac{z^2}{z^2+\eta x^-}. \tag{3.20}$$

Following [46], the above result is another way to see that the YB model provides a deformation that is marginal exactly in the deformation parameter.

In order to find deformations that at least potentially are not TsT, one should look at the class of non-Abelian R-matrices. The full list of non-Abelian R-matrices of $\mathfrak{sl}(2,\mathbb{R})_{\rm L} \oplus \mathfrak{sl}(2,\mathbb{R})_{\rm R}$ (up to $SL(2,\mathbb{R})_{\rm L} \times SL(2,\mathbb{R})_{\rm R}$ automorphisms) is given in table 1. They are special cases of the R-matrices for $\mathfrak{sl}(2,\mathbb{R})_{\rm L} \oplus \mathfrak{sl}(2,\mathbb{R})_{\rm R} \oplus \mathfrak{su}(2)_{\rm L} \oplus \mathfrak{su}(2)_{\rm R}$ classified in appendix C. Analysing the first example $R_1 = S_0 \wedge S_+$ one finds that, after the field redefinition $x^+ \to x^+ - \frac{\eta}{2}\log z$, not only the leading order in η vanishes in the action, but also all the higher orders. This is obvious from equation (3.12) and the fact that R with an anti-chiral index vanishes. In other words the deformation is trivial, since its effect is to give the undeformed AdS_3 background in new (η-dependent) coordinates. To be more precise, from R_1 we get back undeformed AdS_3 *up to* a non-vanishing $K = -\eta\partial_{x^+}$, from (3.4). This is of course a 'trivial solution' of generalised supergravity (i.e. one that solves also standard supergravity upon dropping K, notice that K is null)[24].

The above result for $R_1 = S_0 \wedge S_+$ turns out to be useful to analyse some of the following examples in the table. It is easy to see that R_2 and R_3 in table 1 are of the form $R = S_0 \wedge S_+ + R'$ and that in both cases R' is compatible with the field redefinition that trivialises the effect of the piece $S_0 \wedge S_+$, see (3.8). Therefore these two R-matrices are equivalent to two of the TsT transformations already discussed[25], generated by $S_+ \wedge \bar{S}_-$ and $S_+ \wedge \bar{S}_0$. Alternatively it is easy to see this directly from the Lagrangian in (3.12) and (A.25).

The last two R-matrices in table 1, instead, give backgrounds that are not of this type. From (3.4) we find that they have $K = -\eta(\partial_{x^+} \pm \partial_{x^-})$ and $K = -\eta(\partial_{x^+} \mp \partial_{x^-})$ respectively, neither of which is null. The only way they can give solutions of standard supergravity is then, as shown in appendix E, if $\mathrm{X} = \mathrm{d}\phi + \tilde{K}$ with ϕ the dilaton and $\tilde{K}$ an independent Killing vector field. We can extract $\tilde{K}$ from the equation $dK + i_{\tilde{K}}H = 0$ in (E.1) and for R_4 one finds $\tilde{K} = \eta(\partial_{x^+} \mp \partial_{x^-}) \pm \eta^2 z^{-1}\partial_z + \mathcal{O}(\eta^3)$ which, at lowest order, differs from K only in the sign of the ∂_{x^+}-term. However from the lowest order correction to the action $J_+\bar{J}_0 \pm J_0\bar{J}_-$ we read off the deformed metric

$$z^{-2}(-\mathrm{d}x^-\mathrm{d}x^+ + \mathrm{d}z\mathrm{d}z) - \eta z^{-4}(z\mathrm{d}z(\mathrm{d}x^- \mp \mathrm{d}x^+) + (\pm x^+ - x^-)\mathrm{d}x^+\mathrm{d}x^-) + \mathcal{O}(\eta^2), \tag{3.21}$$

which is only invariant under $\delta x^+ = \eta\epsilon$, $\delta x^- = \pm\eta\epsilon$ which is generated by K, and not under $\delta x^+ = \eta\epsilon$, $\delta x^- = \mp\eta\epsilon$, $\delta z = \pm\eta^2\epsilon z^{-1}$ which is generated by $\tilde{K}$. Therefore $\tilde{K}$ is not Killing and the R_4-background is not a solution to standard supergravity. Note that it only fails to be a solution at order η^2 which is consistent with the fact that the leading-order deformation satisfies the weak CS condition. We will not consider this background further here since our interest is mainly in string theory applications. For R_5, instead, one can show that $\tilde{K}$ is a Killing vector of the deformed metric and it satisfies (E.1), meaning that we get a 'trivial solution' of

[24] Obviously a similar discussion holds also for $R = \bar{S}_0 \wedge \bar{S}_-$.

[25] Also in this case the equivalence holds up to a non-vanishing $K = -\eta\partial_{x^+}$ that is null and decouples from the equations of generalised supergravity.

Table 2. Rank-2 non-Abelian R-matrices of $\mathfrak{sl}(2)_{\rm L}\oplus\mathfrak{sl}(2)_{\rm R}\oplus\mathfrak{su}(2)_{\rm L}\oplus\mathfrak{su}(2)_{\rm R}$ up to $SL(2,\mathbb{R})_{\rm L}\times SL(2,\mathbb{R})_{\rm R}\times SU(2)_{\rm L}\times SU(2)_{\rm R}$ inner automorphisms and swaps of $\rm L\leftrightarrow R$. With T we denote a generic linear combination of $T_1,\bar{T}_2$.

$R=R^{AB}\mathbf{T}_A\wedge\mathbf{T}_B$	Deformation?
$\mathbf{R}_1=(S_0-\bar{S}_0+T)\wedge(S_+\pm\bar{S}_-)$	$R_4+\text{TsT}\Longrightarrow$ not SUGRA
$\mathbf{R}_2=(S_0-a\bar{S}_0+T)\wedge S_+$	$R_3+\text{TsT}\Longrightarrow\text{TsT}$
$\mathbf{R}_3=(S_0\mp\bar{S}_-+T)\wedge S_+$	$R_2+\text{TsT}\Longrightarrow\text{TsT}$

R	$c^{ab}J_a\bar{J}_b$	Strong CS?
$\mathbf{R}_1$	$\pm(J_0+aJ_1)\bar{J}_-+J_+(\bar{J}_0+b\bar{J}_2)$	No
$\mathbf{R}_2$	$J_+(a\bar{J}_0+b\bar{J}_2)$	Yes
$\mathbf{R}_3$	$J_+(\pm\bar{J}_-+b\bar{J}_2)$	Yes

generalised supergravity for generic λ. Actually, using (3.12) and (A.25), for R_5 we find the Lagrangian[26]

$$L=L_0+\frac{\eta z^2(\eta-4\lambda)}{2\left(\eta(\eta-4\lambda)+4z^2\right)}J_+\bar{J}_-,\tag{3.22}$$

which is exactly that of the TsT example in (3.16) with $\eta\to\eta\lambda-\eta^2/4$. This background therefore provides an example of the more general kind of trivial solution of the generalized supergravity equations discussed in appendix E for which $K^2\neq 0$.

This example also shows how the identification of the deformation parameters can be non-trivial. In fact, although at leading order the marginal deformation is given only by $\eta\lambda J_+\bar{J}_-$, the deformation exact in η shows that the deformation parameter of the TsT is instead $\eta\lambda-\eta^2/4$, and in particular it does not vanish even when $\lambda=0$.

It is interesting to look at the marginal operators of the type $c^{ab}J_a\bar{J}_b$ that are generated by each R-matrix. We write them for convenience in table 1. While they all solve the weak CS condition, they all solve also the strong CS condition (which guarantees that they are exactly marginal) except for the fourth one (which we have seen fails to be marginal beyond lowest order).

3.5. YB deformations of $\mathrm{AdS}_3\times\mathrm{S}^3$

Looking at deformations of $\mathrm{AdS}_3\times\mathrm{S}^3$ gives a richer set of possibilities. In this case we want to solve the CYBE on the algebra $\mathfrak{sl}(2)_{\rm L}\oplus\mathfrak{sl}(2)_{\rm R}\oplus\mathfrak{su}(2)_{\rm L}\oplus\mathfrak{su}(2)_{\rm R}$.

The simplest example to start with is $R=S_+\wedge\bar{T}_2$. This is an Abelian R-matrix and therefore corresponds to a TsT mixing AdS_3 and S^3. In [48, 49] it was argued that this background is dual to the single-trace $T\bar{J}$ deformation of the CFT_2. From the YB procedure one explicitly finds

$$\begin{aligned}\mathrm{d}s^2&=\mathrm{d}s^2_{\mathrm{AdS}_3}+\mathrm{d}s^2_{\mathrm{S}^3}+\frac{\eta}{4z^2}\mathrm{d}x^-(\mathrm{d}\phi_2+2\sin\phi_3\mathrm{d}\phi_1),\\ B&=B_0+\frac{\eta\mathrm{d}x^-\wedge(\mathrm{d}\phi_2+2\sin\phi_3\mathrm{d}\phi_1)}{8z^2},\qquad \mathrm{e}^{-2\Phi}=1.\end{aligned}\tag{3.23}$$

[26] For concreteness we take $R_5=S_0\wedge S_++\bar{S}_0\wedge\bar{S}_-+\lambda S_+\wedge\bar{S}_-$, but similar results apply also for $R_5=S_0\wedge S_+-\bar{S}_0\wedge\bar{S}_-+\lambda S_+\wedge\bar{S}_-$.

From (2.9) we see that the Noether current $\bar{\mathcal{J}}_2$ differs from the antichiral current $\bar{J}_2$, and therefore field redefinitions are needed in order to put the action in the form that makes the chirality structure manifest. After the field redefinition $x^+ \to x^+ - \frac{\eta}{4}\phi_2$, or equivalently by looking directly at (A.22), (A.16) and (A.23) the Lagrangian becomes

$$L = L_0 - \frac{1}{2}\eta J_+ \bar{J}_2. \tag{3.24}$$

This deformation is special since the leading linear order is exact. Obviously $\mathrm{d}S/\mathrm{d}\eta = -\frac{T}{2}\int J_+\bar{J}_2$.

We will now focus on non-Abelian R-matrices. These are classified for $\mathfrak{sl}(2)_\mathrm{L} \oplus \mathfrak{sl}(2)_\mathrm{R} \oplus \mathfrak{su}(2)_\mathrm{L} \oplus \mathfrak{su}(2)_\mathrm{R}$ in appendix C. Since the CYBE on the two copies of $\mathfrak{su}(2)$ implies that the subset of generators from this part of the algebra should be Abelian, our classification is useful also to study deformations of the full $\mathrm{AdS}_3 \times \mathrm{S}^3 \times T^4$ or the more generic $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$. Every time an $\mathfrak{su}(2)$ generator appears, it may as well be replaced by another compact generator, as long as all compact generators involved form an Abelian subalgebra. For concreteness we will only look at deformations of $\mathrm{AdS}_3 \times \mathrm{S}^3$. The rank-2 non-Abelian R-matrices are collected in table 2. Since they are special cases of the rank-4 R-matrices, listed in table 3, we will not consider them separately.

To compute the Killing vector K that appears in the generalized supergravity equations we note that equation (3.4) implies that only the non-Abelian generators matter and we can set $T = T' = T'' = 0$ (where these are generic linear combinations of $T_1, \bar{T}_2$). We find therefore the same answer as for the $SL(2,\mathbb{R})$ case in the previous section and comparing to that analysis we see that only $\mathbf{R}_4$ (and therefore $\mathbf{R}_1$) does not give a supergravity solution. Since we are interested in string theory applications we will focus on the analysis of $\mathbf{R}_5$ through $\mathbf{R}_{10}$. Recall that in this way we automatically consider also the rank-2 R-matrices $\mathbf{R}_2$ and $\mathbf{R}_3$. From (A.22), (A.16) and (A.23) it follows that (after the field redefinitions) the Lagrangian takes the form

$$L = L_0 - \frac{1}{2}\eta \hat{J}_a[(1+\eta RM')^{-1}R]^{a\bar{a}}\hat{\bar{J}}_{\bar{a}}, \tag{3.25}$$

where for $\mathbf{R}_5$–$\mathbf{R}_8$

$$\hat{J}_a = J_a + \eta\delta_a^0 y_i Y_A^i R^{A+} J_+, \qquad \hat{\bar{J}}_{\bar{a}} = \bar{J}_{\bar{a}} - \eta\delta_{\bar{a}}^{\bar{0}} Y_A^i R^{A\bar{-}} y_i \bar{J}_-, \tag{3.26}$$

while for $\mathbf{R}_9$

$$\hat{J}_a = \mathrm{e}^{\delta_a} J_a, \qquad \hat{\bar{J}}_{\bar{a}} = \mathrm{e}^{\bar{\delta}_{\bar{a}}}\bar{J}_{\bar{a}}, \qquad \delta_+ = -\bar{\delta}_{\bar{-}} = -\eta Y_A^i R^{A0} y_i. \tag{3.27}$$

It is not hard to show that $\mathbf{R}_5$, $\mathbf{R}_7$ and $\mathbf{R}_8$ are in fact equivalent to TsT transformations. Consider the first one. The deformed Lagrangian is

$$L = L_0 - \frac{1}{2}\eta[J_a + \eta\delta_a^0 y_i Y_A^i R^{A+} J_+][(1+\eta RM')^{-1}R]^{a\bar{a}}\bar{J}_{\bar{a}}. \tag{3.28}$$

However, due to the form of the R-matrix and the fact that $M'_{+A} = 0$ this is equal to

$$L = L_0 - \frac{1}{2}\eta J_a[(1+\eta RM')^{-1}R]^{a\bar{a}}\bar{J}_{\bar{a}}, \tag{3.29}$$

where furthermore we can replace R by the Abelian R-matrix obtained by dropping the $S_0 \wedge S_+$-term in $\mathbf{R}_5$. This deformation therefore reduces to a sequence of commuting TsT transformations. The same conclusion applies to $\mathbf{R}_8$ as is easily seen from the form of the R matrix. For $\mathbf{R}_7$ we have

Table 3. Rank-4 non-Abelian R-matrices of $\mathfrak{sl}(2)_{\mathrm{L}} \oplus \mathfrak{sl}(2)_{\mathrm{R}} \oplus \mathfrak{su}(2)_{\mathrm{L}} \oplus \mathfrak{su}(2)_{\mathrm{R}}$ up to $SL(2,\mathbb{R})_{\mathrm{L}} \times SL(2,\mathbb{R})_{\mathrm{R}} \times SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}}$ inner automorphisms and swaps of $\mathrm{L} \leftrightarrow \mathrm{R}$. With T, T', T'' we denote generic linear combinations of $T_1, \bar{T}_2$.

$R = R^{AB}\mathbf{T}_A \wedge \mathbf{T}_B$	Deformation?
$\mathbf{R}_4 = \mathbf{R}_1 + aT_1 \wedge \bar{T}_2$	$\mathbf{R}_1$+TsT$\Longrightarrow$not SUGRA
$\mathbf{R}_5 = \mathbf{R}_2 + bT_1 \wedge \bar{T}_2$	$\mathbf{R}_2$+TsT$\Longrightarrow$TsT
$\mathbf{R}_6 = (S_0 + T) \wedge S_+ + T' \wedge (\bar{S}_0 + T'')$	SUGRA
$\mathbf{R}_7 = (S_0 + T) \wedge S_+ + T' \wedge (\bar{S}_- + T'')$	TsT
$\mathbf{R}_8 = \mathbf{R}_3 + aT_1 \wedge \bar{T}_2$	$\mathbf{R}_3$+TsT$\Longrightarrow$TsT
$\mathbf{R}_9 = (S_0 + \bar{S}_0 + T) \wedge T' + S_+ \wedge \bar{S}_-$	SUGRA, not TsT
$\mathbf{R}_{10} = (S_0 + T) \wedge S_+ \pm (\bar{S}_0 + T') \wedge \bar{S}_- + \lambda S_+ \wedge \bar{S}_-$	TsT

R	$c^{ab}J_a\bar{J}_b$	Strong CS?
$\mathbf{R}_4$	$(J_0 + aJ_1)\bar{J}_- + J_+(\bar{J}_0 + b\bar{J}_2) + aJ_1\bar{J}_2$	No
$\mathbf{R}_5$	$J_+(a\bar{J}_0 + b\bar{J}_2) + bJ_1\bar{J}_2$	Yes
$\mathbf{R}_6$	$aJ_+\bar{J}_2 + bJ_1\bar{J}_0 + cJ_1\bar{J}_2$	Yes
$\mathbf{R}_7$	$aJ_+\bar{J}_2 + bJ_1\bar{J}_- + cJ_1\bar{J}_2$	Yes
$\mathbf{R}_8$	$J_+(\bar{J}_- + b\bar{J}_2) + aJ_1\bar{J}_2$	Yes
$\mathbf{R}_9$	$aJ_0\bar{J}_2 + bJ_1\bar{J}_0 + cJ_1\bar{J}_2 + J_+\bar{J}_-$	No ($a \neq 0$ or $b \neq 0$)
$\mathbf{R}_{10}$	$cJ_+\bar{J}_2 + dJ_1\bar{J}_- + \lambda J_+\bar{J}_-$	Yes

$$L = L_0 - \frac{1}{2}\eta[J_a + \eta\delta_a^0 y_i Y_A^i R^{A+} J_+][(1+\eta RM')^{-1}R]^{a\bar{a}}[\bar{J}_{\bar{a}} - \eta\delta_{\bar{a}}^{\bar{0}} Y_A^i R^{A\bar{-}} y_i \bar{J}_-], \tag{3.30}$$

but again the terms with $a = 0$ and $\bar{a} = \bar{0}$ drop out due to the form of the R-matrix and the fact that $M_{+A} = 0$ and this reduces to

$$L = L_0 - \frac{1}{2}\eta J_a[(1+\eta RM')^{-1}R]^{a\bar{a}}\bar{J}_{\bar{a}}. \tag{3.31}$$

It is also not hard to see, using again the form of the matrices M and R, that one can again replace R by the Abelian R-matrix obtained by dropping the $S_0 \wedge S_+$-term in $\mathbf{R}_7$. The resulting background is therefore also a TsT. The fact that $\mathbf{R}_5$, $\mathbf{R}_7$ and $\mathbf{R}_8$ are equivalent to TsT backgrounds may be argued also from the fact that the R-matrices are Abelian up to the $S_0 \wedge S_+$-term, and that (3.8) holds.

For $\mathbf{R}_6$ the deformed Lagrangian is

$$L = L_0 - \frac{1}{2}\eta[J_a + \eta\delta_a^0 y_i Y_A^i R^{A+} J_+][(1+\eta RM')^{-1}R]^{a\bar{a}}\bar{J}_{\bar{a}}. \tag{3.32}$$

Again, using the form of R and the fact that $M_{+A} = 0$, this simplifies to

$$L = L_0 - \frac{1}{2}\eta J_a[(1+\eta RM')^{-1}R]^{a\bar{a}}\bar{J}_{\bar{a}}. \tag{3.33}$$

However, in this case we cannot get an equivalent background simply by dropping the $S_0 \wedge S_+$-term in $\mathbf{R}_6$[27]. Let us work out the background for the simplest case, $T = T'' = 0$ and $T' = aT_1 + b\bar{T}_2$. One finds

$$L = L_0 - \eta f^{-1}\Big(2b\eta J_+\bar{J}_2 + 2ab\eta(1-\eta\frac{x^-}{z^2})J_1\bar{J}_2 - \frac{1}{2}\eta^2(a^2+b^2-2ab\sin\phi_3)J_+\bar{J}_0 + 8aJ_1\bar{J}_0\Big) \tag{3.34}$$

where $f = 16 + \eta^2(a^2+b^2-2ab\sin\phi_3)[1-\eta x^-/z^2]$. One can show that in fact the (anti) chiral currents entering this action J_+, J_1, $\bar{J}_0$ and $\bar{J}_2$ extend to chiral currents to all orders in η, i.e. the corresponding isometries are not broken by the deformation. Since this is a characteristic feature of TsT backgrounds it is natural to guess that this background can be generated in that way. It is not hard to show this explicitly in the special cases $a = 1$, $b = 0$ and $a = 0$, $b = 1$ in which cases it is equivalent to the backgrounds generated by

$$R = T_1 \wedge \bar{S}_0 - \frac{\eta^2}{16}S_+ \wedge \bar{S}_0 \qquad \text{and} \qquad R = \frac{4\eta}{16+\eta^2}S_+ \wedge \bar{T}_2 - \frac{\eta^2}{16+\eta^2}S_+ \wedge \bar{S}_0, \tag{3.35}$$

respectively.

$\mathbf{R}_9$ is the only unimodular example, i.e. the only one with $K = 0$. The deformed Lagrangian is

$$L = L_0 - \frac{1}{2}\eta\hat{J}_a[(1+\eta RM')^{-1}R]^{a\bar{a}}\hat{\bar{J}}_{\bar{a}}, \tag{3.36}$$

with

$$\hat{J}_a = \mathrm{e}^{\delta_a}J_a, \qquad \hat{\bar{J}}_{\bar{a}} = \mathrm{e}^{\bar{\delta}_{\bar{a}}}\bar{J}_{\bar{a}}, \qquad \delta_+ = -\bar{\delta}_{\bar{-}} = -\eta Y^i_A R^{A0} y_i. \tag{3.37}$$

For simplicity we will set $T = 0$ and $T' = T_1$. We can argue that this example is not equivalent to a TsT as follows. To order η^2 the Lagrangian is

$$L = L_0 - \frac{1}{2}\eta(1+\frac{\eta}{z^2})J_+\bar{J}_- + \frac{1}{2}\eta J_1\bar{J}_0 - \frac{1}{8}\eta^2 J_0\bar{J}_0 - \frac{1}{2}\eta^2\frac{x^-}{z^2}J_1\bar{J}_- + \mathcal{O}(\eta^3). \tag{3.38}$$

The action is clearly not invariant under the isometry corresponding to constant shifts of x^- and therefore the corresponding chiral current J_+ does not extend to the deformed theory. Instead the equations of motion lead to chiral currents

$$J^\eta_+ = (1+\frac{\eta}{z^2})J_+ - \eta\frac{x^-}{z^2}J_1 + \mathcal{O}(\eta^2), \qquad J^\eta_1 = J_1 + \mathcal{O}(\eta^2), \tag{3.39}$$

while the remaining equations of motion read

$$\partial[(1+\frac{\eta}{z^2})\bar{J}_-] - \eta J_1\bar{J}_- = \mathcal{O}(\eta^2), \qquad \partial[\bar{J}_0 + \eta\frac{x^-}{z^2}\bar{J}_-] - \eta J_+\bar{J}_- = \mathcal{O}(\eta^2). \tag{3.40}$$

At the same time we have

$$\frac{\mathrm{d}S}{\mathrm{d}\eta} = -\frac{T}{2}\int\left(J^\eta_+(1+\frac{\eta}{z^2})\bar{J}_- - J_1\bar{J}_0 + 3\eta\frac{x^-}{z^2}J_1\bar{J}_- + \frac{1}{2}\eta J_0\bar{J}_0 + \mathcal{O}(\eta^2)\right), \tag{3.41}$$

[27] One way to see this is that $\mathbf{R}_6 - R_1$ is not compatible with the coordinate redefinition needed to undo the transformation with $R_1 = S_0 \wedge S_+$ (see equation (3.8)), therefore one does not expect to be able to undo the effect of the R_1 piece of the R-matrix.

which clearly cannot be written as a bilinear in deformed chiral currents. Explicitly, we obtain the background[28]

$$
\begin{aligned}
\mathrm{d}s^2 =& -\frac{\eta \mathrm{d}\phi_1(x^+\mathrm{d}x^- + x^-\mathrm{d}x^+) + 4\mathrm{d}x^-\mathrm{d}x^+ + \left(\eta - z^2\right)\mathrm{d}\phi_1^2}{\eta(\eta x^- x^+ - 4) + 4z^2} + \frac{\mathrm{d}z^2}{z^2} \\
&+ \frac{\mathrm{d}\phi_2^2 + \mathrm{d}\phi_3^2}{4} - \frac{2\mathrm{d}\phi_2 \sin\phi_3 \left(\eta x^- \mathrm{d}x^+ + \left(\eta - z^2\right)\mathrm{d}\phi_1\right)}{\eta(\eta x^- x^+ - 4) + 4z^2}, \\
B =& \frac{-4\mathrm{d}x^- \wedge \mathrm{d}x^+ - 2\sin\phi_3 \left(\mathrm{d}\phi_2 \wedge \left(\eta x^- \mathrm{d}x^+ + \left(\eta - z^2\right)\mathrm{d}\phi_1\right)\right) + \eta \mathrm{d}\phi_1 \wedge (x^+\mathrm{d}x^- - x^-\mathrm{d}x^+)}{2\eta(\eta x^- x^+ - 4) + 8z^2}, \\
\mathrm{e}^{-2\Phi} =& 1 + \frac{\eta(-4 + \eta x^+ x^-)}{4z^2}.
\end{aligned} \tag{3.42}
$$

Finally, $\mathbf{R}_{10} = R_5 + T \wedge S_+ + T' \wedge \bar{S}_-$. Since the $\mathfrak{sl}(2,\mathbb{R})$ R-matrix R_5 does not break the isometries generated by $S_+, \bar{S}_-$, we conclude that the additional terms in $\mathbf{R}_{10}$ have the only effect of adding further TsT transformations on top of the background generated by R_5, which is itself a TsT background.

Interestingly, the matrices $\mathbf{R}_4$ (which may be decomposed in terms of $\mathbf{R}_1$) and $\mathbf{R}_9$ give rise (at leading order in the deformation parameter) to marginal deformations of the WZW model that obviously satisfy the weak CS condition, but not the strong one.

4. Discussion

In this paper we have constructed YB deformations of strings on the pure NSNS $\mathrm{AdS}_3 \times \mathrm{S}^3 \times T^4$ background. Together with Abelian YB deformations, which are known to reproduce TsT transformations, we also constructed non-Abelian YB deformations. While some non-Abelian R-matrices give rise to backgrounds that cannot be obtained simply from TsT transformations, we found that others generate again TsT backgrounds, or even no deformation at all[29]. We expect this to be related to the fact that the initial $G - B$ is degenerate.

For example, the Jordanian R-matrix $R_1 = S_0 \wedge S_+$ gives back the undeformed AdS_3 background up to an η-dependent field redefinition (and up to a non-vanishing $K = -\eta\partial_{x^+}$). Recalling that the YB deformation is equivalent to a shift of the B-field plus NATD, this observation suggests that AdS_3 with NSNS flux has a certain property of self T-duality, when we dualise the non-Abelian algebra of isometries generated by S_0, S_+ and we also regularise the action by performing a B-field gauge transformation.

Although we used our classification of R-matrices to deform, for concreteness, only the $\mathrm{AdS}_3 \times \mathrm{S}^3$ part of the background, our results may also be used to obtain deformations involving the T^4 of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times T^4$, or even deformations of the more general $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ background. Indeed, in all our expressions of the R-matrices the generators $T_1, \bar{T}_2$ may be substituted with any other two commuting generators of the compact part of the isometry algebra[30]. Let us also mention that the string on these AdS_3 backgrounds is integrable [50, 51] and that our deformations preserve the classical integrability.

To leading order in the deformation parameter, all our YB deformations reduce to the marginal current–current deformations of the type considered by Chaudhuri and Schwartz in [8]. While they are all marginal to lowest order, since they satisfy what we called the 'weak CS

[28] Here we are writing the background that is obtained *before* doing the field redefinition.
[29] Except for the introduction of a (decoupled) non-vanishing vector K in the equations of generalised supergravity.
[30] That is because the CYBE implies that the restriction of an R-matrix to a compact algebra must be Abelian. We are therefore free to choose which Abelian subalgebra we wish to consider.

condition', some of them do not satisfy the 'strong CS condition' and the celebrated 'no simple-pole condition' which guarantee exact marginality. Indeed we found examples which failed to be marginal beyond lowest order (and at one-loop in $1/k$), all involving the R-matrix R_4 in table 1, and one example, $\mathbf{R}_9$ in table 3, which remains marginal to all orders in η at least up to one loop in $1/k$. The relation between the space of solutions of the CYBE and that of the weak CS condition is an interesting question. The former is a quadratic equation for R, while the latter is a quartic equation for the coefficients c^{ab} of the current–current deformation, related to the left-right block of the R-matrix simply as $c^{ab} = R^{ab}_{\rm LR}$. While we expect all solutions of the CYBE to generate solutions of the weak CS condition (including the trivial ones) it seems hard to prove this statement for a generic Lie algebra. In appendix D we took a digression from the setup of the paper and we considered the CYBE on the $\mathfrak{sl}_3$ algebra, finding again that it generates non-trivial solutions to the weak CS condition (that do not solve the strong one). We do not rule out the possibility of having solutions to the weak CS condition that cannot be 'completed' to a solution of the CYBE equation. In section 3 we actually discussed a more generic criterion (related to the solvability of the subalgebras involved) not requiring CYBE, to construct solutions of the weak CS condition.

In [52] a marginal deformation constructed out of Abelian currents (a TsT transformation) was interpreted in terms of spectral flow. It would be interesting to understand if this can be generalised to the non-Abelian set-up.

We would like to stress that we worked out the YB deformations in the sigma model description. It would be very interesting to understand how to formulate the YB deformation directly at the level of the WZW action. Such a construction was performed in [53–55] for R a solution of the *modified* CYBE[31]. The formulation of the deformation of the WZW action may be obtained from the construction in terms of NATD, in the spirit of [32] and [33]. An alternative may be to use the language of $\mathcal{E}$ models [56–58], see [59] for a recent application in similar contexts.

One motivation to carry out this work came from the recent developments on the $T\bar{T}$ deformation [60, 61] and its generalisations. The components $T, \bar{T}$ of the stress-energy tensor of a (quite generic) two-dimensional relativistic field theory may be used to construct a 'double-trace' operator generating an irrelevant perturbation of the theory. The deformation is solvable in the sense that the spectrum of the deformed theory may be computed *exactly* in the deformation parameter as a function of the spectrum of the original undeformed theory. In [45] a 'single-trace' version of the $T\bar{T}$ deformation of the symmetric product orbifold CFT was considered. It was argued that the irrelevant deformation of the 'spacetime' CFT governed by[32] $\mathcal{O}(x) \propto \sum_{i=1}^{N} T_i(x)\bar{T}_i(\bar{x})$, where i labels each copy in the symmetric product, corresponds to a marginal deformation of the dual WZW model that infinitesimally is just the current–current deformation $J_+(\sigma)\bar{J}_-(\bar{\sigma})$, where $J_+, \bar{J}_-$ are the left and right $SL(2,\mathbb{R})$ currents generating shifts of the boundary coordinates x^+, x^-. Another deformation, similar in spirit to the above one, was studied in [48] and [49] after replacing the $\bar{T}$ with an antichiral $U(1)$ current of the compact factor[33]. It was argued in [48, 49] that the deformation of the dual WZW model is governed again by a marginal deformation bilinear in the currents (where now the antichiral current belongs to the compact part of the algebra). Such marginal deformations of the WZW

[31] There the special propriety $R^3 = -R$ was used, so that we do not expect their results to be immediately applicable to the case of the homogeneous CYBE. Moreover, here we want R to be a solution of the CYBE on $\mathfrak{f}_{\rm L} \oplus \mathfrak{f}_{\rm R}$ that also couples the left and right copy of the algebra.

[32] We refer to [45] for the connection to Little String Theories. The above operator should be compared to the original double-trace version studied in [60, 61] and given by $T(x)\bar{T}(\bar{x})$, where $T(x) = \sum_{i=1}^{N} T_i(x)$ and $\bar{T}(\bar{x}) = \sum_{i=1}^{N} \bar{T}_i(\bar{x})$.

[33] This deformation is in fact the single-trace version of the one first constructed in [62].

model may be completed to finite values of the deformation parameter in terms of TsT (or equivalently certain $O(d,d)$) transformations. Since TsT deformations are a subclass of YB ones, it would be interesting to understand if it is possible to provide a holographic interpretation also for the YB deformations of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times T^4$ considered here. (The connection to YB models was also pointed out the recent paper [44].) We expect our marginal deformations of the WZW model to correspond to deformations of the dual CFT_2 which generalise the (single trace version of the) $T\bar{T}$ construction. It would be very interesting to understand for example the case of the non-Abelian R-matrix $\mathbf{R}_9$, which gives rise to the marginal deformation $aJ_0\bar{J}_2 + bJ_1\bar{J}_0 + cJ_1\bar{J}_2 + J_+\bar{J}_-$. The non-Abelianity of the generators involved forbids the usual iteration of the infinitesimal deformation in order to obtain the exact one. The YB deformation, despite the non-Abelianity, provides the realisation of the finite deformation on the worldsheet of the string.

Acknowledgments

We thank R Conti, B Hoare and D Thompson for useful discussions. LW thanks the participants of the workshop 'A fresh look at $\mathrm{AdS}_3/\mathrm{CFT}_2$' in Villa Garbald, Castasegna, for stimulating discussions. The work of RB is supported by the Maria de Maeztu Unit of Excellence MDM-2016-0692, by FPA2017-84436-P, by Xunta de Galicia (ED431C 2017/07), and by FEDER. His work was supported by the ERC advanced grant No 341222 while at Nordita.

Appendix A. Details on the field redefinition

The matrix M in (3.11) is a direct sum of the AdS and sphere part, $M = M_\mathrm{a} \oplus M_\mathrm{s}$. They take the form (we restrict to $A, B = \{0, +, \bar{-}, \bar{0}\}$ and $A, B = \{1, \bar{2}\}$ respectively, which is all we need since M always comes multiplied with R on both sides)

$$M_\mathrm{a} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ -\frac{x^+}{z^2} & -\frac{1}{z^2} & 0 & 0 \\ -\frac{1}{4} + \frac{x^+x^-}{z^2} & \frac{x^-}{z^2} & 0 & \frac{1}{4} \end{pmatrix}, \qquad M_\mathrm{s} = \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{1}{2}\sin\phi_3 & \frac{1}{4} \end{pmatrix}. \tag{A.1}$$

We will make an ansatz for the field redefinition based on the isometry transformations used to construct the model, but with the transformation parameters depending linearly on $y = (\ln z, \phi_1, \phi_2)$ (since we want to cancel terms involving ∂y and $\bar{\partial} y$). We therefore consider

$$x^\pm \to x'^\pm = \mathrm{e}^{\eta b^i_\pm y_i}[x^\pm + \eta a^i_\pm y_i], \qquad z \to z' = \mathrm{e}^{\eta b^i y_i} z, \qquad \phi_{1,2} \to \phi'_{1,2} = \phi_{1,2} + \eta a^i_{1,2} y_i, \tag{A.2}$$

where $i = 1, 2, 3$ and $b^i = (b^i_+ + b^i_-)/2$. For the right-moving Noether currents $\mathcal{J}_{A+}$ we get

$$\mathcal{J}'_{0+} = J_0 + \eta a^i_+ y_i J_+ - \frac{1}{2}\partial \ln z + \partial y_i[\eta b^i_+ M'_{00} - \eta b^i_- M'_{0\bar{0}} + \eta a^i_- \mathrm{e}^{\eta b^i_- y_i} M'_{\bar{-}0}], \tag{A.3}$$

$$\mathcal{J}'_{++} = \mathrm{e}^{-\eta b^i_+ y_i} J_+ + \partial y_i[\eta a^i_- \mathrm{e}^{\eta b^i_- y_i} M'_{\bar{-}+} - \eta b^i_- M'_{\bar{0}+}], \tag{A.4}$$

$$\mathcal{J}'_{\bar{0}+} = -\frac{1}{2}\partial \ln z - \frac{1}{2}\eta b^i \partial y_i, \tag{A.5}$$

$$\mathcal{J}'_{1+} = J_1 + \frac{1}{4}\partial\phi_1 + \partial y_i[\eta a_2^i M'_{21} - \eta a_1^i M_{11}], \tag{A.6}$$

$$\mathcal{J}'_{2+} = \frac{1}{4}\partial\phi_2 + \frac{1}{4}\eta a_2^i \partial y_i. \tag{A.7}$$

These expressions take a more natural form if we further assume that

$$a_1^i = -Y_A^i R^{A1}, \quad a_2^i = Y_A^i R^{A\bar{2}}, \quad a_+^i = Y_A^i R^{A+}, \quad a_-^i = Y_A^i R^{A\bar{-}}, \quad b_+^i = Y_A^i R^{A0}, \quad b_-^i = -Y_A^i R^{A\bar{0}}, \tag{A.8}$$

for some constants Y_A^i to be determined. Then we have

$$\mathcal{J}'_{A+} = \mathrm{e}^{\delta_A} J_A + \partial y_i Y_B^i [\eta R M']^B{}_A + \Delta_A \tag{A.9}$$

with $\delta_+ = -\eta Y_A^i R^{A0} y_i$ and

$$\Delta_0 = -\frac{1}{2}\partial \ln z + \eta y_i Y_A^i R^{A+} J_+ + [\mathrm{e}^{\eta b_-^i y_i} - 1]\partial y_i Y_A^i \eta R^{A\bar{-}} M'_{\bar{-}0}, \tag{A.10}$$

$$\Delta_+ = [\mathrm{e}^{\eta b_-^i y_i} - 1]\partial y_i Y_A^i \eta R^{A\bar{-}} M'_{\bar{-}+}, \tag{A.11}$$

$$\Delta_{\bar{0}} = -\frac{1}{2}\partial \ln z, \qquad \Delta_1 = \frac{1}{4}\partial\phi_1, \qquad \Delta_{\bar{2}} = \frac{1}{4}\partial\phi_2, \tag{A.12}$$

the other components vanishing. Using these expressions the transformed Lagrangian becomes

$$L' = L'_0 - \frac{1}{2}\eta \mathrm{e}^{\delta_a} J_a[(1+\eta R M')^{-1}R]^{aB}\mathcal{J}'_{B-} + \frac{1}{2}\eta(\partial y_i Y_A^i - \Delta_A)[(1+\eta R M')^{-1}R]^{AB}\mathcal{J}'_{B-} - \frac{1}{2}\eta \partial y_i Y_A^i R^{AB}\mathcal{J}'_{B-}. \tag{A.13}$$

Picking Y_A^i such that $\partial y_i Y_A^i - \Delta_A$ vanishes to lowest order in η we get that its non-zero components should be

$$Y_0^1 = Y_{\bar{0}}^1 = -\frac{1}{2}, \qquad Y_1^2 = Y_{\bar{2}}^3 = \frac{1}{4} \tag{A.14}$$

and the Lagrangian becomes

$$L' = L'_0 - \frac{1}{2}\eta \partial y_i Y_A^i R^{AB}\mathcal{J}'_{B-} - \frac{1}{2}\eta \hat{J}_a[(1+\eta R M')^{-1}R]^{aB}\mathcal{J}'_{B-} \tag{A.15}$$

where we have defined

$$\hat{J}_a = \mathrm{e}^{\delta_a} J_a + \eta\delta_a^0 y_i Y_A^i R^{A+} J_+ + \eta[\mathrm{e}^{-\bar{\delta}_-} - 1]\partial y_i Y_A^i R^{A\bar{-}} M'_{\bar{-}a}, \tag{A.16}$$

with $\bar{\delta}_- = \eta Y_A^i R^{A\bar{0}} y_i$. A short calculation shows that the transformed undeformed Lagrangian is (up to total derivatives)

$$\begin{aligned} L'_0 = L_0 &+ \frac{1}{2}\eta Y_A^i R^{Aa}\left(J_a + \eta\delta_a^0 y_j Y_B^j R^{B+} J_+\right)\bar{\partial} y_i + \frac{1}{2}\eta \partial y_i Y_A^i R^{A\bar{a}} \bar{J}'_{\bar{a}} \\ &+ \frac{1}{2}\eta[\mathrm{e}^{-\bar{\delta}_-} - 1] Y_A^i R^{A\bar{-}} \partial y_i \bar{J}'_{-} - \frac{1}{2}\eta \partial y_i Y_A^i R^{AB} Y_B^j \bar{\partial} y'_j \end{aligned} \tag{A.17}$$

and using this together with the fact that $\mathcal{J}_{B-} = \bar{J}_B - Y_B^i \bar{\partial} y_i$ we get

$$\begin{aligned} L' = L_0 &+ \frac{1}{2}\eta Y_A^i R^{Aa}(J_a + \eta\delta_a^0 y_j Y_B^j R^{B+} J_+)\bar{\partial} y_i - \frac{1}{2}\eta \hat{J}_a[(1+\eta R M')^{-1}R]^{aB}\mathcal{J}'_{B-} \\ &+ \frac{1}{2}\eta[\mathrm{e}^{\eta b_-^i y_i} - 1]\partial y_i Y_A^i R^{A\bar{-}}\mathcal{J}'_{\bar{-}-}. \end{aligned} \tag{A.18}$$

Finally we use the fact that

$$\mathcal{J}'_{B-} = \mathrm{e}^{\bar{\delta}_B}\bar{J}_B - [1+\eta M'R]_B{}^A Y^i_A \bar{\partial} y_i + \bar{\Delta}_B \tag{A.19}$$

with $\bar{\delta}_{\bar{-}} = \bar{\delta}_-$ was defined above and

$$\bar{\Delta}_{\bar{-}} = -\eta[\mathrm{e}^{-\delta_+} - 1]M'_{\bar{-}+}R^{+A}Y^i_A\bar{\partial}y_i, \tag{A.20}$$

$$\bar{\Delta}_{\bar{0}} = -\eta Y^i_A R^{A\bar{-}} y_i \bar{J}_- - \eta[\mathrm{e}^{-\delta_+} - 1]Y^i_A M'_{\bar{0}+}R^{+A}\bar{\partial}y_i, \tag{A.21}$$

and the remaining components vanishing. Finally the Lagrangian becomes

$$\begin{aligned} L' = L_0 &- \frac{1}{2}\eta \hat{J}_a[(1+\eta RM')^{-1}R]^{a\bar{a}}\hat{\bar{J}}_{\bar{a}} + \frac{1}{2}\eta[\mathrm{e}^{\delta_+}-1]J_+R^{+A}Y^i_A\bar{\partial}y_i - \frac{1}{2}\eta[\mathrm{e}^{\bar{\delta}_-}-1]\partial y_i Y^i_A R^{A\bar{-}}\bar{J}_- \\ &- \frac{1}{2}\eta^2[\mathrm{e}^{-\bar{\delta}_-}-1]\partial y_i Y^i_A R^{A\bar{-}} M'_{\bar{-}+}[\mathrm{e}^{-\delta_+}-1]Y^j_B R^{+B}\bar{\partial}y_j, \end{aligned} \tag{A.22}$$

with (recall that $\delta_+ = -\eta Y^i_A R^{A0} y_i$ and $\bar{\delta}_{\bar{-}} = \eta Y^i_A R^{A\bar{0}} y_i$)

$$\hat{\bar{J}}_{\bar{a}} = \mathrm{e}^{\bar{\delta}_{\bar{a}}}\bar{J}_{\bar{a}} - \eta\delta^{\bar{0}}_{\bar{a}} Y^i_A R^{A\bar{-}} y_i \bar{J}_- - \eta M'_{\bar{a}+}[\mathrm{e}^{-\delta_+}-1]R^{+A}Y^i_A\bar{\partial}y_i. \tag{A.23}$$

The terms involving $\mathrm{e}^{\delta_+}-1$ and $\mathrm{e}^{\bar{\delta}_-}-1$ are not expressed in terms of the chiral currents but they only appear at order η^2, so they do not interfere with the comparison to infinitesimal current–current deformations. In fact they vanish for most of the deformations of $\mathrm{AdS}_3 \times \mathrm{S}^3$, e.g. for AdS_3 deformations we have $R^{0\bar{0}}R^{A+} = R^{0\bar{0}}R^{A\bar{-}} = 0$ and using this we find

$$L = L_0 - \frac{1}{2}\eta\hat{J}_a[(1+\eta RM')^{-1}R]^{a\bar{a}}\hat{\bar{J}}_{\bar{a}}, \tag{A.24}$$

with

$$\hat{J}_a = J_a - \frac{1}{2}\eta\delta^0_a(R^{0+}+R^{\bar{0}+})\ln z J_+, \qquad \hat{\bar{J}}_{\bar{a}} = \bar{J}_{\bar{a}} + \frac{1}{2}\eta\delta^{\bar{0}}_{\bar{a}}(R^{0\bar{-}}+R^{\bar{0}\bar{-}})\ln z\bar{J}_-. \tag{A.25}$$

Appendix B. On-shell equivalence

Here we will demonstrate the on-shell equivalence of the YB deformed sigma models to the original ones by deriving the explicit (non-local) field redefinition that relates them. We will do that by following the NATD transformation and following field redefinition that are used to get the action of the YB model as in [36]. In the notation of [36], let us start from the action

$$\begin{aligned} S' = &\frac{T}{2}\int_\Sigma \Big(A^I \wedge (G_{IJ} * - B_{IJ})A^J + 2\mathrm{d}z^m \wedge (G_{mI} * - B_{mI})A^I \\ &+ \mathrm{d}z^m \wedge (G_{mn} * - B_{mn})\mathrm{d}z^n\Big), \end{aligned} \tag{B.1}$$

where $A = g^{-1}dg$ and we have set fermions to zero for simplicity. When going to the NATD model one relates the original degrees of freedom to the new ones encoded in the Lagrange multiplier ν as

$$(1\pm *)A^I = -(1\pm *)\left(\mathrm{d}\nu_J + \mathrm{d}z^m[\mp G - B]_{mJ}\right)N^{JI}_{\mp}, \qquad N^{IJ}_{\pm} = \left(\pm G_{IJ} - B_{IJ} - \nu_K f_{IJ}^K\right)^{-1}. \tag{B.2}$$

The YB model appears after the redefinition[34]

[34] It is assumed that the initial B-field is actually shifted as $B_{IJ} \to B_{IJ} - \eta^{-1}R^{-1}_{IJ}$.

$$\nu_I = \eta^{-1}\mathrm{tr}\left(T_I \frac{1-\mathrm{Ad}_{\tilde{g}}^{-1}}{\log \mathrm{Ad}_{\tilde{g}}} R^{-1}\log\tilde{g}\right), \tag{B.3}$$

which implies

$$\mathrm{d}\nu_I = \eta^{-1}\left(R_{\tilde{g}}^{-1}(\tilde{g}^{-1}\mathrm{d}\tilde{g})\right)_I, \qquad N = \eta R_{\tilde{g}}\,(1+\eta(G-B)R_{\tilde{g}})^{-1} = \eta\,(1+\eta R_{\tilde{g}}(G-B))^{-1}\,R_{\tilde{g}}, \tag{B.4}$$

where $N = N_+ = -N_-^T$. Combining all redefinitions we find that the original $A = g^{-1}dg$ (which will depend on some coordinates x^i) is related to the new degrees of freedom of the YB model (i.e. the coordinates $\tilde{x}^i$ parameterising $\tilde{g}$, together with the coordinates z^m that remain spectators) as

$$(1\pm *)A^I = (1\pm *)[(1\pm\eta R_{\tilde{g}}(G\mp B))^{-1}]^I{}_J\left((\tilde{g}^{-1}\mathrm{d}\tilde{g})^J \mp \eta R_{\tilde{g}}^{JK}(G\mp B)_{Km}\mathrm{d}z^m\right). \tag{B.5}$$

Using formula (4.21) of [36] we find that it can be written as

$$(1\pm *)A = (1\pm *)(1\pm\eta R_{\tilde{g}}(G\mp B))^{-1}\tilde{V}, \qquad \tilde{V}^M \equiv \delta_I^M(\tilde{g}^{-1}\mathrm{d}\tilde{g})^I + \delta_m^M\mathrm{d}z^m. \tag{B.6}$$

This can *almost* be written as a relation involving only the derivatives of the redefined coordinates

$$(1\pm *)W = (1\pm *)(1\pm\eta\Theta(G\mp B))^{-1}\mathrm{d}\tilde{X}, \qquad \mathrm{d}\tilde{X}^M \equiv \delta_i^M d\tilde{x}^i + \delta_m^M\mathrm{d}z^m, \tag{B.7}$$

where

$$W^M = \delta_i^M \ell_I^i \tilde{\ell}_j^I \mathrm{d}x^j + \delta_m^M\mathrm{d}z^m, \qquad g^{-1}\mathrm{d}g = \mathrm{d}x^i\ell_i^I T_I, \quad \tilde{g}^{-1}\mathrm{d}\tilde{g} = \mathrm{d}\tilde{x}^i\tilde{\ell}_i^I T_I \tag{B.8}$$

and where all indices that have been omitted above are curved indices $M = \{i, m\}$. In general $\ell_I^i\tilde{\ell}_j^I \neq \delta_j^i$ and G, B, Θ may depend on $\tilde{x}$. Thanks to the above field redefinition we can argue that solutions to the classical equations of the YB model can be mapped to solutions of the undeformed model, and vice versa.

Appendix C. All non-Abelian *R*-matrices of $\mathfrak{sl}(2,\mathbb{R})^2 \oplus \mathfrak{su}(2)^2$

We will classify non-Abelian R-matrices solving the classical Yang–Baxter equation relevant to deformations of $\mathrm{AdS}_3\times\mathrm{S}^3$. The R-matrix is an anti-symmetric matrix with indices in the isometry algebra, in our case

$$\mathfrak{sl}(2,\mathbb{R})_{\mathrm{L}} \oplus \mathfrak{sl}(2,\mathbb{R})_{\mathrm{R}} \oplus \mathfrak{su}(2)_{\mathrm{L}} \oplus \mathfrak{su}(2)_{\mathrm{R}}. \tag{C.1}$$

The classical Yang–Baxter equation (CYBE)

$$R^{[A|B|}R^{C|D|}f_{BD}^{E]} = 0, \tag{C.2}$$

implies that R^{AB} is non-degenerate on a subalgebra and zero elsewhere. Calling the inverse ω_{AB} the CYBE is equivalent to $\omega_{A[B}f_{CD]}^{A} = 0$, i.e. ω is a Lie algebra 2-cocycle on the (dual of the) subalgebra where R defined. Since it is also invertible this subalgebra is a quasi-Frobenius (sometimes also called symplectic) subalgebra[35]. Therefore R-matrices solving the CYBE on

[35] If ω is exact, i.e. $\omega_{AB} = f_{AB}^C X_C$ for some X_C, the algebra is Frobenius.

some Lie algebra are in one-to-one correspondence with quasi-Frobenius subalgebras of this Lie algebra [63].

Semi-simple Lie algebras cannot be quasi-Frobenius and therefore such algebras of dimension 4 (or 2) must be solvable [64]. In our case things are particularly simple since we have a sum of four 3 dimensional Lie algebras. When can therefore restrict our attention to subalgebras of the maximal solvable subalgebra of the isometry algebra which we take to be

$$\mathfrak{s} = \text{span}\{S_0,\, S_+,\, \bar{S}_0,\, \bar{S}_-,\, T_1,\, \bar{T}_2\}. \tag{C.3}$$

The only non-Abelian solvable Lie algebra of dimension 2 is $\mathfrak{r}_2$ with Lie bracket $[e_1, e_2] = e_2$. The possible R-matrices of rank 2 are given by embeddings of $\mathfrak{r}_2$ into $\mathfrak{s}$ and up to $SL^2(2) \times SU^2(2)$ automorphisms they are

$$\begin{aligned} \mathbf{R}_1 &= (S_0 - \bar{S}_0 + T) \wedge (S_+ \pm \bar{S}_-) \\ \mathbf{R}_2 &= (S_0 - a\bar{S}_0 + T) \wedge S_+ \\ \mathbf{R}_3 &= (S_0 \mp \bar{S}_- + T) \wedge S_+ \end{aligned} \tag{C.4}$$

where T is any linear combination of T_1 and $\bar{T}_2$.

For the rank 4 case we need to find 4-dimensional solvable subalgebras of $\mathfrak{s}$, which are furthermore quasi-Frobenius (symplectic). The complete list of 4-dimensional quasi-Frobenius algebras can be found in [65]. Taking into account the fact that $[\mathfrak{s}, \mathfrak{s}] = \text{span}\{S_+,\, \bar{S}_-\}$ is 2-dimensional we can rule out any algebra with more than two independent linear combinations of generators arising from commutators. It is also trivial to see that the Heisenberg algebra with non-trivial Lie bracket $[e_1, e_2] = e_3$ is not a subalgebra, $\mathfrak{h}_3 \not\subset \mathfrak{s}$. Using these two facts the list of 4-dimensional solvable subalgebras is reduced to (note that $\mathfrak{r}_{4,-1,0} = \mathbb{R} \oplus \mathfrak{r}_{3,-1}$)

$$\mathbb{R} \oplus \mathfrak{r}_3, \quad \mathbb{R} \oplus \mathfrak{r}_{3,\lambda}, \quad \mathbb{R} \oplus \mathfrak{r}'_{3,\gamma}, \quad \mathfrak{r}_2 \oplus \mathfrak{r}_2, \quad \mathfrak{r}'_2, \quad \mathfrak{d}_{4,1}. \tag{C.5}$$

The same paper lists the symplectic ones which are

$$\begin{array}{l|l} \mathbb{R} \oplus \mathfrak{r}_{3,0} & [e_1, e_2] = e_2 \\ \mathbb{R} \oplus \mathfrak{r}_{3,-1} & [e_1, e_2] = e_2, \quad [e_1, e_3] = -e_3 \\ \mathbb{R} \oplus \mathfrak{r}'_{3,0} & [e_1, e_2] = -e_3, \quad [e_1, e_3] = e_2 \\ \mathfrak{r}_2 \oplus \mathfrak{r}_2 & [e_1, e_2] = e_2, \quad [e_3, e_4] = e_4 \\ \mathfrak{r}'_2 & [e_1, e_3] = e_3, \quad [e_1, e_4] = e_4, \quad [e_2, e_3] = e_4, \quad [e_2, e_4] = -e_3 \\ \mathfrak{d}_{4,1} & [e_1, e_2] = e_3, \quad [e_4, e_3] = e_3, \quad [e_4, e_1] = e_1 \end{array} . \tag{C.6}$$

It is not hard to show, using the fact that the elements arising from commutators are of the form $aS_+ + b\bar{S}_-$ for some a, b, that $\mathfrak{r}'_{3,0}$ is not a subalgebra of $\mathfrak{s}$ and therefore neither is $\mathfrak{r}'_2$. It is also not hard to show that neither is $\mathfrak{d}_{4,1}$. For the remaining ones we find the embeddings (again up to automorphism)

$$\begin{array}{l|l} \mathbb{R} \oplus \mathfrak{r}_{3,0} & e_1 = S_0 - b\bar{S}_0 + e'_1, \quad e_2 = S_+ + a\bar{S}_- \qquad (a = 0 \quad \text{or} \quad a = b = 1) \\ \mathbb{R} \oplus \mathfrak{r}_{3,-1} & e_1 = S_0 + \bar{S}_0 + e'_1, \quad e_2 = S_+, \quad e_3 = \bar{S}_- \\ \mathfrak{r}_2 \oplus \mathfrak{r}_2 & e_1 = S_0 + e'_1, \quad e_2 = S_+, \quad e_3 = -\bar{S}_0 + e'_3, \quad e_4 = \bar{S}_- \end{array} . \tag{C.7}$$

Primed generators denote any linear combination that commutes with the remaining generator. Only the second algebra is unimodular (i.e. the trace of its structure constants vanish) and is contained in the classification of unimodular R-matrices in [43].

The rank 4 R-matrices can then be read off from the classification in [65]. Up to inner automorphisms and exchanging the left and right copy of the algebra they are

$$
\begin{aligned}
\mathbf{R}_4 &= (S_0 - \bar{S}_0 + T) \wedge (S_+ \pm \bar{S}_-) + aT_1 \wedge \bar{T}_2 \\
\mathbf{R}_5 &= (S_0 - a\bar{S}_0 + T) \wedge S_+ + bT_1 \wedge \bar{T}_2 \\
\mathbf{R}_6 &= (S_0 + T) \wedge S_+ + T' \wedge (\bar{S}_0 + T'') \\
\mathbf{R}_7 &= (S_0 + T) \wedge S_+ + T' \wedge (\bar{S}_- + T'') \\
\mathbf{R}_8 &= (S_0 \mp \bar{S}_- + T) \wedge S_+ + aT_1 \wedge \bar{T}_2 \\
\mathbf{R}_9 &= (S_0 + \bar{S}_0 + T) \wedge T' + S_+ \wedge \bar{S}_- \\
\mathbf{R}_{10} &= (S_0 + T) \wedge S_+ \pm (\bar{S}_0 + T') \wedge \bar{S}_- + \lambda S_+ \wedge \bar{S}_-
\end{aligned} \tag{C.8}
$$

where T, T', T'' are linear combinations of T_1 and $\bar{T}_2$. The rank 2 R-matrices are contained as special cases of $\mathbf{R}_4, \mathbf{R}_5, \mathbf{R}_8$ when the last term vanishes.

A rank 6 R-matrix is only possible if $\mathfrak{s}$ is itself quasi-Frobenius. It is in this case but the resulting R-matrix is just $\mathbf{R}_{10} + aT_1 \wedge \bar{T}_2$ and therefore leads just to a TsT transformation of the $\mathbf{R}_{10}$ deformation. Therefore we will not consider it further here.

Appendix D. *R*-matrix on parabolic subalgebra of $\mathfrak{sl}_3$

Let $\mathfrak{f}$ denote the 6-dimensional parabolic Lie subalgebra of $\mathfrak{sl}_3$ [66] with basis (see example 3.6 of [67])

$$
(e_1, \ldots, e_6) = (E_{12}, E_{13}, E_{21}, E_{23}, E_{11} - E_{22}, E_{22} - E_{33}). \tag{D.1}
$$

The Lie brackets are given by

$$
\begin{aligned}
&[e_1, e_3] = e_5, \quad [e_1, e_4] = e_2, \quad [e_1, e_5] = -2e_1, \quad [e_1, e_6] = e_1, \quad [e_2, e_3] = -e_4, \\
&[e_2, e_5] = -e_2, \quad [e_2, e_6] = -e_2, \quad [e_3, e_5] = 2e_3, \quad [e_3, e_6] = -e_3, \quad [e_4, e_5] = e_4, \quad [e_4, e_6] = -2e_4.
\end{aligned} \tag{D.2}
$$

This algebra is not unimodular since $f_{i6}{}^{i} = -3 \neq 0$ and also not solvable since $\mathrm{tr}([e_1, e_3]e_5) \neq 0$ but it is quasi-Frobenius with 2-cocycle

$$
\begin{aligned}
\omega =& a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4 - 2a_{16}e^1 \wedge e^5 + a_{16}e^1 \wedge e^6 + a_{23}e^2 \wedge e^3 - a_{14}e^2 \wedge e^5 - a_{14}e^2 \wedge e^6 \\
&- 2a_{36}e^3 \wedge e^5 + a_{36}e^3 \wedge e^6 - a_{23}e^4 \wedge e^5 + 2a_{23}e^4 \wedge e^6.
\end{aligned} \tag{D.3}
$$

One may take e.g.

$$
\omega = e^1 \wedge (2e^5 - e^6) + e^2 \wedge e^3 + e^4 \wedge (2e^6 - e^5) \tag{D.4}
$$

the inverse being

$$
R = -\frac{1}{3}e_1 \wedge (2e_5 + e_6) - e_2 \wedge e_3 - \frac{1}{3}e_4 \wedge (2e_6 + e_5). \tag{D.5}
$$

For the $SL(3)$ WZW model we can get a deformation by embedding this algebra in the diagonal $SL(3) \subset SL(3) \times SL(3)$. After identifying the coefficients of CS with the off-diagonal components of the R-matrix and using the definitions in (2.19) we find

$$
\begin{aligned}
&C^{12\bar{3}} = -1, \quad C^{13\bar{2}} = -1, \quad C^{15\bar{1}} = -\frac{2}{3}, \quad C^{16\bar{1}} = -\frac{1}{3}, \quad C^{23\bar{4}} = 1, \quad C^{25\bar{5}} = \frac{2}{3}, \\
&C^{26\bar{5}} = \frac{1}{3}, \quad C^{34\bar{2}} = 1, \quad C^{45\bar{4}} = -\frac{1}{3}, \quad C^{46\bar{4}} = -\frac{2}{3}, \quad C^{56\bar{2}} = \frac{1}{3}.
\end{aligned} \tag{D.6}
$$

Once again we are using a bar for indices in the right copy of the algebra. Using the fact that the non-zero components of the Killing metric are

$$K_{13} = K_{31} = 1, \quad K_{55} = K_{66} = 2, \quad K_{56} = K_{65} = -1, \tag{D.7}$$

one finds that $C^{ab\bar{c}} K_{\bar{c}\bar{d}} C^{ef\bar{d}} \neq 0$ but the weak CS condition is satisfied since $C^2 = 0$.

Appendix E. More general 'trivial' solutions of generalized supergravity

In [42] it was shown that the generalized supergravity equations can have 'trivial' solutions, i.e. ones that are also solutions of standard supergravity even though the Killing vector K is non-zero. Writing[36] $\mathrm{X} = \mathrm{d}\phi + i_K B$ it was shown that for ϕ to have a gauge invariant meaning K must be null. While this is a natural condition it is not strictly necessary. It is possible to have a situation where K is not null that still leads to a standard supergravity solution, as we will show here. However in that case one must pick the correct gauge for the B-field to read off the dilaton from the expression $\mathrm{X} = \mathrm{d}\phi + i_K B$, as in a different gauge one may find a different ϕ which will not be the correct dilaton of a standard supergravity solution. To avoid having to deal with gauge-transformations of the B-field we will write $\mathrm{X} = \mathrm{d}\phi + \tilde{K}$ where ϕ should be identified with the dilaton. Below we derive the conditions on $\tilde{K}$ for a trivial solution of the supergravity equations. Note that if we pick a gauge so that B is invariant under the isometry generated by K we have $\tilde{K} = i_K B + \mathrm{d}\phi'$ for some ϕ'.

We will ignore the RR fields in our discussion. Looking at the generalized supergravity equations in [42] it follows from the generalized Einstein equation that for (G, H, ϕ) to solve the standard supergravity $\tilde{K}$ must be a Killing vector (of the metric, other fields do not *a priori* have to be invariant). The remaining equations lead to the following conditions to have a trivial solution

$$\begin{aligned} \mathrm{d}\tilde{K} + i_K H &= 0, \qquad \mathcal{L}_K \phi + K \cdot \tilde{K} = 0, \\ dK + i_{\tilde{K}} H &= 0, \qquad \mathcal{L}_{\tilde{K}} \phi + K^2 + \tilde{K}^2 = 0. \end{aligned} \tag{E.1}$$

If $\tilde{K}$ is proportional to K we get precisely the solutions considered in [42], in particular K is null. But it can also happen that $\tilde{K}$ and K are linearly independent, as the example in (3.22) shows. Such solutions are clearly much less generic than the solutions considered in [42] since they require at least two Killing vectors. They are also harder to identify since they require first extracting the correct dilaton and $\tilde{K}$ and then verifying the equations above.

In [42] it was argued that the analysis based on the generalized supergravity equations agrees with what one gets by looking at the non-local terms in the sigma model action induced by non-Abelian T-duality on a non-unimodular group [38, 39]. The analysis was done for standard YB sigma models where the more general possibility of an independent Killing vector $\tilde{K}$ does not arise. However, the general form of the non-local terms proposed there, namely

$$L_\sigma = \alpha' \mathrm{d}\sigma \wedge K - \alpha' \mathrm{d}\sigma \wedge *\mathrm{X} + \mathcal{O}(\alpha'^2), \tag{E.2}$$

is consistent with the general analysis above since, up to total derivatives, this is equal to

$$\alpha' \sigma (dK + i_{\tilde{K}} H) - \alpha' \sigma (d * \tilde{K} + i_{\tilde{K}} H) + \alpha' \phi R^{(2)} \tag{E.3}$$

and the first term vanishes by (E.1) and the second is proportional to the equations of motion projected along the Killing vector $\tilde{K}$. The order α'^2 terms were also considered in [42] and, given the present analysis, these will be modified so that they now involve a combination of the second and last equation in (E.1) rather than just K^2.

[36] We recall that this was denoted just by X in [42], but we prefer to avoid confusion with the vector $X = \mathrm{X} + K$ of [41].

ORCID iDs

Riccardo Borsato https://orcid.org/0000-0002-0156-441X
Linus Wulff https://orcid.org/0000-0002-9168-3470

References

[1] Maldacena J M 1999 The Large N limit of superconformal field theories and supergravity *Int. J. Theor. Phys.* **38** 1113–33
[2] Giveon A, Kutasov D and Seiberg N 1998 Comments on string theory on AdS_3 *Adv. Theor. Math. Phys.* **2** 733–80
[3] Kutasov D and Seiberg N 1999 More comments on string theory on AdS(3) *J. High Energy Phys.* JHEP04(1999)008
[4] Maldacena J M and Ooguri H 2001 Strings in AdS_3 and SL(2,R) WZW model 1: the Spectrum *J. Math. Phys.* **42** 2929–60
[5] Maldacena J M, Ooguri H and Son J 2001 Strings in AdS_3 and the SL(2,R) WZW model. Part 2. Euclidean black hole *J. Math. Phys.* **42** 2961–77
[6] Maldacena J M and Ooguri H 2002 Strings in AdS(3) and the SL(2,R) WZW model. Part 3. Correlation functions *Phys. Rev.* D **65** 106006
[7] Eberhardt L, Gaberdiel M R and Gopakumar R 2019 The worldsheet dual of the symmetric product CFT *J. High Energy Phys.* JHEP04(2019)103
[8] Chaudhuri S and Schwartz J A 1989 A criterion for integrably marginal operators *Phys. Lett.* B **219** 291–6
[9] Chaudhuri S and Lykken J D 1993 String theory, black holes and SL(2,R) current algebra *Nucl. Phys.* B **396** 270–302
[10] Hassan S F and Sen A 1993 Marginal deformations of WZNW and coset models from O(d,d) transformation *Nucl. Phys.* B **405** 143–65
[11] Kiritsis E 1993 Exact duality symmetries in CFT and string theory *Nucl. Phys.* B **405** 109–42
[12] Tseytlin A A 1994 On a 'universal' class of WZW type conformal models *Nucl. Phys.* B **418** 173–94
[13] Forste S and Roggenkamp D 2003 Current–current deformations of conformal field theories and WZW models *J. High Energy Phys.* JHEP05(2003)071
[14] Israel D, Kounnas C, Orlando D and Petropoulos P M 2005 Electric/magnetic deformations of S∗∗3 and AdS(3) and geometric cosets *Fortsch. Phys.* **53** 73–104
[15] Israel D, Kounnas C, Orlando D and Petropoulos P M 2005 Heterotic strings on homogeneous spaces *Fortsch. Phys.* **53** 1030–71
[16] Orlando D 2007 String theory: exact solutions, marginal deformations and hyperbolic spaces *Fortsch. Phys.* **55** 161–282
[17] Detournay S, Orlando D, Petropoulos P M and Spindel P 2005 Three-dimensional black holes from deformed anti-de Sitter *J. High Energy Phys.* JHEP07(2005)072
[18] Fredenhagen S, Gaberdiel M R and Keller C A 2007 Symmetries of perturbed conformal field theories *J. Phys. A: Math. Theor.* **40** 13685–710
[19] Henningson M and Nappi C R 1993 Duality, marginal perturbations and gauging *Phys. Rev.* D **48** 861–8
[20] Lunin O and Maldacena J M 2005 Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals *J. High Energy Phys.* JHEP05(2005)033
[21] Frolov S A, Roiban R and Tseytlin A A 2005 Gauge-string duality for superconformal deformations of $N = 4$ super Yang–Mills theory *J. High Energy Phys.* JHEP07(2005)045
[22] Frolov S 2005 Lax pair for strings in Lunin–Maldacena background *J. High Energy Phys.* JHEP05(2005)069
[23] Alday L F, Arutyunov G and Frolov S 2006 Green–Schwarz strings in TsT-transformed backgrounds *J. High Energy Phys.* JHEP06(2006)018
[24] Klimcik C 2009 On integrability of the Yang–Baxter sigma-model *J. Math. Phys.* **50** 043508
[25] Delduc F, Magro M and Vicedo B 2014 An integrable deformation of the $AdS_5 \times S^5$ superstring action *Phys. Rev. Lett.* **112** 051601
[26] Kawaguchi I, Matsumoto T and Yoshida K 2014 Jordanian deformations of the $AdS_5 \times S^5$ superstring *J. High Energy Phys.* JHEP04(2014)153

[27] Matsumoto T and Yoshida K 2014 Integrability of classical strings dual for noncommutative gauge theories *J. High Energy Phys.* JHEP06(2014)163
[28] van Tongeren S J 2016 Yang–Baxter deformations, AdS/CFT and twist-noncommutative gauge theory *Nucl. Phys.* B **904** 148–75
[29] van Tongeren S J 2017 Almost Abelian twists and AdS/CFT *Phys. Lett.* B **765** 344–51
[30] Araujo T, Bakhmatov I, Colgáin E O, Sakamoto J, Sheikh-Jabbari M M and Yoshida K 2017 Yang–Baxter σ-models, conformal twists and noncommutative Yang–Mills theory *Phys. Rev.* D **95** 105006
[31] Osten D and van Tongeren S J 2017 Abelian Yang–Baxter deformations and TsT transformations *Nucl. Phys.* B **915** 184–205
[32] Hoare B and Tseytlin A A 2016 Homogeneous Yang–Baxter deformations as non-Abelian duals of the AdS_5 sigma-model *J. Phys. A: Math. Theor.* **49** 494001
[33] Borsato R and Wulff L 2016 Integrable Deformations of T-Dual σ Models *Phys. Rev. Lett.* **117** 251602
[34] Lüst D and Osten D 2018 Generalised fluxes, Yang–Baxter deformations and the $O(d,d)$ structure of non-Abelian T-duality *J. High Energy Phys.* JHEP05(2018)165
[35] Sakamoto J-I and Sakatani Y 2018 Local β-deformations and Yang–Baxter sigma model *J. High Energy Phys.* JHEP06(2018)147
[36] Borsato R and Wulff L 2018 Non-Abelian T-duality and Yang–Baxter deformations of Green–Schwarz strings *J. High Energy Phys.* JHEP08(2018)027
[37] Bakhmatov I, Kelekci Ö, Colgáin E O and Sheikh-Jabbari M M 2018 Classical Yang–Baxter equation from supergravity *Phys. Rev.* D **98** 021901
[38] Alvarez E, Alvarez-Gaume L and Lozano Y 1994 On nonAbelian duality *Nucl. Phys.* B **424** 155–83
[39] Elitzur S, Giveon A, Rabinovici E, Schwimmer A and Veneziano G 1995 Remarks on non-Abelian duality *Nucl. Phys.* B **435** 147–71
[40] Arutyunov G, Frolov S, Hoare B, Roiban R and Tseytlin A A 2016 Scale invariance of the η-deformed $AdS_5 \times S^5$ superstring, T-duality and modified type II equations *Nucl. Phys.* B **903** 262–303
[41] Wulff L and Tseytlin A A 2016 Kappa-symmetry of superstring sigma model and generalized 10d supergravity equations *J. High Energy Phys.* JHEP06(2016)174
[42] Wulff L 2018 Trivial solutions of generalized supergravity versus non-Abelian T-duality anomaly *Phys. Lett.* B **781** 417–22
[43] Borsato R and Wulff L 2016 Target space supergeometry of η and λ-deformed strings *J. High Energy Phys.* JHEP10(2016)045
[44] Araujo T, Colgáin E, Sakatani Y, Sheikh-Jabbari M M and Yavartanoo H 2019 Holographic integration of $T\bar{T}$ & $J\bar{T}$ via $O(d,d)$ *J. High Energy Phys.* JHEP03(2019)168
[45] Giveon A, Itzhaki N and Kutasov D 2017 $\mathrm{T\bar{T}}$ and LST *J. High Energy Phys.* JHEP07(2017)122
[46] Forste S 1994 A Truly marginal deformation of SL(2, R) in a null direction *Phys. Lett.* B **338** 36–9
[47] Israel D, Kounnas C and Petropoulos M P 2003 Superstrings on NS5 backgrounds, deformed AdS(3) and holography *J. High Energy Phys.* JHEP10(2003)028
[48] Chakraborty S, Giveon A and Kutasov D 2018 $J\bar{T}$ deformed CFT2 and string theory *J. High Energy Phys.* JHEP10(2018)057
[49] Apolo L and Song W 2018 Strings on warped AdS_3 via $\mathrm{T\bar{J}}$ deformations *J. High Energy Phys.* JHEP10(2018)165
[50] Cagnazzo A and Zarembo K 2012 B-field in AdS_3/CFT_2 Correspondence and Integrability *J. High Energy Phys.* JHEP11(2012)133
[51] Sundin P and Wulff L 2012 Classical integrability and quantum aspects of the AdS(3) $\times$ S(3) $\times$ S(3) $\times$ S(1) superstring *J. High Energy Phys.* JHEP10(2012)109
[52] Azeyanagi T, Hofman D M, Song W and Strominger A 2013 The Spectrum of Strings on Warped $AdS_3 \times S^3$ *J. High Energy Phys.* JHEP04(2013)078
[53] Delduc F, Magro M and Vicedo B 2015 Integrable double deformation of the principal chiral model *Nucl. Phys.* B **891** 312–21
[54] Delduc F, Hoare B, Kameyama T and Magro M 2017 Combining the bi-Yang–Baxter deformation, the Wess–Zumino term and TsT transformations in one integrable σ-model *J. High Energy Phys.* JHEP10(2017)212
[55] Delduc F, Hoare B, Kameyama T, Lacroix S and Magro M 2019 Three-parameter integrable deformation of $\mathbb{Z}_4$ permutation supercosets *J. High Energy Phys.* JHEP01(2019)109

[56] Klimcik C and Severa P 1996 Poisson–Lie T duality and loop groups of Drinfeld doubles *Phys. Lett.* B **372** 65–71

[57] Stern A 1999 Hamiltonian approach to Poisson Lie T—duality *Phys. Lett.* B **450** 141–8

[58] Klimcik C 2015 η and λ deformations as $\mathcal{E}$-models *Nucl. Phys.* B **900** 259–72

[59] Demulder S, Hassler F and Thompson D C 2019 Doubled aspects of generalised dualities and integrable deformations *J. High Energy Phys.* JHEP02(2019)189

[60] Smirnov F A and Zamolodchikov A B 2017 On space of integrable quantum field theories *Nucl. Phys.* B **915** 363–83

[61] Cavaglià A, Negro S, Szécsényi I M and Tateo R 2016 $T\bar{T}$-deformed 2D quantum field theories *J. High Energy Phys.* JHEP10(2016)112

[62] Guica M 2018 An integrable Lorentz-breaking deformation of two-dimensional CFTs *SciPost Phys.* **5** 048

[63] Stolin A 1999 Rational solutions of the classical Yang–Baxter equation and quasi frobenius lie algebras *J. Pure Appl. Algebra* **137** 285–93

[64] Chu B-Y 1974 Symplectic homogeneous spaces *Trans. Am. Math. Soc.* **197** 145–59

[65] Ovando G 2006 Four dimensional symplectic lie algebras *Beitr. Algebra Geometrie* **47** 419–34

[66] Stolin A 1991 Constant solutions of Yang–Baxter equation for $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ *Math. Scand.* **69** 81–8

[67] Burde D and Moens W A 2016 Commutative post-Lie algebra structures on lie algebras *J. Algebra* **467** 183–201

Published for SISSA by Springer

Received: *June 26, 2018*
Accepted: *August 6, 2018*
Published: *August 8, 2018*

Non-abelian T-duality and Yang-Baxter deformations of Green-Schwarz strings

Riccardo Borsato[a] **and Linus Wulff**[b]

[a] *Nordita, Stockholm University and KTH Royal Institute of Technology, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden*

[b] *Department of Theoretical Physics and Astrophysics, Masaryk University, 611 37 Brno, Czech Republic*

E-mail: riccardo.borsato@su.se, wulff@physics.muni.cz

Abstract: We perform non-abelian T-duality for a generic Green-Schwarz string with respect to an isometry (super)group G, and we derive the transformation rules for the supergravity background fields. Specializing to G bosonic, or G fermionic but abelian, our results reproduce those available in the literature. We discuss also continuous deformations of the T-dual models, obtained by adding a closed B-field before the dualization. This idea can also be used to generate deformations of the original (un-dualized) model, when the 2-cocycle identified from the closed B is invertible. The latter construction is the natural generalization of the so-called Yang-Baxter deformations, based on solutions of the classical Yang-Baxter equation on the Lie algebra of G and originally constructed for group manifolds and (super)coset sigma models. We find that the deformed metric and B-field are obtained through a generalization of the map between open and closed strings that was used also in the discussion by Seiberg and Witten of non-commutative field theories. When applied to integrable sigma models these deformations preserve the integrability.

Keywords: Sigma Models, String Duality, Superstrings and Heterotic Strings, Integrable Field Theories

ArXiv ePrint: 1806.04083

https://doi.org/10.1007/JHEP08(2018)027

JHEP08(2018)027

Contents

1 Introduction

While ordinary abelian T-duality is an exact symmetry of string perturbation theory, its non-abelian generalization [1] is not [2, 3]. It should be rather viewed as a solution-generating technique in supergravity, since it (typically) maps one string background to another, inequivalent one. Starting with the work of [4], which gave a prescription for the transformation of the RR fields, it has been successfully applied to construct several interesting supergravity solutions, e.g. [5–10].

Like its abelian version, non-abelian T-duality (NATD) can be understood as a canonical transformation [11–13], so that the dualization preserves the (classical) integrability of the sigma model (when present). To be more precise, starting from a sigma model whose equations of motion are equivalent to the flatness of a Lax connection, one obtains a dual model whose equations of motion can also be put into Lax form. Here we want to exploit this property in order to generate integrable deformations of sigma models, following the ideas of [14–16].[1] The deformations are interesting also because they (partially) break the initial isometries. We remark that integrability is not essential for the construction, and the deformations can be carried out also for non-integrable models. Some of the deformations constructed here may be viewed as continuous interpolations between the "original" model and the "dual" one obtained after applying NATD.

Starting from a *generic* type II Green-Schwarz superstring whose isometries contain a (super)group G, we work out the transformation rules for the supergravity background fields under NATD with respect to G. The derivation is performed in section 3, where all orders in fermions are taken into account by working in superspace. When choosing a bosonic G and focusing on the bosonic supergravity fields, the transformation rules reproduce those of [4, 20], including the Ramond-Ramond (RR) fields whose transformations were conjectured by analogy with the abelian case [21]. Moreover, when the Lie algebra of G consists of only (anti)commuting fermionic generators, we also reproduce the rules for fermionic T-duality derived in [22] from the pure spinor string. As expected, we show that after NATD one still obtains a kappa symmetric Green-Schwarz superstring. It follows from [23] that the target space is therefore a solution of the generalized supergravity equations of [23, 24]. When G is unimodular (i.e. the structure constants of its Lie algebra satisfy $f_{IJ}^{J} = 0$) the background fields satisfy the standard type II supergravity equations, and the (dualized) sigma model is Weyl invariant. When G is not unimodular there is typically an anomaly which breaks Weyl invariance and obstructs the interpretation of the dual model as a string [25, 26]. We will also discuss exceptions to this, given by the "trivial solutions" of [27].

Deformations of the non-abelian T-dual backgrounds may be generated by adding a closed B-field before dualizing. The deformation will be controlled by one or more continuous parameters that enter the definition of this B. From the point of view of the original model, adding a B-field with $dB = 0$ does not affect the local physics, since this term does not change the equations of motion. We will nevertheless obtain a non-trivial deformation and a dependence on B in the equations of motion after applying NATD, since this transformation involves a *non-local* field redefinition.[2] Writing $B = \frac{1}{2}(g^{-1}dg)^J \wedge (g^{-1}dg)^I \, \omega_{IJ}$ with $g \in G$, the condition $dB = 0$ is equivalent to ω being a 2-cocycle on the Lie algebra of G. The resulting models were dubbed deformed T-dual (DTD) models in [15], and we refer to section 4.1 for more details.

In [15, 16] it was proved that a DTD model constructed from a principal chiral model (PCM) or supercoset sigma model with ω invertible is actually equivalent (thanks to a local field redefinition) to the so-called Yang-Baxter (YB) sigma models [28–33] based on an R-matrix solving the classical Yang-Baxter equation.[3] The R-matrix is related to

[1] Another class of integrable deformations related to NATD are the so-called λ-deformations of [17–19].

[2] If B is not just closed but also exact, it contributes to the action of the original model as a total derivative and it can be dropped. Even if kept, the dependence on this B can be removed by a (local) field redefinition even after applying NATD. Therefore an exact B generates a trivial deformation of the dual model.

[3] These are sometimes called "homogeneous" YB models. In the "inhomogeneous" YB models R solves the *modified* classical Yang-Baxter equation. They were first introduced in [28, 29] and later generalized to the supercoset case in [34], where the so-called η-deformation of $AdS_5 \times S^5$ was constructed. The inhomogeneous YB models are not related in such a simple way to NATD and we will not consider them further here.

the 2-cocycle simply as $R = \omega^{-1}$. The equivalence was first proposed and checked on various examples in [14].[4] When the R-matrix acts only on an abelian subalgebra YB deformations are simply TsT (T-duality-shift-T-duality) transformations [36], so that we can think of YB deformations as the "non-abelian" generalization of TsT transformations. Here we propose to use the connection to NATD in order to extend the applicability of YB deformations, from just PCM and supercoset models to a generic sigma model with isometries. We do this in section 4.2 by carrying out the field redefinition which leads from the DTD model to the YB model in the case of invertible ω. Although the construction comes from a deformation of the dual model, when sending the continuous deformation parameter of the YB model to zero we recover the *original* model. These deformations may be particularly interesting for the AdS/CFT correspondence, and in section 4.3.1 we use our results to "uplift" a YB deformation of $\mathrm{AdS}_5 \times S^5$ — that cannot be interpreted as (a sequence of) TsT transformations — to a deformation of the full D3-brane background, of which $\mathrm{AdS}_5 \times S^5$ is the near-horizon limit.

For YB deformations of the PCM or (super)coset models, it is easy to see that the background metric and B-field are related to the metric of the original model by a map that coincides with the open/closed string map used also by Seiberg and Witten in [37]. For YB deformations the open string non-commutativity parameter is identified with the R-matrix itself [38]. Based on this observation it was suggested in [39] that this map could be used to generate solutions to (generalized) supergravity.[5] Our results, based on the construction of [15], generalize this to cases with a non-vanishing B-field in the original model. Our derivation also ensures that the YB backgrounds are automatically solutions of the (generalized) supergravity equations. Yet another approach to such general (homogeneous) YB deformations was proposed in the context of doubled field theory, since known YB deformations were shown to be equivalent to so-called β-shifts [41–43]. In section 4.3.2 we check in an example that a recent solution generated in [43] coincides with the one obtained from our method based on NATD.

In the next section we collect the transformation rules for the background fields under NATD and under a generic YB deformation.

2 Summary of the transformation rules

In this section we wish to present and summarize in a self-contained way the transformation rules derived in the paper, so that the reader may consult them without the need of going through the whole derivation.

2.1 Rules of (bosonic) NATD

Here we summarize the NATD transformation rules for the bosonic supergravity fields only, when we take G to be an ordinary (i.e. non-super) Lie group. The general transformations

[4] An equivalent construction, applying NATD on a centrally extended algebra, was used there. See also [35].

[5] In [40] it was shown that the map generates solutions of the generalized supergravity equations if the non-commutativity parameter satisfies the classical Yang-Baxter equation.

can be found in section 3 (for the case of G a supergroup see footnote 11). It is convenient to rewrite the background fields in a way that makes the G isometry manifest. The metric, for example, will be written in the following block form

$$G_{\mu\nu} = \begin{pmatrix} G_{mn} & G_{mj} \\ G_{in} & G_{ij} \end{pmatrix}, \qquad G_{in} = \ell_i^I G_{In}, \qquad G_{ij} = \ell_i^I \ell_j^J G_{IJ}\,. \tag{2.1}$$

We have chosen coordinates such that we can split indices into (i, m), where i takes $\dim G$ values and m labels the remaining spectator fields which do not transform under G. We have collected our conventions in appendix A. It is also convenient to rewrite certain blocks by extracting ℓ_i^I, defined by $g^{-1}\partial_i g = \ell_i^I T_I$, where $g \in G$ and $I = 1, \dots, \dim G$ is an index in $\mathfrak{g}$ (the Lie algebra of G) so that $[T_I, T_J] = f_{IJ}^K T_K$. The dependence on the coordinates x^i (i.e. the coordinates to be dualized) is all in ℓ_i^I, so that G_{IJ}, G_{Im}, G_{mn} only depend on the spectators x^m. The transformation rules will be presented in terms of these objects, and we will continue to call them "metric" and "B-field" also when writing them with indices (m, I) instead of (m, i). In order to have a uniform derivation and presentation, we do not restrict further the range of the index I even when a local symmetry is present.[6] We refer to section 3 for more details. Setting fermions to zero the transformation rules for the metric and B-field in (3.6)–(3.8) read[7]

$$\tilde{G}_{mn} = G_{mn} - \big[(G-B)N(G-B)\big]_{(mn)}\,, \tag{2.2}$$

$$\tilde{G}_{mI} = \frac{1}{2}\big[(G-B)N\big]_{mI} - \frac{1}{2}\big[N(G-B)\big]_{Im}\,, \qquad \tilde{G}_{IJ} = N_{(IJ)}\,,$$

$$\tilde{B}_{mn} = B_{mn} + \big[(G-B)N(G-B)\big]_{[mn]}\,, \tag{2.3}$$

$$\tilde{B}_{mI} = -\frac{1}{2}\big[(G-B)N\big]_{mI} - \frac{1}{2}\big[N(G-B)\big]_{Im}\,, \qquad \tilde{B}_{IJ} = -N_{[IJ]}\,,$$

where $N_{IJ} = \delta_{IK} N^{KL} \delta_{LJ}$ etc. and

$$N^{IJ} = \big(G_{IJ} - B_{IJ} - \nu_K f_{IJ}^K\big)^{-1}\,. \tag{2.4}$$

The transformation of the RR fields, encoded in the bispinor (for more on the conventions see [23, 44])

$$\mathcal{S}^{12} = \begin{cases} \mathcal{F}^{(0)} - \frac{1}{2}\mathcal{F}^{(2)}_{ab}\Gamma^{ab} + \frac{1}{4!}\mathcal{F}^{(4)}_{abcd}\Gamma^{abcd} & \text{IIA} \\ -\mathcal{F}^{(1)}_a\Gamma^a - \frac{1}{3!}\mathcal{F}^{(3)}_{abc}\Gamma^{abc} - \frac{1}{2\cdot 5!}\mathcal{F}^{(5)}_{abcde}\Gamma^{abcde} & \text{IIB} \end{cases}, \tag{2.5}$$

given in (3.30) is given by the action of a Lorentz transformation $\Lambda \in O(1,9)$ as

$$\tilde{\mathcal{S}}^{12} = \hat{\Lambda}\mathcal{S}^{12}\,, \qquad \Lambda^{ab} = \eta^{ab} - 2E_I{}^a N^{IJ} E_J{}^b\,, \tag{2.6}$$

[6] Therefore the range of (m, I) can exceed ten. Both the original and the final action are still written only in terms of ten physical coordinates thanks to the local symmetry that survives NATD and removes the additional degrees of freedom, see the discussion in section 3.1.

[7] The coordinates ν_I that result from the dualization naturally have lower indices, since they parameterize the dual space. To have the standard upper placement of indices also in the dualized model we declare that those indices are raised with the Kronecker delta $\nu^I = \delta^{IJ}\nu_J$, and the total set of coordinates is (x^m, ν^I).

where $G_{IJ} = E_I{}^a E_J{}^b \eta_{ab}$, and we denote by $\hat{\Lambda}$ the Lorentz transformation acting on spinor indices that multiplies $\mathcal{S}^{12}$, defined such that $\Lambda^a{}_b \Gamma^b = \hat{\Lambda}^T \Gamma^a \hat{\Lambda}$. Finally the generalized supergravity fields K and X given in (3.17)–(3.18) become[8]

$$K^m = 0\,, \quad K^I = n^I\,, \quad X_m = \partial_m \left(\phi + \frac{1}{2} \ln \det N\right) - \tilde{B}_{mI} n^I\,, \quad X_I = -\tilde{B}_{IJ} n^J\,. \tag{2.7}$$

They involve the trace of the structure constants[9] of $\mathfrak{g}$, $n^I = \delta^{IJ} n_J$ with $n_I = f_{IJ}^J$. As already mentioned, in the generic case we must write the results in terms of the fields K and X. Indeed when $n_I \neq 0$ the background solves the generalized type II supergravity equations [23, 24] but not the standard ones, and the sigma model is scale but not Weyl invariant at one loop. When $\mathfrak{g}$ is unimodular, $n_I = 0$ and we get a solution of standard type II supergravity consistent with the results of [25, 26]. In that case, since X is a total derivative we can write $X = d\tilde{\phi}$ in terms of a dual dilaton

$$\tilde{\phi} = \phi + \frac{1}{2} \ln \det N\,. \tag{2.8}$$

It was shown in [27] that there exist special "trivial" solutions of the generalized supergravity equations which solve the standard supergravity equations although K is not zero. For this to happen K must be null and, in addition to a condition involving the RR fields which we ignore here, it should satisfy $dK = i_K H$. Using the rules of NATD presented here the latter condition can be written as

$$n(\tilde{G} - \tilde{B}) = 0\,. \tag{2.9}$$

Since $(\tilde{G} - \tilde{B})_{IJ} = N_{IJ}$ is invertible by assumption, it has no zero-eigenvector and therefore it would seem that no trivial solution can be generated by NATD. However, the condition written above is not invariant with respect to B-field gauge transformations, so that the conclusion can change. This will actually play a role in the discussion of the closely related YB models.

2.2 Rules of YB deformations

For YB deformations the rules are a bit simpler in the sense that we do not have to write the background fields in the block-form as previously. The result can be phrased in different ways, see section 4.2. Here we will describe the results in terms of Killing vectors of the original background. The final result of our derivation is that in order to apply a YB deformation one should first construct

$$\Theta^{\mu\nu} = k_I^\mu R^{IJ} k_J^\nu\,, \tag{2.10}$$

where R^{IJ} solves the classical Yang-Baxter equation (4.6) and k_I^μ are a collection of Killing vectors labeled by I that are properly normalized so that they satisfy (4.24). Then the

[8]Here we drop the tilde since these fields are not present before dualization. Also note that we have raised the index on K with $\tilde{G}^{-1}$ in order to get a simpler expression. We assume the original dilaton ϕ to respect the G isometry, so that is depends only on the spectators, but this assumption can be relaxed.

[9]The identification of K with the trace of the structure constants was suggested earlier in [45].

background metric and B-field of the YB model are simply obtained by the following generalization of the open/closed string map

$$\tilde{G} - \tilde{B} = (G - B)[1 + \eta\Theta(G - B)]^{-1}\,, \tag{2.11}$$

where we have omitted indices μ, ν. The RR bispinor transforms as

$$\tilde{\mathcal{S}}^{12} = \hat{\Lambda}\mathcal{S}^{12}\,, \qquad \Lambda^{ab} = \eta^{ab} - 2\eta E_\mu{}^a \hat{N}^\mu{}_\nu \Theta^{\nu\rho} E_\rho{}^b\,, \tag{2.12}$$

where $\hat{N}^\nu{}_\mu = \left[\delta^\mu{}_\nu + \eta\Theta^{\mu\rho}(G_{\rho\nu} - B_{\rho\nu})\right]^{-1}$. We further have

$$K^\mu = \eta\Theta^{\mu\nu} n_\nu\,, \qquad X_\mu = \partial_\mu\left(\phi - \frac{1}{2}\ln\det[1 + \eta\Theta(G - B)]\right) - \eta\tilde{B}_{\mu\nu}\Theta^{\nu\rho}n_\rho\,, \tag{2.13}$$

and, when the Killing vectors used to construct Θ define a unimodular algebra $f_{IJ}^J = 0$, we find the deformed dilaton

$$\tilde{\phi} = \phi - \frac{1}{2}\ln\det[1 + \eta\Theta(G - B)]\,. \tag{2.14}$$

We refer to section 4.2 for the derivation and a discussion of trivial solutions for YB deformations.

3 NATD of Green-Schwarz strings

In this section we apply NATD to a generic Green-Schwarz string with isometries. To perform NATD we assume that we can bring the supervielbein to the form

$$E^A = (g^{-1}dg)^I E_I{}^A(z) + dz^M E_M{}^A(z)\,, \qquad (A = (a, \alpha)\,, \quad a = 0, \ldots, 9\,, \quad \alpha = 1, \ldots, 32)\,, \tag{3.1}$$

with $g \in G$ encoding the coordinates we want to dualize and $z^M = (x^m, \theta^{\underline{\alpha}})$ denoting the remaining (spectator) coordinates. The isometry (sub)group G to be dualized acts as $g \to ug$, $z \to z$ for a constant element $u \in G$. To avoid extra awkward signs, we will present the derivation when G is an ordinary Lie group, but we will write the end result for the dualized geometry such that it applies also to the case when G is a super Lie group. The index I takes $\dim G$ values and since we want to include the case in which a local symmetry of the sigma model (which we do not fix) is a subgroup of G, we allow the possibility that the total range of indices (m, I) is greater than ten. In that case the local symmetry can be used at the end to remove the spurious coordinates and leave the ten physical ones. In that case $E_I{}^a$ also involves a projection matrix [20], the simplest example being a supercoset geometry where $E_I{}^a$ is proportional to the projector on the coset directions (usually denoted by $P^{(2)}$).

The (classical) Green-Schwarz string action is

$$S = T\int_\Sigma \left(\frac{1}{2}E^a \wedge *E^b\eta_{ab} + B\right)\,, \tag{3.2}$$

where we are using worldsheet form notation and the supervielbein E^a and NSNS two-form potential B are understood to be pulled back to the worldsheet Σ. To perform NATD we write this action in first order form using (3.1), replacing $g^{-1}dg \to A$ and adding a Lagrange multiplier term to enforce the flatness of A

$$S' = \frac{T}{2}\int_\Sigma \Big(A^I \wedge (G_{IJ} * - B_{IJ}) A^J + 2dz^M \wedge (G_{MI} * - B_{MI}) A^I \\ + (-1)^{\deg N} dz^M \wedge (G_{MN} * - B_{MN}) dz^N + \nu_I (2dA^I - f^I_{JK} A^J \wedge A^K) \Big) . \quad (3.3)$$

The components of the (super) metric are $G_{IJ} = E_I{}^a E_J{}^b \eta_{ab}$, $G_{IM} = G_{MI} = E_I{}^a E_M{}^b \eta_{ab}$ and $G_{MN} = E_M{}^a E_N{}^b \eta_{ab}$. Integrating out A gives[10]

$$(1 \pm *) A^I = -(1 \pm *) \left(d\nu_J + dz^M [\mp G - B]_{MJ} \right) N^{JI}_\mp , \qquad N^{IJ}_\pm = \left(\pm G_{IJ} - B_{IJ} - \nu_K f^K_{IJ} \right)^{-1} \quad (3.4)$$

and the dual action

$$\begin{aligned} \tilde{S} &= \frac{T}{4} \int_\Sigma \Big\{ \left(d\nu_I + dz^M [G - B]_{MI} \right) N^{IJ}_+ \wedge (1 + *) \left(d\nu_J - dz^M [G + B]_{MJ} \right) \\ &\qquad + \left(d\nu_I - dz^M [G + B]_{MI} \right) N^{IJ}_- \wedge (1 - *) \left(d\nu_J + dz^M [G - B]_{MJ} \right) \\ &\qquad + 2(-1)^N dz^M \wedge (G_{MN} * - B_{MN}) dz^N \Big\} \\ &= T \int_\Sigma \left(\frac{1}{2} \tilde{E}^a_\pm \wedge * \tilde{E}^b_\pm \eta_{ab} + \tilde{B} \right) . \end{aligned} \quad (3.5)$$

In the last step we have written the dualized action in Green-Schwarz form by defining two possible sets of dual supervielbeins[11]

$$\tilde{E}^A_\pm = dz^M E_M{}^A - \left(d\nu_I + dz^M [\mp G - B]_{MI} \right) N^{IJ}_\mp E_J{}^A . \quad (3.6)$$

The dual B-field can also be written in two equivalent ways

$$\tilde{B} = \frac{1}{2} dz^N \wedge dz^M B_{MN} + \frac{1}{2} \left(d\nu_I + dz^M [\pm G - B]_{MI} \right) \wedge N^{IJ}_\pm \left(d\nu_J - dz^M [\pm G + B]_{MJ} \right) . \quad (3.7)$$

We choose $\tilde{E}^a_+$ to be the dual bosonic supervielbein, while $\tilde{E}^a_-$ is related to it by a Lorentz transformation as follows

$$\tilde{E}^a = \tilde{E}^a_+ , \qquad \tilde{E}'^a = \tilde{E}^a_- = \tilde{E}^b \Lambda_b{}^a , \qquad \Lambda_b{}^a = \delta^a_b - 2 E_{Ib} N^{IJ}_+ E_J{}^a . \quad (3.8)$$

This is easily seen to follow from the useful identity

$$\left(d\nu_I + dz^M [G - B]_{MI} \right) N^{IJ}_+ = \left(d\nu_I - dz^M [G + B]_{MI} \right) N^{IJ}_- + 2 \tilde{E}^a E_{Ia} N^{IJ}_+ . \quad (3.9)$$

[10] These solutions and the following action are written so that they hold also when G is a supergroup.

[11] When G is a supergroup the correct expressions are obtained by writing things in a form which is symmetric between N_+ and N_- (and where contracted indices are adjacent), e.g.

$$\tilde{E}^a_\pm = dz^M E_M{}^a - \frac{1}{2} \left(d\nu_I + dz^M [\mp G - B]_{MI} \right) N^{IJ}_\mp E_J{}^a + \frac{1}{2} E_I{}^a N^{IJ}_\pm \left(d\nu_J + dz^M [\mp G - B]_{MJ} \right) .$$

This will be true also for the expressions for Λ, K, X and $\tilde{\mathcal{S}}$ to be derived below.

It is interesting to compute the determinant of the Lorentz transformation Λ. We have (suppressing the indices)

$$\begin{aligned}\det\Lambda &= \exp(\mathrm{tr}\,\ln\Lambda) = \exp\left(-\mathrm{tr}\sum_{n=1}^{\infty}\frac{(2EN_+E)^n}{n}\right) = \exp\left(-\mathrm{tr}\sum_{n=1}^{\infty}\frac{(2GN_+)^n}{n}\right)\\ &= \exp[\mathrm{tr}\,\ln(1-2GN_+)] = \exp[\mathrm{tr}\,\ln(1-(N_+^{-1}+N_+^{-T})N_+)] = \det(-N_+^{-T}N_+)\\ &= (-1)^{\dim G}.\end{aligned} \tag{3.10}$$

This shows that this Lorentz transformation is an element of SO(1, 9) only when $\dim G$ is even. When $\dim G$ is odd, i.e. one dualizes on an odd number of directions, the Lorentz transformation involves a reflection. In the latter case its action on spinors contains an odd number of gamma matrices, which means that one goes from type IIA to type IIB or vice versa, cf. (3.28).

3.1 The case with local symmetry

Here we wish to give more details on the case when the original sigma model has a local symmetry that is a subgroup of G. We will explain how the results of the previous section apply also in that case. We will assume that the action (3.2) is invariant under a local group $H \subset G$ that acts on g from the right as[12] $g \to gh$, $h \in H$. Our goal will be to show that if the local H invariance is not fixed before the dualization, NATD can still be applied in the usual way and the dual action naturally inherits the local H symmetry. Therefore this ensures that the additional degrees of freedom can be removed also in the dual model, and that we are left only with physical ones of the correct number.

The action (3.2) is invariant under $g \to gh$ if the couplings are H invariant and project out $\mathfrak{h}$, the Lie algebra of H

$$\begin{aligned}(\mathrm{Ad}_h^{-1})^K{}_I(G_{KL}*-B_{KL})(\mathrm{Ad}_h^{-1})^L{}_J = G_{IJ}*-B_{IJ}, &\qquad y^I(G_{IJ}*-B_{IJ}) = 0 = (G_{IJ}*-B_{IJ})y^J,\\ (G_{MJ}*-B_{MJ})(\mathrm{Ad}_h^{-1})^J{}_I = G_{MI}*-B_{MI}, &\quad (G_{MI}*-B_{MI})y^I = 0.\end{aligned} \tag{3.11}$$

Here $y \in \mathfrak{h}$. This local symmetry may be used to remove $\dim H$ degrees of freedom from the parametrization of g, so that the total number of physical bosonic fields (including spectators) is ten. We do not fix this local invariance yet, since this allows us to gauge the whole G isometry and fix the gauge $g = 1$ to arrive at the action (3.3). This first order action is still invariant under a local H which is now implemented as

$$A \to h^{-1}Ah + h^{-1}dh, \qquad\qquad \nu \to h^{-1}\nu h\,. \tag{3.12}$$

Here $\nu = \nu_I T^I$ is taken to be an element of $\mathfrak{g}^*$, the dual of the Lie algebra of G. We refer to section 4.1 for our conventions regarding $\mathfrak{g}^*$. At the moment of integrating out A^I from (3.3) one may worry about the invertibility of the relevant linear operators, given that the couplings project out the components in $\mathfrak{h}$ as assumed above. We consider cases when the operators $\pm G_{IJ} - B_{IJ} - \nu_K f_{IJ}^K$ are invertible on the whole algebra $\mathfrak{g}$, so that also the

[12] One may equivalently discuss this local invariance by introducing a vector valued in the Lie algebra of H, so that integrating out such vector the original action is obtained.

components of A in $\mathfrak{h}$ can be integrated out. Obviously, since $\pm G_{IJ} - B_{IJ}$ are degenerate, the invertibility of the operators must be ensured by the term $\nu_K f_{IJ}^K$. We recall that ν has not been gauged-fixed yet, and that we have a total of $\dim G$ such fields. In general the invertibility will hold only locally, meaning that there may be values of ν_I such that the operators $N_\pm^{IJ}$ become singular. Those loci will correspond to singularities in target space that we cannot remove. It is easy to check that the dual action (3.5) is still invariant under the local H symmetry, which is now simply implemented by $\nu \to h^{-1}\nu h$. We can then fix the local symmetry at the level of the dual action, at the same time making sure that we have the correct number of degrees of freedom and that the gauge fixing is done correctly.

Our reasoning is completely analogous to that of [20, 46]. There the degenerate matrices $\pm G_{IJ} - B_{IJ}$ are regulated by taking $\pm G_{IJ} - B_{IJ} + \lambda\,(\mathrm{Id}_{\mathfrak{h}})_{IJ}$, where $\mathrm{Id}_{\mathfrak{h}}$ is the identity on $\mathfrak{h}$. The parameter λ is kept during the dualization and sent to zero only at the end. It is clear that the $\lambda \to 0$ limit is non-singular only if the degeneracies of $\pm G_{IJ} - B_{IJ}$ are lifted by the additional term $\nu_K f_{IJ}^K$. Therefore the way coset models are treated in [20, 46] is analogous to ours. For concreteness we work out an explicit example in appendix B.

3.2 Extracting X and K from anomaly terms

The easiest way to extract the generalized supergravity fields X and K is to look at the terms in the action induced at the quantum level by the NATD change of variables $g^{-1}dg \to A$ in the path integral measure [26].[13] It was shown in [27] that these non-local terms take the form

$$\tilde{S}_\sigma = \frac{1}{2\pi}\int_\Sigma \left(d\sigma \wedge K - d\sigma \wedge *X - \frac{1}{2}\alpha' d\sigma \wedge *d\sigma\, |K|^2 \right) , \tag{3.13}$$

where $\sigma = \partial^{-2}\sqrt{g}R^{(2)}$ is the conformal factor. From the first two terms it is easy to read off X and K. To compute $\tilde{S}_\sigma$ we include the σ-dependent terms in the first order action (3.3). They are [26]

$$S_\sigma = \frac{1}{2\pi}\int_\Sigma \left(\sigma n_I d * A^I - \Phi d * d\sigma \right) , \tag{3.14}$$

where $n_I = f_{IJ}^J$, the trace of the structure constants, $d * d\sigma = d^2\xi\sqrt{g}R^{(2)}$ and Φ is the dilaton superfield of the original background. Integrating out A as before but now including these terms, and keeping track of the $\det N$ from the measure, we obtain

$$(1 \pm *)A^I = -(1 \pm *)\left(d\nu_J + dz^M [\mp G - B]_{MJ} \mp \alpha' n_J d\sigma \right) N_\mp^{JI} \tag{3.15}$$

and

$$\begin{aligned}\tilde{S}_\sigma = \frac{1}{4\pi}\int_\Sigma \Big(& n_I d\sigma N_+^{IJ} \wedge (1+*)(d\nu_J - dz^M [G+B]_{MJ}) \\ & - n_I d\sigma N_-^{IJ} \wedge (1-*)(d\nu_J + dz^M [G-B]_{MJ}) - 2 d\sigma \wedge * d\left(\Phi + \frac{1}{2}\ln\det N_+ \right) \Big) + \mathcal{O}(\alpha') .\end{aligned} \tag{3.16}$$

[13] A more direct, but lengthier, approach uses the superspace constraints as we do below to extract the RR fields, see for example [47].

Comparing to (3.13) we find

$$K=\frac{1}{2}\left\{(d\nu_J+dz^M[G-B]_{MJ})N_+^{JI}-(d\nu_J-dz^M[G+B]_{MJ})N_-^{JI}\right\}n_I\,, \tag{3.17}$$

$$X=d\left(\Phi+\frac{1}{2}\ln\det N_+\right)+\frac{1}{2}\left\{(d\nu_J+dz^M[G-B]_{MJ})N_+^{JI}+(d\nu_J-dz^M[G+B]_{MJ})N_-^{JI}\right\}n_I\,. \tag{3.18}$$

These expressions simplify when written in terms of $\tilde{G}$ and $\tilde{B}$ as in (2.7).

3.3 Extracting the RR fields

The simplest way to find the RR fields is to compute the superspace torsion $T^A=dE^A+E^B\wedge\Omega_B{}^A$ and compare to the superspace torsion constraints of [23, 44], see e.g. [47]. In particular the $E^a\wedge E^{\alpha 1}$-term in $T^{\alpha 2}$ takes the form[14]

$$T^2=-\frac{1}{8}E^a\,(E^1\Gamma_a\mathcal{S}^{12})+\ldots \tag{3.19}$$

from which we can read off the RR bispinor $\mathcal{S}$. Here E^a is the bosonic supervielbein and $E^{\alpha 1}$, $E^{\alpha 2}$ with $\alpha=1,\ldots,16$ are the two fermionic supervielbeins, corresponding to the two Majorana-Weyl spinors of type II supergravity. For convenience of the presentation we will use type IIA notation so that $E^1=\frac{1}{2}(1+\Gamma_{11})E^1$ and $E^2=\frac{1}{2}(1-\Gamma_{11})E^2$ but the type IIB expressions are essentially identical.

To compute $\tilde{T}^2$ and then extract the RR fields of the dualized model, we must first find the form of the fermionic supervielbeins $\tilde{E}^1$, $\tilde{E}^2$. We therefore start with the constraint on the bosonic torsion

$$T^a=-\frac{i}{2}E\Gamma^aE=-\frac{i}{2}E^1\Gamma^aE^1-\frac{i}{2}E^2\Gamma^aE^2\,, \tag{3.20}$$

and we can compute $\tilde{T}^a$ from $\tilde{E}^a$.[15] By assumption the constraint on T^a holds in the original model before dualization. In our adapted coordinates (3.1) it takes the form[16]

$$2\partial_{[M}E_{N]}{}^a+2\Omega_{[M|b|}{}^aE_{N]}{}^b=(-1)^N iE_M\Gamma^aE_N\,, \tag{3.21}$$

$$\partial_ME_I{}^a+\Omega_{Mb}{}^aE_I{}^b-\Omega_{Ib}{}^aE_M{}^b=iE_M\Gamma^aE_I\,, \tag{3.22}$$

$$f_{IJ}^KE_K{}^a+2\Omega_{[J|b|}{}^aE_{I]}{}^b=iE_J\Gamma^aE_I\,. \tag{3.23}$$

We will also need the constraints on $H=dB$ which are

$$H=-\frac{i}{2}E^a\,E\Gamma_a\Gamma_{11}E+\frac{1}{6}E^cE^bE^aH_{abc}=-\frac{i}{2}E^a\,E^1\Gamma_aE^1+\frac{i}{2}E^a\,E^2\Gamma_aE^2+\frac{1}{6}E^cE^bE^aH_{abc}\,. \tag{3.24}$$

In our adapted coordinates we have

$$\begin{aligned}3\partial_{[M}B_{NP]}&=H_{MNP}\,, & 2\partial_{[M}B_{N]I}&=H_{MNI}\,,\\ \partial_MB_{IJ}+f_{IJ}^KB_{MK}&=H_{MIJ}\,, & 3f_{[IJ}^LB_{K]L}&=H_{IJK}\,,\end{aligned} \tag{3.25}$$

[14]To improve the readability we suppress the spinor index α and drop the explicit $\wedge$'s from now on.

[15]It might appear that one needs to know the spin connection to do this but this is not the case. Instead the fermionic vielbeins and spin connection can be read off by computing $d\tilde{E}^a$ as we will see.

[16]The anti-symmetrization is graded, e.g. $Y_{[M}Z_{N]}=\frac{1}{2}(Y_MZ_N-(-1)^{MN}Y_NZ_M)$.

where $H_{IJK} = E_K{}^C E_J{}^B E_I{}^A H_{ABC}$ etc. Using these relations we can compute the exterior derivative of $\tilde{E}^a_\pm$ in (3.6) and we find

$$d\tilde{E}^a_\pm = -\frac{i}{2}\tilde{E}_\pm\Gamma^a\tilde{E}_\pm + \frac{i}{2}\tilde{E}_\pm\Gamma_b(1\pm\Gamma_{11})\tilde{E}_\pm(\pm E_I{}^a N_\pm^{IJ} E_J{}^b) - \tilde{E}^b_\pm\tilde{E}^C_\pm\Omega_{Cb}{}^a$$
$$\pm\, i\tilde{E}^b_\pm\,\tilde{E}_\pm\Gamma_b(1\mp\Gamma_{11})E_I N_\mp^{IJ} E_J{}^a \pm \tilde{E}^c_\pm\tilde{E}^b_\pm\left(\Omega_{Ibc}\pm\frac{1}{2}E_I{}^d H_{bcd}\right)N_\mp^{IJ}E_J{}^a\,. \qquad (3.26)$$

Using our definition of the dualized bosonic supervielbein, $\tilde{E}^a = \tilde{E}^a_+$, this can be recast, using the definition (3.6), as[17]

$$\tilde{T}^a = d\tilde{E}^a + \tilde{E}^b\tilde{\Omega}_b{}^a = -\frac{i}{2}\Lambda^a{}_b\tilde{E}^1_+\Gamma^b\tilde{E}^1_+ - \frac{i}{2}\tilde{E}^2_-\Gamma^a\tilde{E}^2_-\,. \qquad (3.27)$$

Comparing to the standard form (3.20) we can read off the fermionic supervielbeins of the dualized model[18]

$$\tilde{E}^1 = \hat{\Lambda}\tilde{E}^1_+\,, \qquad \tilde{E}^2 = \tilde{E}^2_-\,, \qquad (3.28)$$

where the action of the Lorentz transformation on spinors is defined by $\Lambda^a{}_b\Gamma^b = \hat{\Lambda}^T\Gamma^a\hat{\Lambda}$. We are now ready to compute the fermionic torsion and extract the dualized RR fields by comparing to (3.19). Following the same lines as above we find

$$d\tilde{E}^2_- = \frac{1}{4}(\Gamma_{ab}\tilde{E}^2_-)\,\tilde{E}^C_-\Omega_C{}^{ab} + \frac{1}{2}\tilde{E}^B_-\tilde{E}^A_- T^2_{AB} - i\tilde{E}^1_-\Gamma_a\tilde{E}^1_-(E^2_I N_-^{IJ} E_J{}^a)$$
$$-\,2i\tilde{E}^a_-\,\tilde{E}^1_-\Gamma_a E^1_I(N_+^{IJ}E^2_J) - \tilde{E}^b_-\tilde{E}^a_-\left(\Omega_{Iab}-\frac{1}{2}H_{abc}E_I{}^c\right)N_+^{IJ}E^2_J\,. \qquad (3.29)$$

Extracting the $\tilde{E}^a\tilde{E}^1$-terms we can read of the RR bispinor which takes the form

$$\tilde{\mathcal{S}}^{12} = \hat{\Lambda}\mathcal{S}^{12} + 16i\hat{\Lambda}E^1_I N_+^{IJ}E^2_J\,. \qquad (3.30)$$

The first term is a Lorentz transformation acting on one side of the original bispinor in agreement with the NATD transformation rules first proposed in [4], by analogy with the abelian case. The second term starts at quadratic order in fermions if one dualizes on a bosonic algebra. However, in cases involving fermionic T-dualities the bosonic background is affected by the second term. In the case of a single fermionic T-duality it reproduces the transformation rule derived in [22].[19]

[17]Note that (3.9) implies $\tilde{E}_- = \tilde{E}_+ - 2\tilde{E}^a E_{Ia}N_+^{IJ}E_J$.

[18]We also find the spin connection of the dualized background

$$\tilde{\Omega}^{ab} = \tilde{E}^C_+\Omega_C{}^{ab} - 4i\tilde{E}^2_+\Gamma^{[a}E^2_I N_-^{IJ}E_J{}^{b]} - 2\tilde{E}^c\left(\Omega_{Ic}{}^{[a}-\frac{1}{2}E_I{}^d H_{cd}{}^{[a}\right)N_-^{IJ}E_J{}^{b]} + \tilde{E}^c\left(\Omega_I{}^{ab}+\frac{1}{2}E_I{}^d H_d{}^{ab}\right)N_-^{IJ}E_{Jc}$$
$$-4i\tilde{E}^c E_I{}^{[a}N_+^{IJ}E^2_J\Gamma^{b]}E^2_K N_-^{KL}E_{Lc} - 2i\tilde{E}^c E_I{}^a N_+^{IJ}E^2_J\Gamma_c E^2_K N_-^{KL}E_L{}^b\,.$$

[19]In the pure spinor formalism used there one does not directly see the Lorentz transformation acting on half of the fermionic directions since the pure spinor description has a larger symmetry with independent Lorentz transformations for bosons and the two fermionic directions. However, setting the fermions to zero Λ becomes trivial and all transformations, including those of the RR fields, match.

To be sure that the sigma model after NATD still has kappa symmetry, or equivalently that the background solves the generalized supergravity equations [23], one must also verify that $\tilde{H} = d\tilde{B}$ satisfies the correct constraints (3.24) (up to dimension zero). A direct calculation using (3.7) and (3.9) shows that $\tilde{H}$ is indeed of the right form (3.24).[20] This proves that the dual model is indeed a Green-Schwarz string invariant under the standard kappa symmetry transformations, and it completes the derivation of the dualized target space fields which therefore solve the equations of (generalized) supergravity [23].

4 Deformations

NATD may be viewed as a solution-generating technique for supergravity backgrounds. Here we slightly modify the procedure to generate *continuous* deformations of the dual model, which will be called deformed T-dual (DTD) models. Later we will show that a subclass of DTD models may be recast in the form of a deformation that reduces to the original sigma model when sending the deformation parameter to zero. This subclass will be identified with a generalization of YB deformations.

4.1 Deformed T-dual models

In order to define DTD models, we start from the original sigma-model, before applying NATD, and we shift the B-field as

$$B_{IJ} \to B_{IJ} - \zeta\, \omega_{IJ}\,. \tag{4.1}$$

Here ω_{IJ} is constant and anti-symmetric in its indices. We use ζ as a parameter to keep track of the shift, or in other words the deformation. The shift affects only the components of the B-field along $\mathfrak{g}$, and it does not spoil the global G isometry. We demand that the new term appearing in the action (i.e. $\zeta(g^{-1}dg)^I \wedge \omega_{IJ}(g^{-1}dg)^J$) should not modify the theory *on-shell*, in other words that it should be a closed B-field. It is easy to see that this happens if and only if ω_{IJ} satisfies the 2-cocycle condition

$$\omega_{I[J} f_{KL]}^I = 0\,, \tag{4.2}$$

where the antisymmetrization involves all three indices J, K, L. We further demand that the B-field $\zeta(g^{-1}dg)^I \wedge \omega_{IJ}(g^{-1}dg)^J$ is closed but not exact, i.e. the shift should not be a gauge transformation. Thanks to this additional condition, after applying NATD the resulting deformation is non-trivial, i.e. the ζ-dependence cannot be removed by a field redefinition. The non-exactness of B is equivalent to ω_{IJ} not being a coboundary, i.e. $\omega_{IJ} \neq c_K f_{IJ}^K$ for any constant vector c_K. Non-trivial deformations are therefore classified by elements of the second Lie algebra cohomology group $H^2(\mathfrak{g})$.

[20] One also finds

$$\tilde{H}_{abc} = -\frac{1}{2} H_{abc} + \frac{3}{2}\Lambda_{[a}{}^d H_{bc]d} - 6E_{I[a} N_+^{IJ} \Omega_{|J|bc]} - 12i(E_{I[a} N_+^{IJ} E^2_{|J|})\Gamma_b(E_{|K|c]} N_+^{KL} E^2_L)\,.$$

We can view the 2-cocycle as an element of $\mathfrak{g}^* \otimes \mathfrak{g}^*$ by writing $\omega = \omega_{IJ} T^I \wedge T^J$. Alternatively we may view it as a map from $\mathfrak{g}$ to the dual vector space (we continue to call this ω without fear of creating confusion) $\omega : \mathfrak{g} \to \mathfrak{g}^*$, whose action is given by

$$\omega(T_K) = \omega_{IJ} T^I \mathrm{tr}(T^J T_K) = \omega_{IK} T^I \,. \tag{4.3}$$

To proceed further we will endow the dual vector space $\mathfrak{g}^*$ with a Lie algebra structure with structure constants $\tilde{f}_K^{IJ}$ so that $\mathfrak{g}$ has a bialgebra structure. Therefore $\mathfrak{g} \oplus \mathfrak{g}^*$ becomes a Lie algebra with Drinfel'd double commutation relations[21]

$$[T_I, T_J] = f_{IJ}^K T_K \,, \qquad [T^I, T^J] = \tilde{f}_K^{IJ} T^K \,, \qquad [T_I, T^J] = f_{KI}^J T^K + \tilde{f}_I^{JK} T_K \,. \tag{4.4}$$

This is always possible since we can always take $\mathfrak{g}^*$ to be abelian with $\tilde{f}_K^{IJ} = 0$. In general this construction is far from unique and there exist many possible choices of Lie algebra structure on $\mathfrak{g}^*$, however this choice will have no effect in what follows. The 2-cocycle condition (4.2) can now be written

$$\omega[T_I, T_J] = P^T([\omega T_I, T_J] + [T_I, \omega T_J]) \,, \tag{4.5}$$

where P^T projects on $\mathfrak{g}^*$. Note that if we take $\mathfrak{g}^*$ to be abelian we can drop the projector and this equation just says that ω is a derivation on the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$. This is the choice that is most useful for the general discussion here.[22]

Apart from the shift $B_{IJ} \to B_{IJ} - \zeta\omega_{IJ}$, nothing changes in the derivation of the transformation of the action and of the background fields under NATD. Therefore, the transformation rules derived in section 3 and presented in section 2.1 are valid also for DTD if we shift $B_{IJ} \to B_{IJ} - \zeta\omega_{IJ}$. The resulting DTD background is a deformation of the NATD background, and it reduces to it when $\zeta = 0$. We refer to [15, 16] for some explicit examples of DTD models obtained from PCM or from the superstring on $\mathrm{AdS}_5 \times S^5$.

4.2 Yang-Baxter deformations

We will now construct deformations of the original background, rather than its NATD. We introduce a deformation parameter η such that $\eta = 0$ gives back the original sigma model. These deformations will be obtained from the DTD construction, where we identify $\eta = \zeta^{-1}$. We identify them with Yang-Baxter deformations, since they are generated by solutions of the classical Yang-Baxter equation and they generalize the original construction for PCM and (super)cosets to generic (Green-Schwarz) sigma models.

The construction is possible when ω_{IJ} is invertible. Writing $R = \omega^{-1} : \mathfrak{g}^* \to \mathfrak{g}$ it is easy to verify that the 2-cocycle condition for ω implies that R solves the classical Yang-Baxter equation

$$[Rx, Ry] - R([Rx, y] + [x, Ry]) = 0 \,, \quad \forall x, y \in \mathfrak{g}^* , \qquad \Longleftrightarrow \qquad R^{L[I} R^{|M|J} f_{LM}^{K]} = 0 \,, \tag{4.6}$$

[21] This is very similar to how one realizes NATD as a special case of Poisson-Lie T-duality [48] and it would be interesting to consider the extension of our construction to the Poisson-Lie case.

[22] In the PCM case considered in [15] or the supercoset model case considered in [16] there is a natural Lie algebra structure on $\mathfrak{g}^*$, inherited from the full isometry group. This is the structure that was chosen in [15, 16]. Nevertheless, as already mentioned this choice has no consequence in our construction, and a more natural choice may be for example to take $\mathfrak{g}^*$ abelian.

where the action of the operator is again defined by $R(T^I) = T_K R^{KI}$. The above is equivalent to the more familiar form of the classical Yang-Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0\,, \tag{4.7}$$

written in terms of $r = R^{IJ} T_I \wedge T_J \in \mathfrak{g} \otimes \mathfrak{g}$, where the subscripts of r_{ij} denote the spaces in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ where it acts. To recast the DTD model as a deformation of the original model we need to replace the coordinates ν_I, which parametrize the dual space, by a group element $g \in G$. The invertible map $\omega : \mathfrak{g} \to \mathfrak{g}^*$ allows us to do this by writing [15]

$$\nu_I = \zeta \mathrm{tr}\left(T_I \frac{1 - \mathrm{Ad}_g^{-1}}{\log \mathrm{Ad}_g} \omega \log g \right)\,. \tag{4.8}$$

Using the 2-cocycle condition it can be shown that this implies[23]

$$d\nu_I = \eta^{-1} \left(R_g^{-1} (g^{-1} dg) \right)_I\,, \qquad \nu_K f_{IJ}^K = \eta^{-1} R_{IJ}^{-1} - \eta^{-1} (R_g^{-1})_{IJ}\,, \tag{4.9}$$

where $R_g = \mathrm{Ad}_g^{-1} R \mathrm{Ad}_g$. Using this in the definition of N^{IJ} in (2.4) we get

$$N = \eta R_g \left(1 + \eta (G - B) R_g\right)^{-1} = \eta \left(1 + \eta R_g (G - B)\right)^{-1} R_g\,. \tag{4.10}$$

With these substitution rules it is easy to check that the DTD action is recast into the following form[24]

$$\begin{aligned} S = \frac{T}{2} \int_\Sigma \Big(& (g^{-1} dg)^I \wedge (\tilde{G}_{IJ} * - \tilde{B}_{IJ}) (g^{-1} dg)^J + 2 dz^M \wedge (\tilde{G}_{MI} * - \tilde{B}_{MI}) (g^{-1} dg)^J \\ & + (-1)^N dz^M \wedge (\tilde{G}_{MN} * - \tilde{B}_{MN}) dz^N - \eta^{-1} (dg g^{-1})^I \wedge \omega_{IJ} (dg g^{-1})^J \Big)\,, \end{aligned} \tag{4.11}$$

where we isolated the last term which does not behave well in the $\eta \to 0$ limit. This term is again a closed B-field thanks to the 2-cocycle condition satisfied by ω, and therefore it does not contribute to the equations of motion. We define the action of the YB model as the above one where the closed $B = \eta^{-1} (dg g^{-1})^I \wedge \omega_{IJ} (dg g^{-1})^J$ is removed. Dropping it we do not modify the on-shell theory, so that if the original model is classically integrable this property is inherited also by the YB deformation. In this way we can also achieve a non-singular $\eta \to 0$ limit, which yields the original undeformed model as is clear from the expressions given below. This also implies that YB deformations may be viewed as interpolations between the original model (obtained just by sending $\eta \to 0$) and the dual one (which is recovered in the equivalent DTD formulation after sending $\zeta \to 0$, which is $\eta \to \infty$).

[23] The easiest way to show this is to extend ω to act as a derivation on the universal enveloping algebra of $\mathfrak{g}$. With this definition we can write $\eta\nu = g^{-1}\omega(g) \in \mathfrak{g}^*$. We can now compute $d\nu$ and the two equivalent expressions $\omega(dg) = \omega(g g^{-1} dg) = \eta g \nu g^{-1} dg + g \omega(g^{-1} dg)$ and $\omega(dg) = \omega(dg g^{-1} g) = \omega(dg g^{-1}) g + \eta dg \nu$. This gives us the two equations.

[24] We still use tilde to denote transformed metric and B-field, but now they differ from the ones of NATD. The transformations rules are given below.

Setting fermions to zero and assuming a bosonic group G, we then read off

$$\tilde{G}_{mn} = G_{mn} - \eta\big[(G-B)\hat{N}R_g(G-B)\big]_{(mn)}\,, \tag{4.12}$$

$$\begin{aligned}
\tilde{G}_{mI} &= \frac{1}{2}\big[(G-B)\hat{N}\big]_{mI} + \frac{1}{2}\big[\check{N}(G-B)\big]_{Im}\,, \qquad \tilde{G}_{IJ} = \big[(G-B)\hat{N}\big]_{(IJ)}\,,\\
\tilde{B}_{mn} &= B_{mn} + \eta\big[(G-B)\hat{N}R_g(G-B)\big]_{[mn]}\,,\\
\tilde{B}_{mI} &= -\frac{1}{2}\big[(G-B)\hat{N}\big]_{mI} + \frac{1}{2}\big[\check{N}(G-B)\big]_{Im}\,, \qquad \tilde{B}_{IJ} = -\big[(G-B)\hat{N}\big]_{[IJ]}\,,
\end{aligned} \tag{4.13}$$

while the RR bispinor is again transformed by a Lorentz transformation $\hat{\Lambda}$ acting on spinor indices from the left[25]

$$\tilde{\mathcal{S}}^{12} = \hat{\Lambda}\mathcal{S}^{12}\,, \qquad \Lambda^{ab} = \eta^{ab} - 2\eta E_I{}^a \hat{N}^I{}_J (R_g)^{JK} E_K{}^b\,. \tag{4.15}$$

In the above we have also defined

$$\begin{aligned}
\hat{N}^J{}_I &= \big[\delta^I{}_J + \eta(R_g)^{IK}(G_{KJ} - B_{KJ})\big]^{-1}\,,\\
\check{N}_I{}^J &= \big[\delta_J{}^I + \eta(G_{JK} - B_{JK})(R_g)^{KI}\big]^{-1} = \big[R_g^{-1}\hat{N}R_g\big]_I{}^J\,.
\end{aligned} \tag{4.16}$$

Using (4.9) in (3.17) and (3.18) we find[26]

$$K^m = 0\,, \quad K^I = \eta[R_g n]^I\,, \quad X_m = \partial_m\left(\phi + \frac{1}{2}\ln\det\hat{N}\right) - \eta\tilde{B}_{mI}[R_g n]^I\,, \quad X_I = -\eta\tilde{B}_{IJ}[R_g n]^J\,. \tag{4.17}$$

At this point we wish to comment on the possibility of having "trivial" solutions of the generalized supergravity equations, namely ones that solve the more restricting standard supergravity equations while K does not vanish. This is possible if [27]

$$0 = K^I(\tilde{G}-\tilde{B})_{IJ} = -\eta[nR_g(G-B)\hat{N}]_J = [n(\hat{N}-1)]_J \iff K^I(G-B)_{IJ} = 0\,, \tag{4.18}$$

i.e. the *original* $G-B$ must be degenerate. Such trivial solutions are possible for YB deformations since we do not need to assume that $G-B$ is non-degenerate. They are, at least naively, not possible for NATD since there they would imply that the *dual* $\tilde{G}-\tilde{B}$ is degenerate, which is not allowed by assumption, see section 3. This discrepancy has to do with the fact that when going from DTD to YB we did not just change coordinates, we also shifted B by dropping the extra closed term in (4.11). Explicit trivial solutions were found in [50], and more recently in [43] by double field theory β-shifts starting from $\mathrm{AdS}_3 \times S^3 \times T^4$ with non-zero B-field. It is clear from the present discussion that these solutions can be equivalently generated from the construction of YB deformations provided here. An example is provided in section 4.3.2.

[25] For YB deformations $\Lambda \in \mathrm{SO}(1,9)$ and it is therefore useful to parametrize it in terms of an anti-symmetric matrix A^{ab} as $\Lambda = (1+A)^{-1}(1-A)$ which implies $A = (1-\Lambda)(1+\Lambda)^{-1}$, where we lowered one index with η_{ab} to obtain e.g. $\Lambda^a{}_b$. Then the Lorentz transformation on spinor indices $\Lambda_a{}^b\Gamma_b = \hat{\Lambda}^T\Gamma_a\hat{\Lambda}$ can be written as a finite sum [49]

$$\hat{\Lambda} = [\det(\eta+A)]^{-1/2}\,\text{Æ}\left(-\frac{1}{2}A_{ab}\Gamma^{ab}\right), \quad \text{Æ}\left(\frac{1}{2}A_{ab}\Gamma^{ab}\right) \equiv 1 + \sum_{n=1}^{n=5}\frac{1}{n!2^n}A_{a_1b_1}\cdots A_{a_nb_n}\Gamma^{a_1b_1\cdots a_nb_n}\,. \tag{4.14}$$

[26] In the expression for X we have used the fact that $d(\ln\det[\eta R_g]) = \mathrm{tr}(R_g^{-1}dR_g) = 2f_{JI}^I[g^{-1}dg]^J = 2(g^{-1}dg)^I n_I$.

4.2.1 A convenient rewriting

As remarked in the introduction the deformed metric and B-field can be obtained from the original G and B by the following generalization of the open/closed string map used by Seiberg and Witten

$$\tilde{G} - \tilde{B} = (G - B)[1 + \eta R_g(G - B)]^{-1} . \tag{4.19}$$

This is readily seen after noticing that, since R_g has only IJ indices, the following operator is of block form

$$1 + \eta R_g(G - B) = \begin{pmatrix} \delta^m{}_n & 0 \\ \eta[R_g(G-B)]^I{}_n & \delta^I{}_J + \eta[R_g(G-B)]^I{}_J \end{pmatrix} , \tag{4.20}$$

and it is straightforward to invert it giving

$$[1 + \eta R_g(G - B)]^{-1} = \begin{pmatrix} \delta^m{}_n & 0 \\ -\eta[\hat{N} R_g(G-B)]^I{}_n & \hat{N}^I{}_J \end{pmatrix} , \tag{4.21}$$

where we used $\hat{N}^J{}_I = [\delta^I{}_J + \eta(R_g)^{IK}(G_{KJ} - B_{KJ})]^{-1}$. It is easy to check that (4.19) indeed reproduces the formulas (4.12)–(4.13) for the transformed metric and B-field.

So far we have worked with explicit group elements and algebra indices. It is sometimes convenient to translate the results so that the information on the initial isometries of the model is encoded in a set of Killing vectors. Thanks to this rewriting the YB deformation may be applied without the need of introducing an explicit parametrization of the group G. Isometries of the metric and B-field are translated into equations for a family of Killing vectors k_I^μ, where $I = 1, \ldots, \dim(G)$ is the index to enumerate them. In particular, the metric possesses an isometry when shifting infinitesimally the coordinates $X^\mu \to X^\mu + \epsilon^I k_I^\mu + \mathcal{O}(\epsilon^2)$, if k_I^μ satisfy the Killing vector equation

$$\nabla_\mu k_{I\,\nu} + \nabla_\nu k_{I\,\mu} = 0 . \tag{4.22}$$

In order to make a connection with the formulation in terms of the group element g, it is enough to notice that its variation δg under an infinitesimal transformation can be understood in two ways, either as $\delta x^i \partial_i g$, or as $\epsilon^I T_I g$, the latter being the infinitesimal version of the global transformation $g \to \exp(\epsilon^I T_I) g$. We recall that indices i, j are used to label coordinates x^i on the group G. This leads to the identification

$$k_I^J \equiv k_I^\mu \ell_\mu^J = \mathrm{tr}(T^J \mathrm{Ad}_g^{-1} T_I) = (\mathrm{Ad}_g^{-1})_I^J , \quad \text{where } g^{-1} dg = \ell^I T_I . \tag{4.23}$$

Obviously, ℓ_μ^I and k_I^μ are non-zero only for $\mu = i$. The structure constants of the Lie algebra may be recovered by computing

$$\mathcal{L}_{k_I} k_{J\mu} - \mathcal{L}_{k_J} k_{I\mu} = -f_{IJ}^K k_{K\mu} , \tag{4.24}$$

where $\mathcal{L}$ is the Lie derivative. Now let us notice that we can rewrite

$$\Theta^{IJ} \equiv (R_g)^{IJ} = k_K^I R^{KL} k_L^J , \tag{4.25}$$

and that, before fixing any local symmetry (if present), the matrix ℓ_i^I is invertible. Let us denote the inverse by ℓ_I^i so that $\ell_i^I \ell_J^i = \delta_J^I$. This allows us to convert all algebra indices I, J in (4.19) into curved indices i, j. Therefore the YB deformation of the metric and B-field may also be written as

$$\tilde{G} - \tilde{B} = (G - B)[1 + \eta\Theta(G - B)]^{-1} . \tag{4.26}$$

This formula is then equally valid both when we use indices $\{m, I\}$ or $\{m, i\}$. When a local symmetry is present we arrive at the same result since the local invariance can be left unfixed until the end. With a similar reasoning we may rewrite also the transformation rule for the dilaton when $n_I = f_{IJ}^J = 0$. In fact, when computing the determinant of $\hat{N}^I{}_J$ we may as well extend it to all μ, ν indices. Since the (inverse of the) operator is in the block-form (4.20), it is clear that $\det(\hat{N}^\mu{}_\nu) = \det(\hat{N}^I{}_J)$. This also means that we can obtain the deformed dilaton simply by calculating

$$\tilde{\phi} = \phi - \frac{1}{2} \ln \det[1 + \eta\Theta(G - B)] . \tag{4.27}$$

More generally, when $n_I \neq 0$ we may write

$$K^\mu = \eta\Theta^{\mu\nu} n_\nu , \qquad X_\mu = \partial_\mu \tilde{\phi} - \eta \tilde{B}_{\mu\nu}\Theta^{\nu\rho} n_\rho . \tag{4.28}$$

4.3 Two examples of YB deformations

We wish to work out two examples of YB deformations that do not fall under the (super)coset construction. In addition to the intrinsic interest of the following (deformed) backgrounds, the calculations also illustrate the applicability of our method.

4.3.1 YB deformation of the D3-brane background

Our first motivation is to understand a YB deformation of $\mathrm{AdS}_5 \times S^5$ generated by an R-matrix that cannot be interpreted as a sequence of TsT transformations. In particular, we want to use the formula (4.26) to "uplift" the YB deformation from the $\mathrm{AdS}_5 \times S^5$ background to the full D3-brane background, before taking the near-horizon limit. This is in the spirit of [51, 52], where the uplift to the brane background was done for YB deformations that are (sequences of) TsT transformations. For the sake of the discussion we focus on the NS-NS sector, where the dilaton is constant (we set it to zero for simplicity), $B = 0$ and the metric is

$$ds^2 = H^{-1/2}\, dx_i dx^i + H^{1/2}(dr^2 + r^2\, ds^2_{S^5}) , \qquad H = 1 + \frac{(\alpha')^2 L^4}{r^4} , \tag{4.29}$$

where $i = 0, \dots, 3$ and $\eta_{ij} = \mathrm{diag}(-1, 1, 1, 1)$. The above metric has an $\mathrm{ISO}(1, 3)$ Poincaré isometry acting on the x^i coordinates, and an $\mathrm{SO}(6)$ isometry acting on the five-dimensional sphere S^5. We will now deform the background by exploiting the Poincaré part of the isometries. The Killing vectors in this case may be written as

$$\text{Translations: } k^\mu_{[p_i]} = \delta^\mu_i , \qquad \text{Lorentz: } k^\mu_{[J_{ij}]} = -\delta^\mu_i x_j + \delta^\mu_j x_i , \qquad i, j = 0, \dots, 3. \tag{4.30}$$

We wish to "uplift" the YB deformation of $\mathrm{AdS}_5 \times S^5$ worked out in section 6.4 of [47], where the R-matrix was chosen to be

$$R = p_1 \wedge p_3 + (p_0 + p_1) \wedge (J_{03} + J_{13})\,. \tag{4.31}$$

That is possible since this R-matrix is constructed out of generators that are isometries also of the D3-brane background before the near-horizon limit. Following (4.25) we therefore construct

$$\Theta^{\mu\nu} = 2\left[k^{\mu}_{[p_1]}k^{\nu}_{[p_3]} + (k^{\mu}_{[p_0]} + k^{\mu}_{[p_1]})(k^{\nu}_{[J_{03}]} + k^{\nu}_{[J_{13}]})\right] - \mu \leftrightarrow \nu\,. \tag{4.32}$$

More explicitly, in the block with $\mu, \nu = 0, \ldots, 3$ it is

$$\Theta^{\mu\nu} = 2\begin{pmatrix} 0 & 0 & 0 & -x^- \\ 0 & 0 & 0 & -x^- + 1 \\ 0 & 0 & 0 & 0 \\ x^- & x^- - 1 & 0 & 0 \end{pmatrix}, \tag{4.33}$$

where we introduced the standard light-cone coordinates $x^{\pm} = x^0 \pm x^1$. Now, using (4.26) and (4.27) we obtain the following deformed metric, B-field and dilaton

$$\begin{aligned} d\tilde{s}^2 &= -\frac{\hat{\eta}^2\xi_-^2 H^{-1/2} d\xi_-^2}{4\,(H - 4\hat{\eta}^2\xi_-)} - \frac{H^{-1/2}\left(H - 2\hat{\eta}^2\xi_-\right) d\xi_- dx^+}{2\,(H - 4\hat{\eta}^2\xi_-)} - \frac{\hat{\eta}^2 H^{-1/2}(dx^+)^2}{(H - 4\hat{\eta}^2\xi_-)} \\ &\quad + H^{-1/2}dx_2^2 + \frac{H^{1/2}dx_3^2}{H - 4\hat{\eta}^2\xi_-} + H^{1/2}(dr^2 + r^2\, ds^2_{S^5})\,, \\ \tilde{B} &= \frac{\hat{\eta}}{2}\frac{dx^3 \wedge (2dx^+ + \xi_- d\xi_-)}{H - 4\hat{\eta}^2\xi_-}\,, \qquad \exp(-2\tilde{\phi}) = 1 - \frac{4\hat{\eta}^2\xi_-}{H}\,. \end{aligned} \tag{4.34}$$

We chose $\hat{\eta}$ as deformation parameter and to simplify expressions we redefined $\xi_- = 2x^- - 1$. We now want to check that the near-horizon geometry of this YB deformation of the D3-brane background indeed yields the YB deformation of $\mathrm{AdS}_5 \times S^5$ of [47]. In the near-horizon limit one sends $r \to 0$ and $\alpha' \to 0$ while keeping the ratio r/α' fixed. We achieve this by rewriting $r = \alpha' L^2/z$ and $\hat{\eta} = \eta L^{-2}/\alpha'$, and then sending $\alpha' \to 0$. We obtain

$$\begin{aligned} \lim_{\alpha'\to 0}\frac{ds^2}{\alpha' L^2} &= z^{-6}\left(1 - \frac{4\eta^2\xi_-}{z^4}\right)^{-1}\left[z^4 dx_3{}^2 - \eta^2(dx^+)^2 - \frac{1}{4}d\xi_-\left(\eta^2\xi_-^2\,d\xi_- + 2dx^+\left(z^4 - 2\eta^2\xi_-\right)\right)\right] \\ &\quad + \frac{dx_2{}^2 + dz^2}{z^2} + ds^2_{S^5} \\ \lim_{\alpha'\to 0}\frac{B}{\alpha' L^2} &= \frac{\eta}{2}\frac{dx^3 \wedge (2dx^+ + \xi_- d\xi_-)}{z^4 - 4\eta^2\xi_-}\,, \qquad \lim_{\alpha'\to 0} e^{-2\phi} = 1 - \frac{4\eta^2\xi_-}{z^4}\,, \end{aligned} \tag{4.35}$$

which indeed reproduces[27] (the NS-NS sector of) the deformation of $\mathrm{AdS}_5 \times S^5$ appearing in section 6.4 of [47]. Uplifting the YB deformation to the D3-brane background is particularly interesting since it also allows us to go far from the brane and understand how the flat space

[27] In this paper we have a different convention for the sign of the B-field.

JHEP08(2018)027

in which it is embedded has been deformed. In the limit $r \to \infty$ we have simply $H \to 1$

$$
\begin{aligned}
ds^2 &= -\frac{\hat{\eta}^2 \xi_-^2 \, d\xi_-^2}{4\,(1-4\hat{\eta}^2\xi_-)} - \frac{\left(1-2\hat{\eta}^2\xi_-\right) d\xi_- dx^+}{2\,(1-4\hat{\eta}^2\xi_-)} - \frac{\hat{\eta}^2 (dx^+)^2}{(1-4\hat{\eta}^2\xi_-)} \\
&\quad + dx_2^2 + \frac{dx_3^2}{1-4\hat{\eta}^2\xi_-} + ds^2_{R^6} \\
B &= \frac{\hat{\eta}}{2} \frac{dx^3 \wedge (2dx^+ + \xi_- d\xi_-)}{1-4\hat{\eta}^2\xi_-}, \qquad\qquad e^{-2\phi} = 1-4\hat{\eta}^2\xi_- \,.
\end{aligned}
\tag{4.36}
$$

Obviously, the above background may be also obtained directly as a YB deformation of flat space with Θ given by (4.32). In the AdS/CFT correspondence one looks at open strings stretching between D3-branes in flat space, whose low-energy limit produces $\mathcal{N}=4$ super Yang-Mills. In the presence of a B-field as in the case considered here, open strings feel an effective metric $g_{\mu\nu}$ and a non-commutativity parameter $\theta^{\mu\nu}$ that are related to the metric and B-field $G_{\mu\nu}, B_{\mu\nu}$ of the closed string by[28] [37]

$$
g^{\mu\nu} + \frac{\theta^{\mu\nu}}{2\pi\alpha'} = (G_{\mu\nu} - B_{\mu\nu})^{-1} \,, \tag{4.37}
$$

where $g^{\mu\nu}$ is obviously obtained by taking the symmetric part of the right-hand-side, while $\theta^{\mu\nu}$ the antisymmetric part. In general, if we apply the open/closed string map to a background obtained by a YB deformation we get

$$
\begin{aligned}
g^{-1} + \frac{\theta}{2\pi\alpha'} &= (\tilde{G} - \tilde{B})^{-1} = [(G-B)^{-1} + \eta\Theta] \,, \\
\Longrightarrow \; g^{-1} &= (G-B)_s^{-1} \,, \qquad \theta = 2\pi\alpha' [(G-B)_a^{-1} + \eta\Theta] \,,
\end{aligned}
\tag{4.38}
$$

where we directly relate the open-string quantities to the metric and B-field G, B of the original model before the YB deformation, and subscripts s and a indicate the symmetric and antisymmetric parts. In our specific example, before deforming, the brane system is in a flat spacetime with vanishing B-field, meaning that the effective open-string metric will coincide with the flat one, and the non-commutativity parameter will be essentially defined by the YB R-matrix

$$
g_{\mu\nu} = G_{\mu\nu} \,, \qquad \theta^{\mu\nu} = 2\pi\alpha' \hat{\eta}\, \Theta^{\mu\nu} \,. \tag{4.39}
$$

This discussion is obviously generic and is not confined to the current example. Apart from uncovering the non-commutativity structure, at this point one should also take the low-energy limit of open strings in the non-commutative spacetime. Here we are considering a case with an electric B-field, and these instances are known to produce problems when trying to take the low-energy limit [53]. It is therefore not clear whether the low-energy limit yields a non-commutative gauge theory with θ as non-commutativity parameter. The relation between gravity duals of non-commutative gauge theories and YB deformations was first pointed out in [54].

[28] As it is written, this open/closed string map assumes the invertibility of $(G-B)$. The generalization (of the inverse transformation) to the case of degenerate $(G-B)$ is in fact given by our (4.26).

Certain YB deformations of $\mathrm{AdS}_5 \times S^5$ are constructed out of generators that are not isometries of the brane background and that become isometries only after taking the near-horizon limit. For these examples it is not clear how to uplift the YB deformation to the brane background. It would be interesting to see if YB deformations can be extended also to cases without isometries by using Poisson-Lie T-duality.

4.3.2 YB deformation of $\mathrm{AdS}_3 \times S^3 \times T^4$ with H-flux

We now want to apply the YB deformation to a background with degenerate $G-B$, and we will compare our results to those of [43]. There it was indeed shown that YB deformations of $\mathrm{AdS}_5 \times S^5$ are equivalent to local β-transformations of the double theory, and it was proposed that local β-shifts should be the natural way to generalize YB deformations to generic backgrounds, including cases with degenerate $G-B$. The example we consider is that of $\mathrm{AdS}_3 \times S^3 \times T^4$ with non-vanishing H-flux

$$
\begin{aligned}
ds^2 &= \frac{dx_i dx^i + dz^2}{z^2} + ds^2_{S^3} + ds^2_{T^4}\,, \qquad ds^2_{S^3} = \frac{1}{4}\left[d\theta^2 + \sin^2\theta d\varphi^2 + (d\psi + \cos\theta d\varphi)^2\right] \\
B &= \frac{dx^0 \wedge dx^1}{z^2} + \frac{1}{4}\cos\theta d\varphi \wedge d\psi\,.
\end{aligned}
\tag{4.40}
$$

$G-B$ is degenerate because of the rows (or columns) $i = 0, 1$. The dilaton is constant and for simplicity we set it to zero. To generate a YB deformation we will make use of the Killing vectors of the Poincaré isometry

$$
\text{Translations: } k^\mu_{[p_i]} = \delta^\mu_i\,, \qquad \text{Lorentz: } k^\mu_{[J_{ij}]} = -\delta^\mu_i x_j + \delta^\mu_j x_i\,, \qquad i, j = 0, 1\,. \tag{4.41}
$$

In order to compare to the results of section 4.2.2 of [43] we take $R = c^i p_i \wedge J_{01}$ or

$$
\Theta^{\mu\nu} = (c^i k^\mu_{[p_i]}) k^\nu_{[J_{01}]} - \mu \leftrightarrow \nu\,, \tag{4.42}
$$

where we sum over $i = 0, 1$. The classical YB equation is satisfied only when the parameters satisfy $c^0 = \pm c^1$. Now using (4.26) and (4.27) we obtain the YB deformed background

$$
\begin{aligned}
ds^2 &= \frac{dx_i dx^i}{z^2 - 2\eta c_j x^j} + \frac{dz^2}{z^2} + ds^2_{S^3} + ds^2_{T^4}\,, \\
B &= \frac{dx^0 \wedge dx^1}{z^2 - 2\eta c_j x^j} + \frac{1}{4}\cos\theta d\varphi \wedge d\psi\,, \qquad e^{-2\phi} = 1 - \frac{2\eta c_i x^i}{z^2}\,,
\end{aligned}
\tag{4.43}
$$

which agrees with the background obtained in section 4.2.2 of [43]. This confirms in a specific example the expected equivalence of YB deformations and local β-shifts even beyond the standard ($H = 0$) supercoset case. As already noticed in [43] the above background is actually a trivial solution since the vector K decouples from the generalized supergravity equations.

5 Conclusions

We have derived the transformation rules for the supergravity fields under NATD by carrying out the dualization in the general case for the Green-Schwarz string. This generalizes the derivation performed for the case of the supercoset in [16]. If the dualized group G is not unimodular there is in general an anomaly, which is reflected in the fact that the resulting background solves the generalized supergravity equations of [23, 24] rather than the standard ones. We have also discussed a generalization where one adds a closed B-field to the action prior to performing the duality transformation. This leads to so-called DTD models and, in special cases, a generalization of Yang-Baxter models [28, 29]. We have also seen that this gives us an interesting way to find examples that avoid the anomaly from non-unimodularity of G along the lines discussed in [27].

Non-abelian T-duality can be embedded in the even more general framework of Poisson-Lie T-duality [48]. Also this case can be formulated at the path integral level and an anomaly arises in a similar way [55] (see also [56]). It would be interesting to extend our analysis to this case which would also make further contact with [42]. It would also allow us to extend DTD and YB deformations to cases without isometries, and perhaps help to uplift all YB deformations of $AdS_5 \times S^5$ to deformations of the brane background. It would also be interesting to consider the case of open strings along the lines of the recent paper [57].

We have found that a natural way to rephrase YB deformations is in terms of a generalization of (the inverse of) the open/closed string map of Seiberg and Witten, thus extending what was observed in the case of both homogeneous and inhomogeneous YB deformations of PCM or (super)cosets. Since the inhomogeneous case cannot be formulated in terms of our construction we have only considered the homogeneous one here, but it would be interesting to see what happens if we take R in (4.19) to solve the *modified* classical YB equation on the Lie algebra of G. The lessons learned from the supercoset case [24, 47, 58, 59] suggest that the resulting sigma model will possibly be kappa-symmetric, but that the background fields will probably only solve the equations of generalized supergravity rather than the standard ones.

When applied to classically integrable sigma models, the deformations studied here preserve the integrability. It would be interesting to extend the integrability methods developed in the context of the AdS/CFT correspondence [60, 61] also beyond the "abelian" YB deformations considered so far, namely the "diagonal abelian" deformations (considered e.g. in [62] and with an exact spectrum encoded in the equations of [63]), and the "off-diagonal abelian" deformations (addressed e.g. at one loop in [64]).

Acknowledgments

We thank B. Hoare and S. van Tongeren for interesting and valuable discussions. RB also thanks the Department of Theoretical Physics and Astrophysics of Masaryk University for hospitality during part of this work. The work of R.B. was supported by the ERC advanced grant No 341222.

JHEP08(2018)027

A Conventions

Let us summarize our index conventions in the following table

$$\begin{array}{rl} \mu, \nu, \ldots: & \text{labels of all bosonic coordinates} \\ I, J, \ldots: & \text{indices of } \mathfrak{g} \text{ (the Lie algebra of } G\text{) and of the dual } \mathfrak{g}^* \\ i, j, \ldots: & \text{labels of coordinates parameterizing the group } G \\ M, N, \ldots: & \text{labels of spectator coordinates, of which} \\ m, n, \ldots: & \text{labels of bosonic spectator coordinates} \\ \underline{\alpha}, \underline{\beta}, \ldots: & \text{labels of fermionic spectator coordinates} \\ A, B, \ldots: & \text{indices of tangent space, of which} \\ a, b, \ldots: & \text{indices of bosonic tangent space} \\ \alpha, \beta, \ldots: & \text{indices of fermionic tangent space} \end{array} \tag{A.1}$$

When working with (super)forms we define the components as $A_n = \frac{1}{n!} dz^{M_n} \wedge dz^{M_{n-1}} \ldots \wedge dz^{M_1} A_{M_1 M_2 \cdots M_n}$ and we take the exterior derivative to act from the right, so that $d(A_n \wedge A_m) = A_n \wedge dA_m + (-1)^m dA_n \wedge A_m$. The (graded) anti-symmetrization of n indices is denoted by $[\cdots]$ and it includes a factor $1/n!$.

B An example with local symmetry

To make the discussion in section 3.1 more concrete we will here apply the rules of NATD to an explicit example with local symmetry (a case also referred to "with isotropy"). We will follow the discussion in section 3.1 and show that we reproduce an example worked out in section 4.1 of [20]. The starting point is the $\mathrm{AdS}_3 \times S^3 \times T^4$ background with pure RR flux, and the goal is to apply NATD on the SO(4) global isometry of S^3, which has obviously also a local SO(3) symmetry. The metric and the flux are given by

$$ds^2 = ds^2_{\mathrm{AdS}_3} + ds^2_{S^3} + ds^2_{T^4}, \qquad F_3 = 2\left(\mathrm{vol}(\mathrm{AdS}_3) + \mathrm{vol}(S^3)\right). \tag{B.1}$$

We describe S^3 in terms of the coset SO(4)/SO(3), where the generators of $\mathfrak{so}(4)$ satisfy $[J_{ab}, J_{cd}] = \delta_{bc} J_{ad} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac} + \delta_{ad} J_{bc}$ and admit the matrix realisation $J_{ab} = E_{ab} - E_{ba}$, in terms of the matrices $(E_{ab})_{cd} = \delta_{ac}\delta_{bd}$. Following [20] we enumerate the generators of the coset part as $T_I = J_{1,I+1}$ where $I = 1, 2, 3$, and the generators of the subalgebra $\mathfrak{so}(3)$ as $T_4 = J_{23}, T_5 = J_{24}, T_6 = J_{34}$. The metric of the original S^3 comes from the piece of the action $\frac{T}{2}\int A^I \wedge G_{IJ} * A^J$, where $A = g^{-1}dg,\ g \in \mathrm{SO}(4)$ and $G_{IJ} = \mathrm{diag}(1,1,1,0,0,0)$ projects on the coset part of the algebra. We do not need to look at AdS_3 and T^4, since the off-diagonal blocks G_{mI} are 0 and therefore the AdS_3 and T^4 spaces are not affected by the NATD transformations, see (2.2). It is easy to construct $G_{IJ} - \nu_K f_{IJ}^K$ that in this case is[29]

$$\begin{pmatrix} 1 & \nu_4 & \nu_5 & -\nu_2 & -\nu_3 & 0 \\ -\nu_4 & 1 & \nu_6 & \nu_1 & 0 & -\nu_3 \\ -\nu_5 & -\nu_6 & 1 & 0 & \nu_1 & \nu_2 \\ \nu_2 & -\nu_1 & 0 & 0 & \nu_6 & -\nu_5 \\ \nu_3 & 0 & -\nu_1 & -\nu_6 & 0 & \nu_4 \\ 0 & \nu_3 & -\nu_2 & \nu_5 & -\nu_4 & 0 \end{pmatrix}, \tag{B.2}$$

[29]This is the transpose of M of [20].

and invert it to obtain N^{IJ}. Notice that G_{IJ} is not invertible, but we can invert $G_{IJ} - \nu_K f_{IJ}^K$. For special values of the coordinates ν_K also $G_{IJ} - \nu_K f_{IJ}^K$ becomes degenerate. After fixing the gauge, some of these degeneracies will produce singularities in target space. Taking the symmetric and antisymmetric parts of N^{IJ} we can compute the deformed metric and B-field. In the action the contributions are respectively $\frac{T}{2}\int d\nu_I N^{(IJ)} * d\nu_J$ and $-\frac{T}{2}\int d\nu_I N^{[IJ]} d\nu_J$. These are still written in terms of all six dual coordinates ν_K, meaning that we should fix the gauge. We fix it as in [20] setting $\nu_1 = \nu_2 = \nu_6 = 0$, and we also rename $\nu_3 = x_1, \nu_4 = x_2, \nu_5 = x_3$. In agreement with [20] we find that the B-field vanishes and that the metric of the dualised sphere and the dilaton are

$$\begin{aligned} ds^2_{\tilde{S}^3} &= \frac{dx_2^2\left(\left(x_1^2 - x_2^2\right)^2 + x_2^2 x_3^2 + x_2^2\right)}{x_1^2 x_3^2} + \frac{\left(x_2^2 + x_3^2 + 1\right) dx_3^2}{x_1^2} \\ &\quad + \frac{2x_2 dx_2 dx_3\left(-x_1^2 + x_2^2 + x_3^2 + 1\right)}{x_1^2 x_3} + \frac{2dx_1}{x_1}\left(x_2 dx_2 + x_3 dx_3\right) + dx_1^2, \\ e^{-2\phi} &= x_1^2 x_3^2. \end{aligned} \tag{B.3}$$

In order to compute the transformation of the RR fields we first need to compute the Lorentz transformation Λ. Suppose we use labels in tangent space $a = 0, \ldots, 9$ so that $a = 3, 4, 5$ are the labels for the tangent space of the sphere. Then we can take $E_I{}^a$ to be $E_1{}^3 = E_2{}^4 = E_3{}^5 = 1$, and 0 otherwise. Calculating $\Lambda^{ab} = \eta^{ab} - 2E_I{}^a N^{IJ} E_J{}^b$ in the above gauge for ν_I we easily find (for the block with $a, b = 3, 4, 5$) $\Lambda = \mathrm{diag}(1, -1, -1)$. As expected the Lorentz transformation is an element of SO$(1, 9)$, since we have dualized an even-dimensional group. In this case it is a simple reflection along $a = 4$ and $a = 5$. Therefore on spinor indices it is realised just as the product of the two corresponding ten-dimensional gamma matrices. The transformed RR fluxes obtained from $\tilde{\mathcal{S}} = \hat{\Lambda}\mathcal{S}$ then agree with the ones of [20].

References

[1] X.C. de la Ossa and F. Quevedo, *Duality symmetries from nonAbelian isometries in string theory*, *Nucl. Phys.* **B 403** (1993) 377 [hep-th/9210021] [INSPIRE].

[2] A. Giveon and M. Roček, *On nonAbelian duality*, *Nucl. Phys.* **B 421** (1994) 173 [hep-th/9308154] [INSPIRE].

[3] E. Alvarez, L. Álvarez-Gaumé, J.L.F. Barbon and Y. Lozano, *Some global aspects of duality in string theory*, *Nucl. Phys.* **B 415** (1994) 71 [hep-th/9309039] [INSPIRE].

[4] K. Sfetsos and D.C. Thompson, *On non-abelian T-dual geometries with Ramond fluxes*, *Nucl. Phys.* **B 846** (2011) 21 [arXiv:1012.1320] [INSPIRE].

[5] Y. Lozano, E. Ó Colgáin, D. Rodríguez-Gómez and K. Sfetsos, *Supersymmetric AdS_6 via T Duality*, *Phys. Rev. Lett.* **110** (2013) 231601 [arXiv:1212.1043] [INSPIRE].

JHEP08(2018)027

[6] G. Itsios, C. Núñez, K. Sfetsos and D.C. Thompson, *On Non-Abelian T-duality and new $N = 1$ backgrounds*, *Phys. Lett.* **B 721** (2013) 342 [arXiv:1212.4840] [INSPIRE].

[7] G. Itsios, C. Núñez, K. Sfetsos and D.C. Thompson, *Non-Abelian T-duality and the AdS/CFT correspondence:new $N = 1$ backgrounds*, *Nucl. Phys.* **B 873** (2013) 1 [arXiv:1301.6755] [INSPIRE].

[8] Y. Lozano and C. Núñez, *Field theory aspects of non-Abelian T-duality and $\mathcal{N} = 2$ linear quivers*, *JHEP* **05** (2016) 107 [arXiv:1603.04440] [INSPIRE].

[9] Y. Lozano, N.T. Macpherson, J. Montero and C. Núñez, *Three-dimensional $\mathcal{N} = 4$ linear quivers and non-Abelian T-duals*, *JHEP* **11** (2016) 133 [arXiv:1609.09061] [INSPIRE].

[10] Y. Lozano, C. Núñez and S. Zacarias, *BMN Vacua, Superstars and Non-Abelian T-duality*, *JHEP* **09** (2017) 000 [arXiv:1703.00417] [INSPIRE].

[11] Y. Lozano, *NonAbelian duality and canonical transformations*, *Phys. Lett.* **B 355** (1995) 165 [hep-th/9503045] [INSPIRE].

[12] K. Sfetsos, *NonAbelian duality, parafermions and supersymmetry*, *Phys. Rev.* **D 54** (1996) 1682 [hep-th/9602179] [INSPIRE].

[13] K. Sfetsos, *Canonical equivalence of nonisometric σ-models and Poisson-Lie T duality*, *Nucl. Phys.* **B 517** (1998) 549 [hep-th/9710163] [INSPIRE].

[14] B. Hoare and A.A. Tseytlin, *Homogeneous Yang-Baxter deformations as non-abelian duals of the AdS_5 σ-model*, *J. Phys.* **A 49** (2016) 494001 [arXiv:1609.02550] [INSPIRE].

[15] R. Borsato and L. Wulff, *Integrable Deformations of T-Dual σ Models*, *Phys. Rev. Lett.* **117** (2016) 251602 [arXiv:1609.09834] [INSPIRE].

[16] R. Borsato and L. Wulff, *On non-abelian T-duality and deformations of supercoset string σ-models*, *JHEP* **10** (2017) 024 [arXiv:1706.10169] [INSPIRE].

[17] K. Sfetsos, *Integrable interpolations: From exact CFTs to non-Abelian T-duals*, *Nucl. Phys.* **B 880** (2014) 225 [arXiv:1312.4560] [INSPIRE].

[18] T.J. Hollowood, J.L. Miramontes and D.M. Schmidtt, *Integrable Deformations of Strings on Symmetric Spaces*, *JHEP* **11** (2014) 009 [arXiv:1407.2840] [INSPIRE].

[19] T.J. Hollowood, J.L. Miramontes and D.M. Schmidtt, *An Integrable Deformation of the $AdS_5 \times S^5$ Superstring*, *J. Phys.* **A 47** (2014) 495402 [arXiv:1409.1538] [INSPIRE].

[20] Y. Lozano, E. Ó Colgáin, K. Sfetsos and D.C. Thompson, *Non-abelian T-duality, Ramond Fields and Coset Geometries*, *JHEP* **06** (2011) 106 [arXiv:1104.5196] [INSPIRE].

[21] S.F. Hassan, *T duality, space-time spinors and RR fields in curved backgrounds*, *Nucl. Phys.* **B 568** (2000) 145 [hep-th/9907152] [INSPIRE].

[22] N. Berkovits and J. Maldacena, *Fermionic T-duality, Dual Superconformal Symmetry and the Amplitude/Wilson Loop Connection*, *JHEP* **09** (2008) 062 [arXiv:0807.3196] [INSPIRE].

[23] L. Wulff and A.A. Tseytlin, *κ-symmetry of superstring σ-model and generalized* 10d *supergravity equations*, *JHEP* **06** (2016) 174 [arXiv:1605.04884] [INSPIRE].

[24] G. Arutyunov, S. Frolov, B. Hoare, R. Roiban and A.A. Tseytlin, *Scale invariance of the η-deformed $AdS_5 \times S^5$ superstring, T-duality and modified type-II equations*, *Nucl. Phys.* **B 903** (2016) 262 [arXiv:1511.05795] [INSPIRE].

[25] E. Alvarez, L. Álvarez-Gaumé and Y. Lozano, *On nonAbelian duality*, *Nucl. Phys.* **B 424** (1994) 155 [hep-th/9403155] [INSPIRE].

[26] S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer and G. Veneziano, *Remarks on nonAbelian duality*, *Nucl. Phys.* **B 435** (1995) 147 [hep-th/9409011] [INSPIRE].

[27] L. Wulff, *Trivial solutions of generalized supergravity vs non-abelian T-duality anomaly*, *Phys. Lett.* **B 781** (2018) 417 [arXiv:1803.07391] [INSPIRE].

[28] C. Klimčík, *Yang-Baxter σ-models and dS/AdS T duality*, *JHEP* **12** (2002) 051 [hep-th/0210095] [INSPIRE].

[29] C. Klimčík, *On integrability of the Yang-Baxter σ-model*, *J. Math. Phys.* **50** (2009) 043508 [arXiv:0802.3518] [INSPIRE].

[30] F. Delduc, M. Magro and B. Vicedo, *On classical q-deformations of integrable σ-models*, *JHEP* **11** (2013) 192 [arXiv:1308.3581] [INSPIRE].

[31] I. Kawaguchi, T. Matsumoto and K. Yoshida, *Jordanian deformations of the $AdS_5 \times S^5$ superstring*, *JHEP* **04** (2014) 153 [arXiv:1401.4855] [INSPIRE].

[32] T. Matsumoto and K. Yoshida, *Yang-Baxter σ-models based on the CYBE*, *Nucl. Phys.* **B 893** (2015) 287 [arXiv:1501.03665] [INSPIRE].

[33] S.J. van Tongeren, *On classical Yang-Baxter based deformations of the $AdS_5 \times S^5$ superstring*, *JHEP* **06** (2015) 048 [arXiv:1504.05516] [INSPIRE].

[34] F. Delduc, M. Magro and B. Vicedo, *An integrable deformation of the $AdS_5 \times S^5$ superstring action*, *Phys. Rev. Lett.* **112** (2014) 051601 [arXiv:1309.5850] [INSPIRE].

[35] B. Hoare and D.C. Thompson, *Marginal and non-commutative deformations via non-abelian T-duality*, *JHEP* **02** (2017) 059 [arXiv:1611.08020] [INSPIRE].

[36] D. Osten and S.J. van Tongeren, *Abelian Yang-Baxter deformations and TsT transformations*, *Nucl. Phys.* **B 915** (2017) 184 [arXiv:1608.08504] [INSPIRE].

[37] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142] [INSPIRE].

[38] T. Araujo, I. Bakhmatov, E. Ó Colgáin, J. Sakamoto, M.M. Sheikh-Jabbari and K. Yoshida, *Yang-Baxter σ-models, conformal twists and noncommutative Yang-Mills theory*, *Phys. Rev.* **D 95** (2017) 105006 [arXiv:1702.02861] [INSPIRE].

[39] I. Bakhmatov, Ö. Kelekci, E. Ó Colgáin and M.M. Sheikh-Jabbari, *Classical Yang-Baxter Equation from Supergravity*, *Phys. Rev.* **D 98** (2018) 021901 [arXiv:1710.06784] [INSPIRE].

[40] I. Bakhmatov, E. Ó Colgáin, M.M. Sheikh-Jabbari and H. Yavartanoo, *Yang-Baxter Deformations Beyond Coset Spaces (a slick way to do TsT)*, *JHEP* **06** (2018) 161 [arXiv:1803.07498] [INSPIRE].

[41] J.-i. Sakamoto, Y. Sakatani and K. Yoshida, *Homogeneous Yang-Baxter deformations as generalized diffeomorphisms*, *J. Phys.* **A 50** (2017) 415401 [arXiv:1705.07116] [INSPIRE].

[42] D. Lüst and D. Osten, *Generalised fluxes, Yang-Baxter deformations and the $O(d,d)$ structure of non-abelian T-duality*, *JHEP* **05** (2018) 165 [arXiv:1803.03971] [INSPIRE].

[43] J.-I. Sakamoto and Y. Sakatani, *Local β-deformations and Yang-Baxter σ-model*, *JHEP* **06** (2018) 147 [arXiv:1803.05903] [INSPIRE].

[44] L. Wulff, *The type-II superstring to order θ^4*, *JHEP* **07** (2013) 123 [arXiv:1304.6422] [INSPIRE].

[45] M. Hong, Y. Kim and E. Ó Colgáin, *On non-Abelian T-duality for non-semisimple groups*, arXiv:1801.09567 [INSPIRE].

[46] K. Sfetsos, *Duality invariant class of two-dimensional field theories*, Nucl. Phys. **B 561** (1999) 316 [hep-th/9904188] [INSPIRE].

[47] R. Borsato and L. Wulff, *Target space supergeometry of η and λ-deformed strings*, JHEP **10** (2016) 045 [arXiv:1608.03570] [INSPIRE].

[48] C. Klimčík and P. Ševera, *Dual nonAbelian duality and the Drinfeld double*, Phys. Lett. **B 351** (1995) 455 [hep-th/9502122] [INSPIRE].

[49] S.F. Hassan, SO(d,d) *transformations of Ramond-Ramond fields and space-time spinors*, Nucl. Phys. **B 583** (2000) 431 [hep-th/9912236] [INSPIRE].

[50] B. Hoare and S.J. van Tongeren, *On Jordanian deformations of AdS_5 and supergravity*, J. Phys. **A 49** (2016) 434006 [arXiv:1605.03554] [INSPIRE].

[51] S.J. van Tongeren, *Yang-Baxter deformations, AdS/CFT and twist-noncommutative gauge theory*, Nucl. Phys. **B 904** (2016) 148 [arXiv:1506.01023] [INSPIRE].

[52] S.J. van Tongeren, *Almost abelian twists and AdS/CFT*, Phys. Lett. **B 765** (2017) 344 [arXiv:1610.05677] [INSPIRE].

[53] N. Seiberg, L. Susskind and N. Toumbas, *Strings in background electric field, space/time noncommutativity and a new noncritical string theory*, JHEP **06** (2000) 021 [hep-th/0005040] [INSPIRE].

[54] T. Matsumoto and K. Yoshida, *Integrability of classical strings dual for noncommutative gauge theories*, JHEP **06** (2014) 163 [arXiv:1404.3657] [INSPIRE].

[55] E. Tyurin and R. von Unge, *Poisson-lie T duality: The Path integral derivation*, Phys. Lett. **B 382** (1996) 233 [hep-th/9512025] [INSPIRE].

[56] R. Von Unge, *Poisson Lie T plurality*, JHEP **07** (2002) 014 [hep-th/0205245] [INSPIRE].

[57] F. Cordonier-Tello, D. Lüst and E. Plauschinn, *Open-string T-duality and applications to non-geometric backgrounds*, arXiv:1806.01308 [INSPIRE].

[58] G. Arutyunov, R. Borsato and S. Frolov, *Puzzles of η-deformed $AdS_5 \times S^5$*, JHEP **12** (2015) 049 [arXiv:1507.04239] [INSPIRE].

[59] B. Hoare and A.A. Tseytlin, *Type IIB supergravity solution for the T-dual of the η-deformed $AdS_5 \times S^5$ superstring*, JHEP **10** (2015) 060 [arXiv:1508.01150] [INSPIRE].

[60] N. Beisert et al., *Review of AdS/CFT Integrability: An Overview*, Lett. Math. Phys. **99** (2012) 3 [arXiv:1012.3982] [INSPIRE].

[61] D. Bombardelli et al., *An integrability primer for the gauge-gravity correspondence: An introduction*, J. Phys. **A 49** (2016) 320301 [arXiv:1606.02945] [INSPIRE].

[62] N. Beisert and R. Roiban, *Beauty and the twist: The Bethe ansatz for twisted $N = 4$ SYM*, JHEP **08** (2005) 039 [hep-th/0505187] [INSPIRE].

[63] M. de Leeuw and S.J. van Tongeren, *The spectral problem for strings on twisted $AdS_5 \times S^5$*, Nucl. Phys. **B 860** (2012) 339 [arXiv:1201.1451] [INSPIRE].

[64] M. Guica, F. Levkovich-Maslyuk and K. Zarembo, *Integrability in dipole-deformed $\mathcal{N} = 4$ super Yang-Mills*, J. Phys. **A 50** (2017) 394001 [arXiv:1706.07957] [INSPIRE].

JHEP08(2018)027

Published for SISSA by Springer

Received: *August 23, 2017*
Accepted: *September 25, 2017*
Published: *October 4, 2017*

On non-abelian T-duality and deformations of supercoset string sigma-models

Riccardo Borsato[a] **and Linus Wulff**[b]

[a] *Nordita, Stockholm University and KTH Royal Institute of Technology, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden*

[b] *Department of Theoretical Physics and Astrophysics, Masaryk University, 611 37 Brno, Czech Republic*

E-mail: riccardo.borsato@su.se, linus.wulff@gmail.com

Abstract: We elaborate on the class of deformed T-dual (DTD) models obtained by first adding a topological term to the action of a supercoset sigma model and then performing (non-abelian) T-duality on a subalgebra $\tilde{\mathfrak{g}}$ of the superisometry algebra. These models inherit the classical integrability of the parent one, and they include as special cases the so-called homogeneous Yang-Baxter sigma models as well as their non-abelian T-duals. Many properties of DTD models have simple algebraic interpretations. For example we show that their (non-abelian) T-duals — including certain deformations — are again in the same class, where $\tilde{\mathfrak{g}}$ gets enlarged or shrinks by adding or removing generators corresponding to the dualised isometries. Moreover, we show that Weyl invariance of these models is equivalent to $\tilde{\mathfrak{g}}$ being unimodular; when this property is not satisfied one can always remove one generator to obtain a unimodular $\tilde{\mathfrak{g}}$, which is equivalent to (formal) T-duality. We also work out the target space superfields and, as a by-product, we prove the conjectured transformation law for Ramond-Ramond (RR) fields under bosonic non-abelian T-duality of supercosets, generalising it to cases involving also fermionic T-dualities.

Keywords: Integrable Field Theories, String Duality

ArXiv ePrint: 1706.10169

https://doi.org/10.1007/JHEP10(2017)024

Contents

JHEP10(2017)024

1 Introduction

In this paper we investigate further the deformed T-dual (DTD) supercoset sigma models introduced in [1], and we find results that are of interest also when considering the undeformed case, i.e. when applying just non-abelian T-duality (NATD).

The construction of DTD models is equivalent to applying NATD on a centrally extended subalgebra as first suggested in [2].[1] The models are constructed by picking a subalgebra of the (super)isometry algebra $\tilde{\mathfrak{g}} \subset \mathfrak{g}$ — the canonical example is the $AdS_5 \times S^5$ superstring where $\mathfrak{g} = \mathfrak{psu}(2,2|4)$ — and a 2-cocycle, i.e. an anti-symmetric linear map $\omega : \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}} \to \mathbb{R}$ satisfying

$$\omega(X,[Y,Z]) + \omega(Z,[X,Y]) + \omega(Y,[Z,X]) = 0\,, \qquad \forall X,Y,Z \in \tilde{\mathfrak{g}}\,. \tag{1.1}$$

[1]The first hint of the relation of YB models to NATD appeared in [3] for the case of Jordanian deformations.

Together with an element of the corresponding group $\tilde{g} \in \tilde{G}$, the 2-cocycle defines a 2-form $B = \omega(\tilde{g}^{-1}d\tilde{g}, \tilde{g}^{-1}d\tilde{g})$ which is closed, i.e. $dB = 0$, thanks to the 2-cocycle condition. The idea behind the construction is to add this topological term to the supercoset sigma model Lagrangian and then perform NATD on $\tilde{G}$. If ζB is added to the Lagrangian, with ζ a parameter, the resulting model can be thought of as a deformation of the non-abelian T-dual of the original model with deformation parameter ζ. The classical integrability of the original sigma model is preserved by the deformation, since both adding a topological term and performing NATD preserve integrability. We refer to [1] for more details on how this procedure relates to the construction of [2]. Let us remark that DTD models may be constructed starting from a generic σ-model, for example the principal chiral model as in [1], and the starting model does not have to be (classically) integrable. In this paper we will only consider the supercoset case.

It was proven in [1] that the so-called Yang-Baxter (YB) sigma models [4–7], defined by an R-matrix solving the classical Yang-Baxter equation (CYBE), are equivalent to DTD models with invertible ω. This relation was first conjectured and checked for many examples — in the language of T-duality on a centrally extended subalgebra — in [2]. See also [8] for a more detailed discussion of some of the examples. In [1] we used the fact that when ω is invertible its inverse $R = \omega^{-1}$ solves the CYBE, and therefore defines a corresponding YB model; by means of a field redefinition and relating the deformation parameters as $\eta = \zeta^{-1}$ we could prove the equivalence of the two sigma model actions [1].

Note that simply by setting the deformation parameter to zero, DTD models include all non-abelian and abelian T-duals of the original supercoset model, including fermionic T-dualities. Therefore all the statements we prove for DTD models apply also to (non-abelian) T-duals of supercoset models. They are also easily seen to describe all so-called TsT-transformations of the underlying supercoset model. In fact we will argue here that the class of DTD models is closed under the action of NATD, as well as certain deformations, meaning that applying these operations yields a new DTD model. They therefore represent a very broad class of integrable string sigma models.

It was shown in [1] that these models are invariant under kappa symmetry, which is needed to interpret them as Green-Schwarz superstrings. From the results of [9] it follows that their target spaces must solve the generalised supergravity equations of [9, 10] that ensure the one-loop scale invariance of the string sigma model. To have a fully consistent superstring, however, we must require the stronger condition of Weyl invariance, which implies that the target space should be a solution of the more stringent standard supergravity equations. Here we show that Weyl invariance of the DTD model is equivalent to the Lie algebra $\tilde{\mathfrak{g}}$ being unimodular, i.e. its structure constants should satisfy $f_{ij}^{j} = 0$. In fact, this condition is precisely the one found in [11, 12] when analysing the Weyl invariance of bosonic sigma models under NATD by path integral considerations. The presence of ω and the deformation does not modify the supergravity condition. When ω is invertible the condition is also equivalent to unimodularity of the R-matrix $R = \omega^{-1}$, as defined in [13], which was shown there to be the condition for Weyl invariance of YB models. The fact that these conditions are the same was in fact an important hint that the latter should have an interpretation involving NATD [2].

JHEP10(2017)024

Here we give the detailed proof of kappa symmetry for DTD models and extract the target space superfields from components of the torsion as was done for η (i.e. YB) and λ models in [13]. In particular, the RR fields and dilaton are difficult to extract by other means but we find that they are given by the simple expressions

$$e^{-2\phi} = \mathrm{sdet}'\widetilde{\mathcal{O}}\,, \qquad \mathcal{S}^{\alpha 1\beta 2} = -8i[\mathrm{Ad}_h(1+4\mathrm{Ad}_f^{-1}\widetilde{\mathcal{O}}^{-T}\mathrm{Ad}_f)]^{\alpha 1}{}_{\gamma 1}\widehat{\mathcal{K}}^{\gamma 1\beta 2}\,, \tag{1.2}$$

with $\widetilde{\mathcal{O}}$ defined in (2.4) and $\mathcal{S}$ defined in (5.2) — for definitions of the remaining quantities see sections 2 and 5. A by-product of these expressions is a formula for the transformation of RR fields under NATD for the case of supercosets. As we show in section 5 it agrees, for bosonic T-dualities, with the formula conjectured in [14], see also [15], but our formula is valid also when doing fermionic T-dualities.

An advantage of the formulation of DTD models is that many statements about the sigma model boil down to simple algebraic statements about the Lie algebra $\tilde{\mathfrak{g}}$. One example is the Weyl invariance condition already mentioned, while another concerns their transformation under NATD — possibly including additional deformation. The advantages are clear also when discussing the isometries of these models. We show that they fall into two classes; in fact, besides the standard ones, i.e. the unbroken part of the G isometries, there are also certain (abelian) shift isometries. We prove that T-dualising on either type of isometry we get back a DTD model; in particular, T-dualising on the first type of isometries is equivalent to the simple operation of enlarging $\tilde{\mathfrak{g}}$ by the corresponding generators, while T-dualising on the shift isometries removes generators from $\tilde{\mathfrak{g}}$. The latter operation can be used to prove, in this context, that solutions of the generalised supergravity equations are (formally) T-dual to solutions of the standard supergravity equations [10]. For more general NATD, where one applies T-duality on both types of isometries at the same time, we propose that the resulting model is still obtained in a similar way, namely simply by adding to $\tilde{\mathfrak{g}}$ the isometry generators that lie outside of it and removing from it the generators that are inside. We show that this conjecture is indeed consistent, i.e. the resulting model is a well-defined DTD model, which turns out to be quite non-trivial. As already mentioned this suggests that the class of DTD models is closed under (bosonic and fermionic) NATD, including also the deformations considered here.

It was suggested in [1] that it might be possible to think of all DTD models as non-abelian T-duals of YB models. Here we show that this is in fact not true by providing an example of a DTD model which cannot be obtained from a YB model by NATD.

The outline of the paper is as follows. In section 2 we introduce the DTD models based on supercosets, discuss their gauge invariances and the equivalence to YB models when ω is invertible. Section 3 describes the two classes of global symmetries, or isometries, of these models. We also address the question of what happens if one performs NATD and deformation of a DTD model and argue that this gives a new DTD model, proving this in simpler cases. Models which cannot be obtained by NATD of YB models are also discussed. In section 4 we demonstrate the kappa symmetry of DTD models and write the DTD model as a Green-Schwarz superstring. Given these results it is then straightforward to derive the target space fields of the DTD model from components of the superspace torsion, which we

JHEP10(2017)024

do in section 5. This includes a derivation of the Weyl-invariance condition for these models. In section 6 we work out the supergravity background for two examples of DTD models. The first is equivalent to a well known TsT-background but is useful to demonstrate the procedure. The second example is one of the new examples which cannot be obtained from a YB model by NATD. We finish with some conclusions and open problems. Three appendices contain some useful algebraic identities, a derivation of the DTD model action and a proof of integrability.

2 The deformed T-dual models

As described in the introduction the deformed T-dual (DTD) models are constructed as follows. We start with a supercoset sigma model, e.g. the $AdS_5 \times S^5$ superstring [16] or one of the other examples in [17, 18]. We single out a subalgebra $\tilde{\mathfrak{g}} \subset \mathfrak{g}$ of the ($\mathbb{Z}_4$-graded) superisometry algebra and write the group element as $g = \tilde{g}f$ with $\tilde{g} \in \tilde{G}$ and $f \in G$. This parametrization is of course redundant and introduces a corresponding $\tilde{G}$ gauge symmetry $\tilde{g} \to \tilde{g}\tilde{h}^{-1}$ and $f \to \tilde{h}f$ on which we will comment below. The second ingredient, which is responsible for the deformation, is a Lie algebra 2-cocycle ω on $\tilde{\mathfrak{g}}$ satisfying (1.1). We add to the original supercoset sigma model action the term

$$S_\omega = \frac{T}{4}\int_\Sigma \zeta\omega(\tilde{g}^{-1}d\tilde{g}, \tilde{g}^{-1}d\tilde{g})\,, \tag{2.1}$$

where ζ is a parameter introduced to keep track of the deformation — if there exist many 2-cocycles we could introduce a parameter for each.[2] As explained already, this is equivalent to adding a B-field to the action, which is closed by virtue of the 2-cocycle condition. This term is therefore topological and has no effect on local properties of the theory — issues with boundary conditions are more subtle and will not be considered here. The final step is to perform NATD on $\tilde{\mathfrak{g}}$. This is done in the usual way by gauging the global $\tilde{\mathfrak{g}}$ symmetry and integrating out the gauge field. This procedure guarantees that properties like integrability are preserved, see appendix C for an explicit proof. However, since T-duality is a non-local transformation of the fields of the sigma model, ω will now affect *local* properties of the deformed model.

If ω is a coboundary, meaning that $\omega(X,Y) = f([X,Y])$ for some function $f : \tilde{\mathfrak{g}} \to \mathbb{R}$, the B-field is exact; this is equivalent to no deformation at all since B is pure gauge — alternatively a field redefinition can remove the ζ dependent contributions in the deformed model. Therefore non-trivial deformations are classified by the second (Lie algebra) cohomology group $H^2(\tilde{\mathfrak{g}})$. The same group also classifies non-trivial central extensions of $\tilde{\mathfrak{g}}$, consistent with the interpretation of these models as arising from NATD on a centrally extended subalgebra of the isometry algebra [2].

Performing the above procedure one obtains the DTD supercoset model action

$$S = -\frac{T}{2}\int d^2\sigma\, \frac{\gamma^{ij} - \epsilon^{ij}}{2}\mathrm{Str}\big(J_i\hat{d}_f J_j + (\partial_i\nu - \hat{d}_f^T J_i)\widetilde{\mathcal{O}}^{-1}(\partial_j\nu + \hat{d}_f J_j)\big)\,, \quad \gamma^{ij} = \sqrt{-h}h^{ij}\,, \tag{2.2}$$

[2] If ω has mixed Grassmann even-odd components the corresponding deformation parameter ζ would be fermionic. Since the interpretation of such a fermionic deformation is not so clear we will generally assume that ω has only even-even and odd-odd components and that ζ is real.

JHEP10(2017)024

and we refer to appendix B for the details of its derivation. Here $J = dff^{-1}$ encodes the degrees of freedom in f, while $\nu \in \tilde{\mathfrak{g}}^*$ denotes the dualised degrees of freedom coming from $\tilde{g}$. We have further defined

$$\hat{d}_f = \mathrm{Ad}_f \hat{d} \mathrm{Ad}_f^{-1}\,, \qquad \hat{d} = P^{(1)} + 2P^{(2)} - P^{(3)}\,, \quad \hat{d}^T = -P^{(1)} + 2P^{(2)} + P^{(3)}\,, \tag{2.3}$$

where $P^{(i)}$ project onto the corresponding $\mathbb{Z}_4$-graded component of $\mathfrak{g} = \sum_{i=0}^{3} \mathfrak{g}^{(i)}$ and $\widetilde{\mathcal{O}}^{-1}$ is the inverse[3] of the linear operator $\widetilde{\mathcal{O}} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^*$

$$\widetilde{\mathcal{O}} = \tilde{P}^T (\hat{d}_f - \mathrm{ad}_\nu - \zeta\omega)\tilde{P}\,. \tag{2.4}$$

Given a basis $\{T_i\}$ of $\tilde{\mathfrak{g}}$ and using the fact that $\mathfrak{g}$ has a non-degenerate metric given by the supertrace, we define the Lie algebra $\tilde{\mathfrak{g}}^* \subset \mathfrak{g}$ dual to $\tilde{\mathfrak{g}}$ by taking as dual basis $\{T^i\}$, where $\mathrm{Str}(T^j T_i) = \delta_i^j$. Then we have $\tilde{P}$ and $\tilde{P}^T$ which are projectors onto $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}^*$ respectively. At the same time we are thinking of the 2-cocycle ω as a map $\omega : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^*$ so that the cocycle condition takes the form

$$\omega[x,y] = \tilde{P}^T\left([\omega x, y] + [x, \omega y]\right)\,, \qquad \forall x, y \in \tilde{\mathfrak{g}}\,. \tag{2.5}$$

Therefore, modulo the projector on the right-hand-side, ω acts as a derivation with respect to the Lie bracket, similarly to ad_ν which is a derivation thanks to the Jacobi identity.

In general one needs to make sure that the inverse $\widetilde{\mathcal{O}}^{-1}$ exists in order to be able to define the model, and this puts some restrictions on the subalgebra $\tilde{\mathfrak{g}}$. By expanding in the parameter ζ we can think of the DTD model as a deformation of the non-abelian T-dual of the original model, since taking $\zeta = 0$ reduces to ordinary NATD. Therefore, at least for a small deformation parameter the invertibility is guaranteed if one can apply NATD with respect to $\tilde{\mathfrak{g}}$. There may also be cases in which NATD cannot be implemented but the operator is invertible for finite values of ζ, i.e. the cocycle removes the 0-eigenvalues of $\widetilde{\mathcal{O}}$.

We now want to turn to the discussion of the gauge invariances of the action (2.2) of DTD models. Besides the fermionic kappa symmetry, which will be discussed separately in section 4, the action has two types of gauge invariances:

1. *Local Lorentz invariance*:

$$f \to fh\,, \qquad h \in H = G^{(0)}\,. \tag{2.6}$$

2. *Local $\tilde{G}$ invariance*:

$$f \to \tilde{h} f\,, \quad \nu \to \tilde{P}^T\left(\mathrm{Ad}_{\tilde{h}}\nu + \zeta\frac{1 - e^{\mathrm{ad}_x}}{\mathrm{ad}_x}\omega x\right)\,, \qquad \tilde{h} = e^x \in \tilde{G} \subset G\,. \tag{2.7}$$

The former is obvious and, as in the case of supercosets, it boils down to the fact that $P^{(0)}$ is missing in $\hat{d}$. As mentioned at the beginning of this section, the latter comes about from the decomposition of the original group element as $g = \tilde{g}f$ where multiplication of $\tilde{g}$ from the right by an element of $\tilde{G}$ can be compensated for by multiplying f on the left by

[3] Notice that $\widetilde{\mathcal{O}}\widetilde{\mathcal{O}}^{-1} = \tilde{P}^T$ and $\widetilde{\mathcal{O}}^{-1}\widetilde{\mathcal{O}} = \tilde{P}$ rather than 1.

JHEP10(2017)024

the inverse group element. To verify that the action is indeed invariant under the second type of symmetry we use the identities (A.7) and (A.8) that say how the transformations of $\widetilde{\mathcal{O}}$ and $d\nu$ can be rewritten. Then the difference of the actions after and before the transformation (2.7) is proportional to

$$\int d^2\sigma\epsilon^{ij}\mathrm{Str}\Big(2\partial_i\nu\tilde{h}^{-1}\partial_j\tilde{h}+\tilde{h}^{-1}\partial_i\tilde{h}(\mathrm{ad}_\nu+\zeta\omega)(\tilde{h}^{-1}\partial_j\tilde{h})\Big)\,. \tag{2.8}$$

The terms involving ν combine to a total derivative, and the one with ω is closed as already remarked, meaning that it is also a total derivative at least locally. This establishes the invariance of the action under the local transformation (2.7). This gauge invariance is obviously present also in the case of NATD, where the shift of ν is absent since $\zeta=0$.

The classical integrability of DTD models may be argued by the fact that they are obtained by adding a closed B-field and then applying NATD to the action of a supercoset, since neither of these operations breaks classical integrability, see e.g. [19] for the argument in the case of NATD. In appendix C we give a direct proof of the classical integrability of these models by showing that, similarly to what was shown in the case of DTD of PCM in [1], the on-shell equations can be recast into the flatness condition

$$\epsilon^{ij}(\partial_i\mathcal{L}_j+\mathcal{L}_i\mathcal{L}_j)=0\,, \tag{2.9}$$

for the Lax connection

$$\mathcal{L}_i=A_i^{(0)}+zA_i^{(1)}+\frac{1}{2}\left(z^2+z^{-2}\right)A_i^{(2)}+\frac{1}{2}\gamma_{ij}\epsilon^{jk}\left(z^{-2}-z^2\right)A_i^{(2)}+z^{-1}A_i^{(3)}\,, \tag{2.10}$$

where z is the spectral parameter, $A^i=A^i_++A^i_-$ and $A^i_\pm\equiv\mathrm{Ad}_f^{-1}(\tilde{A}^i_\pm+J^i_\pm)$, with $\tilde{A}^i_\pm$ given in (B.5). See appendix B for our notation. Notice that the presence of the Lax connection still implies that we have conserved charges corresponding to the full original $\mathfrak{g}$ symmetry. However, in contrast to the case of supercosets, for DTD models one cannot argue any more that they are all local, see appendix C.

2.1 Relation to Yang-Baxter sigma models

Given a DTD model with a cocycle ω which is non-degenerate on $\tilde{\mathfrak{g}}$, we can show that the action can be recast into the one of a YB model via a field redefinition. This result was first presented in [1] and we collect here more details of the proof.

Given a non-degenerate ω we denote its inverse by $R=\omega^{-1}$. From the cocycle condition for ω it follows that R solves the CYBE on $\tilde{\mathfrak{g}}^*$. Conversely any solution of the CYBE on $\mathfrak{g}$ defines an invertible 2-cocycle on a subalgebra[4] $\tilde{\mathfrak{g}}$, which demonstrates the one-to-one correspondence between DTD models with invertible ω and YB sigma models based on an R-matrix solving the CYBE. The field redefinition that relates the two models is

$$\nu=\zeta\tilde{P}^T\frac{1-\mathrm{Ad}_{\bar{g}}}{\mathrm{ad}_{Rx}}\omega Rx\,,\qquad \bar{g}=e^{Rx}\in\tilde{G}\,, \tag{2.11}$$

[4]This follows from the fact that the subspace on which R is invertible must be a subalgebra due to the CYBE [20]. Since $\omega=R^{-1}$ is a 2-cocycle on this subalgebra the subalgebra is quasi-Frobenius. Note that these results are true also for non-semisimple algebras and superalgebras.

with $x \in \tilde{\mathfrak{g}}^*$ so that $Rx \in \tilde{\mathfrak{g}}$. In fact, using the identities in (A.5) and (A.4) we find

$$d\nu = \tilde{P}^T(\mathrm{ad}_\nu + \zeta\omega)(\bar{g}^{-1}d\bar{g}) , \qquad \tilde{P}^T \mathrm{ad}_\nu \tilde{P} = \zeta \tilde{P}^T \mathrm{Ad}_{\bar{g}}^{-1} \omega \mathrm{Ad}_{\bar{g}} \tilde{P} - \zeta\omega , \tag{2.12}$$

and the action (2.2) becomes, after a bit of algebra,

$$S = -\frac{T}{2}\int d^2\sigma \, \frac{\gamma^{ij}-\epsilon^{ij}}{2} \mathrm{Str}\left(g^{-1}\partial_i g \hat{d} \left(1 - \frac{R_g \hat{d}}{R_g \hat{d} - \zeta} \right) g^{-1}\partial_j g + \bar{g}^{-1}\partial_i \bar{g} (\mathrm{ad}_\nu + \zeta\omega) \bar{g}^{-1}\partial_j \bar{g} \right) , \tag{2.13}$$

where we have defined $g = \bar{g} f$ and $R_g = \mathrm{Ad}_g^{-1} R \mathrm{Ad}_g$. The last term vanishes up to a total derivative and we are left precisely with the action of the YB sigma model [6, 7]

$$S = -\frac{T}{2}\int d^2\sigma \, \frac{\gamma^{ij}-\epsilon^{ij}}{2} \mathrm{Str}\left(g^{-1}\partial_i g \; \hat{d} \; (1 - \eta R_g \hat{d})^{-1} (g^{-1}\partial_j g) \right) , \tag{2.14}$$

with deformation parameter $\eta = \zeta^{-1}$. In the special case when $\tilde{\mathfrak{g}}$ is abelian the DTD model is equivalent to a TsT transformation of the original supercoset sigma model, in agreement with the YB side for abelian R [2, 21].

Let us mention that one can also construct a YB model for an R-matrix solving the modified CYBE, whose action takes essentially the same form as the above one [6]; however, in that case it is not clear how to define the operator corresponding to ω, and the relation to DTD models remains unclear. This case should be related by Poisson-Lie T-duality to the λ-model of [19, 22].

We will argue in the next section that all (bosonic and fermionic) non-abelian T-duals of YB sigma models can be described as DTD models with certain degenerate ω. The converse is not true, in fact it is possible to identify DTD models which are not related to YB models by NATD; we refer to section 3.2 for an example and a discussion on this.

3 Global symmetries

We will now describe the global symmetries, i.e. superisometries, of DTD models. Setting $\zeta = 0$ and ignoring the presence of ω this discussion reduces to what one would have in the case of NATD. In order to identify the global symmetries of these models we study the global transformations that leave the action invariant, *modulo* gauge transformations with a global parameter, since the latter would not produce any Noether charge. We find two types of global symmetries:[5]

1. *Unbroken global G-transformations*:

$$\begin{aligned} &f \to g_0 f , \quad \nu \to \tilde{P}^T \mathrm{Ad}_{g_0} \nu , \qquad g_0 \in G \text{ and } g_0 \notin \tilde{G}, \\ &\text{such that} \quad (1-\tilde{P})\mathrm{Ad}_{g_0}\tilde{P} = 0 , \quad \tilde{P}^T \mathrm{Ad}_{g_0}^{-1} \omega \mathrm{Ad}_{g_0} \tilde{P} = \omega . \end{aligned} \tag{3.1}$$

   The requirement $g_0 \notin \tilde{G}$ comes from the fact that for $g_0 \in \tilde{G}$ a combination of this isometry and the shift isometries described below is equivalent to a global $\tilde{G}$ gauge transformation.

[5]The two sets of transformations do not commute and their commutator is a transformation of the second type.

JHEP10(2017)024

2. *Global shifts of* ν:

$$\nu \to \nu + \lambda \,, \qquad \lambda \in \tilde{\mathfrak{g}}^* \quad \text{such that} \quad \tilde{P}^T \mathrm{ad}_\lambda \tilde{P} = 0 \,. \tag{3.2}$$

Note that the set of such λ's will in general *not* close into a subalgebra, although the corresponding isometry transformations of course commute since they are just shifts of ν.

In the case when ω is invertible, which is equivalent to a YB sigma model with $R = \omega^{-1}$, it is not hard to show that these isometries coincide with the ones of the YB model which are normally written as $t \in \mathfrak{g}$ such that $R\mathrm{ad}_t = \mathrm{ad}_t R$.

Having global symmetries at our disposal means that we can gauge them and implement further NATD. Before discussing the details of this in the next subsection, we would like to exploit this possibility to make a comment regarding Weyl invariance of DTD models. As we prove in section 5, the target spaces of DTD models solve the standard supergravity equations if and only if the Lie algebra $\tilde{\mathfrak{g}}$ is unimodular, i.e. $f_{ab}{}^b = 0$. The standard supergravity equations are equivalent to the Weyl invariance at one-loop for the sigma-model, as opposed to just the scale invariance implied by the generalised supergravity equations [9, 10]. In the non-unimodular case $f_{ab}{}^b \neq 0$, and this defines a distinguished element of $\tilde{\mathfrak{g}}$; we can rotate the basis so that this element is T_1, i.e. $f_{1b}{}^b \neq 0$ and $f_{ab}{}^b = 0$ for $a \neq 1$. The important observation is that the dual of the generator T_1 corresponds to an isometry. In fact, taking the trace of the Jacobi identity we find $f_{ab}{}^1 = 0$ and therefore

$$\mathrm{Str}(T_b \mathrm{ad}_{T^1} T_a) = f_{ab}{}^1 = 0 \,, \tag{3.3}$$

where $T^a \in \tilde{\mathfrak{g}}^*$. This confirms that T^1 satisfies (3.2) and can be used to generate a shift isometry. Using the results of the next subsection, applying T-duality along the isometry direction T^1 one obtains a DTD model where T_1 is removed from $\tilde{\mathfrak{g}}$, so that the subalgebra that is left is now unimodular. Therefore, to each DTD model which is not Weyl invariant we can associate a Weyl invariant one obtained by (formal[6]) T-duality along a particular isometry direction. Obviously this possibility fails if there are obstructions to carrying out the T-duality, e.g. if the isometry in question is a null isometry. More generally, solutions of the *generalised* supergravity equations are formally T-dual to solutions of the *standard* supergravity equations [9, 10], and the above argument shows this relation in the specific context of DTD models.

3.1 DTD of DTD models

It is interesting to start from a DTD model as in (2.2) and further perform NATD, possibly including a deformation by a cocycle. We do this on the one hand to show that the application of these transformations on the sigma model does not require to start from a supercoset formulation, on the other hand to show that after these transformations we

[6]Our discussion of isometries is at the level of the classical sigma model action, where the dilaton only appears in the combination $\mathcal{F} = e^\phi F$ — together with RR fields — and in derivatives $\partial\phi$. When performing the T-duality we ignore the Fradkin-Tseytlin term, which will break the isometry referred to here.

JHEP10(2017)024

obtain a new DTD model. We will also use these results to argue that the example of the next subsection is not related to a YB model by NATD.

We can apply NATD by gauging the global isometries discussed above and dualising the corresponding directions. Obviously, the choice of the type of isometries that we want to dualise will produce qualitative differences. In fact, if we consider isometries of the first type (3.1) and dualise a subalgebra $\hat{\mathfrak{g}}$, we essentially enlarge the subalgebra $\tilde{\mathfrak{g}}$. If instead we consider isometries of the shift type (3.2) and dualise a subspace $\bar{V}^* \subset \tilde{\mathfrak{g}}^*$, then we remove generators from the subalgebra $\tilde{\mathfrak{g}}$. The combination of isometry transformations that we consider here is therefore

$$f = \hat{g} f' \,, \quad \nu = \tilde{P}^T (\mathrm{Ad}_{\hat{g}} \nu' + \bar{\lambda}) \,, \qquad \text{with} \qquad \hat{g} \in \hat{G} \,, \quad \bar{\lambda} \in \bar{V}^* \,. \tag{3.4}$$

After gauging them in the usual way we obtain a sigma model action which is just the one in (2.2), where we replace[7]

$$f \to f' \,, \qquad J \to J' + \hat{A} \,, \qquad d\nu \to d\check{\nu} + \check{P}^T [\hat{A}, \check{\nu}] + \bar{a} \,, \tag{3.5}$$

where $\hat{A} \in \hat{\mathfrak{g}}$ is the non-abelian gauge field corresponding to the $\hat{G}$ isometries and $\bar{a} \in \bar{V}^*$ is the abelian gauge field corresponding to the shift isometries. We add to the action the terms[8]

$$-T \int d^2\sigma \, \mathrm{Str}(\hat{\nu} \hat{F}_{+-} + \bar{\rho} \bar{f}_{+-} - \hat{\zeta} \hat{A}_+ \hat{\omega} \hat{A}_-) \,, \tag{3.6}$$

where $\hat{F}_{+-} = \partial_+ \hat{A}_- - \partial_- \hat{A}_+ + [\hat{A}_+, \hat{A}_-]$ and $\bar{f}_{+-} = \partial_+ \bar{a}_- - \partial_- \bar{a}_+$, $\hat{\nu}$ and $\bar{\rho}$ are two new Lagrange multipliers, and $\hat{\omega}$ is a cocycle on $\hat{\mathfrak{g}}$. Integrating out $\hat{\nu}$ and $\bar{\rho}$ one obtains the action from which we started; to apply NATD we integrate out $\hat{A}$ and $\bar{a}$ instead.

We will now describe what happens when we dualise either $\hat{\mathfrak{g}}$ or $\bar{V}^*$, and then use it to argue what should happen in the most general case where one dualises on both at the same time.[9]

Dualising type 1 isometries. Consider first isometries of type 1 above, where we have $\hat{P} + \check{P} = \tilde{P}$ and $\hat{P}\check{P} = 0$. After a bit of algebra and dropping primes, we find that the new action takes the form $S = -T \int d^2\sigma \mathrm{Str}(J_+ \hat{d}_f J_- + (\partial_+ \nu - \hat{d}_f^T J_+) \mathcal{Q} (\partial_- \nu + \hat{d}_f J_-))$ where $\nu = \check{\nu} + \hat{\nu}$ and $\mathcal{Q}$ is an operator acting on $\tilde{\mathfrak{g}} = \check{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$ which can be written in a 2×2 block form as

$$\mathcal{Q} = \begin{pmatrix} \check{\mathcal{O}}^{-1} + \check{\mathcal{O}}^{-1} (\hat{d}_f - \mathrm{ad}_{\check{\nu}}) U^{-1} (\hat{d}_f - \mathrm{ad}_{\check{\nu}}) \check{\mathcal{O}}^{-1} & -\check{\mathcal{O}}^{-1} (\hat{d}_f - \mathrm{ad}_{\check{\nu}}) U^{-1} \\ -U^{-1} (\hat{d}_f - \mathrm{ad}_{\check{\nu}}) \check{\mathcal{O}}^{-1} & U^{-1} \end{pmatrix} , \tag{3.7}$$

[7]We will now use the notation $\check{\nu} \in \check{\mathfrak{g}}$ for the field and the subalgebra of the DTD model from which we start. Similarly, we will denote the corresponding operators as $\check{P}$, $\check{\mathcal{O}}$, etc. We do this because we want to reserve the usual notation for the DTD model that is obtained at the end, after applying the further deformation of NATD.

[8]For the sake of the discussion here we fix conformal gauge $\gamma^{+-} = \gamma^{-+} = \epsilon^{-+} = -\epsilon^{+-} = 2$ where $\sigma^\pm = \tau \pm \sigma$. In principle it is also possible to add a deformation for the second type of isometry by adding a term $\bar{a} \bar{\omega}' \bar{a}$, but we will not consider this possibility further here.

[9]In the rest of this section we absorb the parameter ζ into ω to simplify the expressions.

where[10] $U = \hat{\mathcal{O}} - \hat{P}^T(\hat{d}_f - \mathrm{ad}_{\check{\nu}})\check{\mathcal{O}}^{-1}(\hat{d}_f - \mathrm{ad}_{\check{\nu}})\hat{P}$. It is straightforward to check that if we take $\omega = \check{\omega} + \hat{\omega}$ and define $\widetilde{\mathcal{O}}$ as in (2.4), then its decomposition in block form is

$$\widetilde{\mathcal{O}} = \begin{pmatrix} \check{\mathcal{O}} & \check{P}^T(\hat{d}_f - \mathrm{ad}_{\check{\nu}})\hat{P} \\ \hat{P}^T(\hat{d}_f - \mathrm{ad}_{\check{\nu}})\check{P} & \hat{\mathcal{O}} \end{pmatrix}, \tag{3.8}$$

and that $\mathcal{Q} = \widetilde{\mathcal{O}}^{-1}$. Therefore performing DTD by exploiting the unbroken isometries of the first type is equivalent to the simple operation of enlarging the dualised subalgebra as $\tilde{\mathfrak{g}} = \check{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$, which is a Lie algebra due to the isometry condition $[\hat{\mathfrak{g}}, \check{\mathfrak{g}}] \subset \check{\mathfrak{g}}$. As for the deformation, we are just adding new contributions, and $\omega = \check{\omega} + \hat{\omega}$ is a 2-cocycle on $\tilde{\mathfrak{g}}$ due to the isometry conditions in (3.1).

Dualising type 2 isometries. For isometries of type 2 we have $\bar{P}^T$ that projects on the space $\bar{V}^*$, so that $\bar{P}\check{P} = \check{P}\bar{P} = \bar{P}$ and $\tilde{P} = \check{P} - \bar{P}$. When integrating out $\bar{a}_\pm$ we get equations where $\bar{P}\check{\mathcal{O}}^{-1}$ appears, so that it is convenient to use the block decomposition on the space $\tilde{\mathfrak{g}} \oplus \bar{V}$

$$\begin{aligned}
\check{\mathcal{O}}^{-1} &\equiv \begin{pmatrix} \widetilde{\mathcal{O}} & \tilde{P}^T(\hat{d}_f - \mathrm{ad}_{\tilde{\nu}} - \check{\omega})\bar{P} \\ \bar{P}^T(\hat{d}_f - \mathrm{ad}_{\tilde{\nu}} - \check{\omega})\tilde{P} & \bar{P}^T(\hat{d}_f - \mathrm{ad}_{\tilde{\nu}} - \check{\omega})\bar{P} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \widetilde{\mathcal{O}}^{-1} + \widetilde{\mathcal{O}}^{-1}(\hat{d}_f - \mathrm{ad}_{\tilde{\nu}} - \check{\omega})U^{-1}(\hat{d}_f - \mathrm{ad}_{\tilde{\nu}} - \check{\omega})\widetilde{\mathcal{O}}^{-1} & -\widetilde{\mathcal{O}}^{-1}(\hat{d}_f - \mathrm{ad}_{\tilde{\nu}} - \check{\omega})U^{-1} \\ -U^{-1}(\hat{d}_f - \mathrm{ad}_{\tilde{\nu}} - \check{\omega})\widetilde{\mathcal{O}}^{-1} & U^{-1} \end{pmatrix},
\end{aligned} \tag{3.9}$$

where $U = \bar{P}^T(\hat{d}_f - \mathrm{ad}_{\tilde{\nu}} - \check{\omega})\bar{P} - \bar{P}^T(\hat{d}_f - \mathrm{ad}_{\tilde{\nu}} - \check{\omega})\widetilde{\mathcal{O}}^{-1}(\hat{d}_f - \mathrm{ad}_{\tilde{\nu}} - \check{\omega})\bar{P}$.

Note that $\tilde{\mathfrak{g}} = \{x \in \check{\mathfrak{g}} \,|\, \mathrm{Str}(x\lambda) = 0\,,\ \forall \lambda \in \bar{V}^*\}$ is indeed a subalgebra since for $x, y \in \tilde{\mathfrak{g}}$ we have $\mathrm{Str}([x,y]\lambda) = -\mathrm{Str}(x\mathrm{ad}_\lambda y) = 0$ as a consequence of (3.2). In fact for $x, y \in \check{\mathfrak{g}}$ we have in the same way $[x,y] \in \tilde{\mathfrak{g}}$. This means in particular that if $\bar{V}$ closes into a subalgebra it must be abelian. Clearly $\check{\omega}$ reduces to a 2-cocycle $\tilde{\omega} = \tilde{P}^T\check{\omega}\tilde{P}$ on $\tilde{\mathfrak{g}}$.

After some algebra and dropping a total derivative $d\nu d\bar{\rho}$-term, the dualised action becomes

$$\begin{aligned}
-T\int d^2\sigma \mathrm{Str}\Big(&(J_+ + \partial_+\bar{\rho})\hat{d}_f(J_- + \partial_-\bar{\rho}) + (\partial_+\tilde{\nu} - \hat{d}_f^T J_+)\widetilde{\mathcal{O}}^{-1}(\partial_-\tilde{\nu} + \hat{d}_f J_-) \\
&+ (\partial_+\tilde{\nu} - \hat{d}_f^T J_+)\widetilde{\mathcal{O}}^{-1}(\hat{d}_f - \mathrm{ad}_{\tilde{\nu}} - \check{\omega})\partial_-\bar{\rho} - \partial_+\bar{\rho}(\hat{d}_f - \mathrm{ad}_{\tilde{\nu}} - \check{\omega})\widetilde{\mathcal{O}}^{-1}(\partial_-\tilde{\nu} + \hat{d}_f J_-) \\
&- \partial_+\bar{\rho}(\hat{d}_f - \mathrm{ad}_{\tilde{\nu}} - \check{\omega})\widetilde{\mathcal{O}}^{-1}(\hat{d}_f - \mathrm{ad}_{\tilde{\nu}} - \check{\omega})\partial_-\bar{\rho} - \partial_+\bar{\rho}(\mathrm{ad}_{\tilde{\nu}} + \check{\omega})\partial_-\bar{\rho}\Big)\,.
\end{aligned} \tag{3.10}$$

As expected $\bar{\nu} = \check{\nu} - \tilde{\nu}$ has dropped out, since we have dualised the corresponding directions. Finally $\bar{\rho}$ can be removed by the field redefinition

$$f \to \bar{h}f\,, \quad \tilde{\nu} \to \tilde{P}^T\left(\mathrm{Ad}_{\bar{h}}\nu + \frac{1 - \mathrm{Ad}_{\bar{h}}}{\mathrm{ad}_{\bar{\rho}}}\check{\omega}\bar{\rho}\right)\,, \qquad \bar{h} = e^{-\bar{\rho}}\,, \tag{3.11}$$

which resembles a $\tilde{G}$ gauge transformation except for the fact that $\bar{h} \notin \tilde{G}$. To check that we match with the DTD action in (2.2) we use the fact that under the above redefinition

[10]The operators $\check{\mathcal{O}}, \hat{\mathcal{O}}$ are obtained from $\widetilde{\mathcal{O}}$ by dressing ν, ω and the projectors with checks or hats.

$\widetilde{\mathcal{O}} \to \check{P}^T \mathrm{Ad}_{\bar{h}} \widetilde{\mathcal{O}} \mathrm{Ad}_{\bar{h}}^{-1} \check{P}$ which follows from[11]

$$\begin{aligned} \check{P}^T \mathrm{ad}_{\tilde{\nu}} \check{P} &\to \check{P}^T \mathrm{Ad}_{\bar{h}} \check{P}^T \mathrm{ad}_{\nu} \check{P} \mathrm{Ad}_{\bar{h}}^{-1} \check{P} + \check{P}^T \mathrm{Ad}_{\bar{h}} \check{\omega} \mathrm{Ad}_{\bar{h}}^{-1} \check{P} - \check{\omega} \,, \\ d\tilde{\nu} &\to \check{P}^T \mathrm{Ad}_{\bar{h}} (d\nu - \mathrm{ad}_{\nu}(\bar{h}^{-1} d\bar{h}) - \check{\omega}(\bar{h}^{-1} d\bar{h}))\,. \end{aligned} \tag{3.12}$$

The calculations are simple when $\bar{V}$ is a (abelian) subalgebra since in that case $\bar{h}^{-1} d\bar{h} = -\mathrm{Ad}_{\bar{h}}^{-1} d\bar{\rho}$ and the last $d\bar{\rho} d\bar{\rho}$ term vanishes up to a total derivative. When $\bar{V}$ is not a subalgebra it is clear that it must still work since these are abelian isometries and we can just T-dualise one at a time. It is nevertheless instructive to show this explicitly. To do this we use the fact that $\bar{h}^{-1} d\bar{h} + \mathrm{Ad}_{\bar{h}}^{-1} d\bar{\rho}$ is in $\tilde{\mathfrak{g}}$ since it involves commutators of elements from $\bar{V}$. This simplifies the left-over terms to $\int d\sigma^2 \epsilon^{ij} \mathrm{Str}(\bar{h}^{-1} \partial_i \bar{h}\, \check{\omega}(\bar{h}^{-1} \partial_j \bar{h}))$ which indeed is a total derivative term and can be dropped. As anticipated, we get that T-dualising on the shift isometries is equivalent to shrinking $\tilde{\mathfrak{g}}$ by removing the generators in $\bar{V}$.

Dualising type 1 and 2 isometries. We have seen that dualising on the isometries outside of $\tilde{\mathfrak{g}}$ has the effect of adding the corresponding generators to $\tilde{\mathfrak{g}}$. Similarly dualising on isometries inside $\tilde{\mathfrak{g}}$ effectively removes the corresponding generators. The natural conjecture is then that dualising on both types of isometries at the same time again just adds/removes the generators outside/inside $\tilde{\mathfrak{g}}$ to give the $\tilde{\mathfrak{g}}$ of the resulting model.

To be more specific, start from a DTD model with a cocycle on the subalgebra[12] $\check{\mathfrak{g}}$ and imagine the most general NATD of this DTD model where we dualise isometries $t_i \notin \check{\mathfrak{g}}$ of type 1 as in (3.1) and $\lambda_I \in \check{\mathfrak{g}}^*$ of type 2 as in (3.2). Our conjecture is that this results in a new DTD model where now

$$\tilde{\mathfrak{g}} = \{x = \check{y} + a_i t_i \,,\, \check{y} \in \check{\mathfrak{g}} \,|\, \mathrm{Str}(\lambda_I \check{y}) = 0 \,,\, \forall \lambda_I \text{ such that } \mathrm{Str}(\lambda_I [t_i, t_j]) = 0 \,, \forall t_i, t_j\}\,. \tag{3.13}$$

In other words, $\tilde{\mathfrak{g}}$ is obtained by adding to $\check{\mathfrak{g}}$ all generators t_i and by removing all elements which are dual to λ_I, except when these are generated in commutators $[t_i, t_j]$. In fact, we want the last condition on λ_I because the commutator of two isometries of type 1 can generate an isometry of type 2, and if we are adding the t_i we want to make sure that they close into an algebra. Here we will not work out explicitly the transformation of the action under this NATD since this is quite involved, we will rather just check that this expectation makes sense and such a DTD model is well-defined.

To start, we must assume that the isometries on which we dualise form a subalgebra of the isometry algebra. This implies the conditions

$$[t_i, t_j] = c_{ij}{}^k t_k + \check{c}_{ij}{}^{K'} \check{t}_{K'} \,, \qquad \check{\omega}(\check{t}_{I'}) = \delta^I_{I'} \lambda_I \,, \qquad \check{P}^T \mathrm{ad}_{t_i} \lambda_I = c_{iI}{}^J \lambda_J \,, \tag{3.14}$$

with some coefficients $c_{ij}{}^k$, $\check{c}_{ij}{}^k$ and $c_{iI}{}^J$. The generators $\check{t}_{K'} \in \check{\mathfrak{g}}$ appear because, as already mentioned, the commutators of two t_i can generate an element in $\check{\mathfrak{g}}$. These must still satisfy the second condition in (3.1) which translates to the second condition above.

[11] These are proved using (A.4), (A.5) and $\tilde{P} \mathrm{Ad}_{\bar{h}} \check{P} = \mathrm{Ad}_{\bar{h}} \tilde{P}$, the last being a consequence of $[x, y] \in \tilde{\mathfrak{g}}$ for any $x, y \in \check{\mathfrak{g}}$.

[12] Also here we prefer to change notation and call $\check{\mathfrak{g}}$ the original subalgebra, so that $\tilde{\mathfrak{g}}$ will be used for the algebra obtained after applying NATD.

JHEP10(2017)024

The first consistency check is to show that $\tilde{\mathfrak{g}}$ defined above indeed forms a subalgebra of $\mathfrak{g}$ so that the corresponding DTD model can be defined. Commuting two elements of $\tilde{\mathfrak{g}}$ we get

$$[\check{y} + a_i t_i, \check{z} + b_j t_j] = [\check{y}, \check{z}] - b_i \mathrm{ad}_{t_i} \check{y} + a_i \mathrm{ad}_{t_i} \check{z} + a_i b_j [t_i, t_j]\,. \tag{3.15}$$

The isometry conditions in (3.1) indeed imply that the second and third term are in $\check{\mathfrak{g}}$. Taking the supertrace with λ_I satisfying $\mathrm{Str}(\lambda_I [t_i, t_j]) = 0$ we get

$$\mathrm{Str}([\check{y}, \check{z}]\lambda_I) + b_i c_{iI}{}^J \mathrm{Str}(\check{y}\lambda_J) - a_i c_{iI}{}^J \mathrm{Str}(\check{z}\lambda_J) = -\mathrm{Str}(\check{y}\mathrm{ad}_{\lambda_I} \check{z}) = 0\,, \tag{3.16}$$

where we used the conditions (3.14) and the fact that $\check{y}, \check{z} \in \tilde{\mathfrak{g}}$ and, in the last step, the isometry condition (3.2) for λ_I. This proves that indeed $\tilde{\mathfrak{g}}$ in (3.13) defines a subalgebra of $\mathfrak{g}$. To define a 2-cocycle on $\tilde{\mathfrak{g}}$ we take $\omega = \tilde{P}^T \check{\omega} \tilde{P}$ — we could also add an additional deformation in the t_i directions but we will not do so here— and we find

$$\begin{aligned}\omega[\check{y} + a_i t_i, \check{z} + b_j t_j] &= \tilde{P}^T \Big([\check{\omega}\check{y}, \check{z} + b_i t_i] + [\check{y} + a_i t_i, \check{\omega}\check{z}] + a_i b_j \check{\omega}[t_i, t_j] \Big) \\ &= \tilde{P}^T [\omega\check{y}, \check{z} + b_i t_i] + \tilde{P}^T [\check{y} + a_i t_i, \omega\check{z}] + a_i b_j \tilde{P}^T \check{\omega}[t_i, t_j]\,,\end{aligned} \tag{3.17}$$

where we used the cocycle condition for $\check{\omega}$, the fact that ad_{t_i} commutes with $\check{\omega}$ (3.1), and in the last step we used (A.1). The first two terms are precisely what we want, it remains to show that the last one vanishes. By the conditions (3.14) this term is proportional to a combination of λ_I and therefore the $\tilde{P}^T$ projection means that this term vanishes unless $\mathrm{Str}([t_k, t_l]\check{\omega}[t_i, t_j]) \neq 0$ for some k, l. However

$$\begin{aligned}\mathrm{Str}([t_k, t_l]\check{\omega}[t_i, t_j]) &= \frac{1}{2}\mathrm{Str}(\check{\omega}[[t_i, t_j], [t_k, t_l]]) \\ &= \frac{1}{2}\mathrm{Str}(\check{P}^T [\check{\omega}[t_i, t_j], [t_k, t_l]]) + \frac{1}{2}\mathrm{Str}(\check{P}^T [[t_i, t_j], \check{\omega}[t_k, t_l]]) \\ &= \frac{1}{2}\check{c}_{ij}{}^I \mathrm{Str}(\check{P}^T \mathrm{ad}_{\lambda_I}[t_k, t_l]) - \frac{1}{2}\check{c}_{kl}{}^I \mathrm{Str}(\check{P}^T \mathrm{ad}_{\lambda_I}[t_i, t_j]) = 0\,,\end{aligned} \tag{3.18}$$

where we used the cocycle condition and the isometry condition in (3.2). Therefore ω is indeed a 2-cocycle on $\tilde{\mathfrak{g}}$ and the corresponding DTD model is well-defined.

3.2 DTD models not related to YB models by NATD

Here we want to present an example of a DTD model which is not related to a YB model by NATD.[13] To argue that this is the case we use two important facts concerning the dualisation of the two types of isometries discussed above. First, when dualising isometries of type 1, thanks to the condition (3.1) the original $\check{\mathfrak{g}}$ will become an ideal of the larger algebra $\tilde{\mathfrak{g}}$ that is obtained by adding the generators t_i, i.e. by applying NATD. That means that starting from a YB model — or, rather, its corresponding DTD model with non-degenerate ω — NATD on isometries of type 1 will produce a DTD model with a cocycle *non-degenerate on an ideal* of $\tilde{\mathfrak{g}}$. When we include also isometries of type 2 it remains true

[13]Let us mention that it is possible to find examples where ω — as well as any 2-cocycle in its equivalence class — is non-degenerate on a space which does not close into an algebra. This corrects a statement in the first version of [1].

that what is left of $\check{\mathfrak{g}}$ forms a proper ideal inside $\tilde{\mathfrak{g}}$, on which, however, ω does not have to be non-degenerate. We also remark that, since they are realised as linear shifts, isometries of type 2 are commuting and are therefore still present even after applying abelian T-duality along them. After the dualisation the corresponding symmetry will be realised as an isometry of type 1.

Consider the following algebra and corresponding 2-cocycle

$$\tilde{\mathfrak{g}} = \text{span}\{p_1,\, p_2,\, p_3,\, J_{12}\}\,, \qquad \omega = k_3 \wedge J_{12}\,, \tag{3.19}$$

where we refer to [13] for our definitions and conventions on the generators of the conformal algebra $\mathfrak{so}(2,4)$. The above 2-cocycle is defined on a space which is not an ideal of $\tilde{\mathfrak{g}}$, and it is clear that adding an exact term to ω cannot change this, since the only terms that we could add are $k_1 \wedge J_{12}$ and $k_2 \wedge J_{12}$. According to the above discussion, this rules out the possibility of this example coming from dualising isometries of type 1 of a YB model. In fact, since there is no proper ideal in $\tilde{\mathfrak{g}}$ that contains the subspace $\{p_3, J_{12}\}$ where ω is defined, a combination of isometries of type 1 and type 2 is also ruled out. This leaves only the possibility that this example is generated by T-dualising isometries of type 2 only. If it were true that it comes from a YB model by dualising isometries of type 2, these should be realised here as isometries of type 1 and we would be able to dualise them back to find a YB model (in DTD form). However, in this example the only isometry of type 1 corresponds to p_0, and adding p_0 to $\tilde{\mathfrak{g}}$ does not help in making the cocycle non-degenerate on the dualised algebra. We therefore conclude that the above example is not related to a YB model by NATD,[14] and we refer to section 6.2 for the corresponding supergravity background.

The above example may be obtained by dropping one of the two terms in R_{11} in table 2 of [13], and similar examples coming from dropping a term in other rank 4 R-matrices of [13] are e.g.

$$\begin{aligned}
&\tilde{\mathfrak{g}} = \text{span}\{p_1,\, p_2,\, p_3,\, p_0 + J_{12}\}\,, && \omega = k_3 \wedge (k_0 {+}\, J_{12})\,, && \text{from } R_{10}\,.\\
&\tilde{\mathfrak{g}} = \text{span}\{p_0,\, p_1,\, p_2,\, J_{12}\}\,, && \omega = k_0 \wedge J_{12}\,, && \text{from } R_{13}\,.\\
&\tilde{\mathfrak{g}} = \text{span}\{p_1,\, p_2,\, J_{12},\, J_{03}\}\,, && \omega = J_{12} \wedge J_{03}\,, && \text{from } R_{14}\,.
\end{aligned} \tag{3.20}$$

In each case it is easy to see that ω cannot be defined on an ideal in $\tilde{\mathfrak{g}}$ even if we add exact terms — in the first case the only terms that we could add are $k_1 \wedge (k_0 + J_{12})$ and $k_2 \wedge (k_0 + J_{12})$, in the second and third case they are $k_1 \wedge J_{12}$ and $k_2 \wedge J_{12}$. In the first case the only isometry of type 1 corresponds to p_0, while in the second and third there is no isometry of type 1. Note that the second case can be embedded into $\mathfrak{so}(2,3)$ and therefore gives a deformation also of AdS_4.

4 Kappa symmetry and Green-Schwarz form

As we will show in a moment the action of DTD models is invariant under kappa symmetry variations, and this will allow us to put it into the Green-Schwarz form. To show invariance

[14] It would be interesting to understand whether this or similar examples are related to YB models in other ways, e.g. contractions.

JHEP10(2017)024

under kappa symmetry we need to consider the variation of the action under the fields ν and f, as well as the worldsheet metric γ^{ij}. The variation of the action with respect to the fields is computed in (C.1). To define a kappa symmetry variation we should also say how δf and $\delta\nu$ are expressed in terms of the kappa symmetry parameters $\tilde{\kappa}_i^{(j)}$, each of them being a local Grassmann parameter of grading j. We define $A^i_\pm \equiv \mathrm{Ad}_f^{-1}(\tilde{A}^i_\pm + J^i_\pm)$, where subscripts $\pm$ indicate that we act with the worldsheet projectors in (B.3) and $\tilde{A}^i_\pm$ is given in (B.5); we take[15]

$$\hat{d}^T(f^{-1}\delta_\kappa f) = \mathrm{Ad}_f^{-1}\delta_\kappa\nu = -\{i\tilde{\kappa}_i^{(1)}, A_-^{(2)i}\} + \{i\tilde{\kappa}_i^{(3)}, A_+^{(2)i}\}\,. \tag{4.1}$$

This relation is fixed by noticing that after we impose it the total variation of the action with respect to the fields simplifies considerably, and we find

$$\begin{aligned}(\delta_f+\delta_\nu)S &= -\frac{T}{2}\int d^2\sigma\; 4\,\mathrm{Str}\left(A_-^{(2)i}A_-^{(2)j}[A^{(1)}_{+i}, i\tilde{\kappa}^{(1)}_j] + A_+^{(2)i}A_+^{(2)j}[A^{(3)}_{-i}, i\tilde{\kappa}^{(3)}_j]\right)\\ &= -\frac{T}{2}\int d^2\sigma\;\frac{1}{2}\Big[\,\mathrm{Str}\left(A_-^{(2)i}A_-^{(2)j}\right)\mathrm{Str}\left(W[A^{(1)}_{+i}, i\tilde{\kappa}^{(1)}_j]\right)\\ &\qquad\qquad + \mathrm{Str}\left(A_+^{(2)i}A_+^{(2)j}\right)\mathrm{Str}\left(W[A^{(3)}_{-i}, i\tilde{\kappa}^{(3)}_j]\right)\Big]\,.\end{aligned} \tag{4.2}$$

Here we used the property $A^i_\pm B^j_\pm = A^j_\pm B^i_\pm$, which follows from the identity $P^{ij}_\pm P^{kl}_\pm = P^{il}_\pm P^{kj}_\pm$, as well as the identity

$$A_\pm^{(2)i}A_\pm^{(2)j} = \frac{1}{8}W\,\mathrm{Str}(A_\pm^{(2)i}A_\pm^{(2)j}) + c^{ij}\mathbb{1}_8\,, \tag{4.3}$$

where c^{ij} is an expression which is not interesting for this calculation, and $W = \mathrm{diag}(1_4, -1_4)$ is the hypercharge. The above variation does not vanish but it can be compensated by the contribution coming from varying the worldsheet metric. In fact, we first notice that the contribution of the terms involving the worldsheet metric to the action may be written as

$$S_\gamma = -\frac{T}{2}\int d^2\sigma\gamma^{ij}\;\mathrm{Str}\left(E_i^{(2)}E_j^{(2)}\right)\,, \tag{4.4}$$

where we have two possible choices for the bosonic vielbein which are related by a local Lorentz transformation, either $E^{(2)} = A_+^{(2)}$ or $E^{(2)} = A_-^{(2)}$, where

$$A_+ = \mathrm{Ad}_f^{-1}(J + \widetilde{\mathcal{O}}^{-T}(d\nu - \hat{d}_f^T J))\,, \qquad A_- = \mathrm{Ad}_f^{-1}(J - \widetilde{\mathcal{O}}^{-1}(d\nu + \hat{d}_f J))\,. \tag{4.5}$$

The subscript on $A_\pm$ is here used only to distinguish the two fields and should not be confused with the $\pm$ used to denote the worldsheet projections; however, we choose this notation since projecting on $A_\pm$ with $P^{ij}_\pm$ after reintroducing worldsheet indices we obtain in fact the $A^i_\pm$ used above.[16] We declare the kappa symmetry variation of the worldsheet metric to be

$$\delta_\kappa\gamma^{ij} = -\frac{1}{2}\left[\mathrm{Str}\left(W[A_+^{(1)i}, i\tilde{\kappa}_+^{(1)j}]\right) + \mathrm{Str}\left(W[A_-^{(3)i}, i\tilde{\kappa}_-^{(3)j}]\right)\right]\,, \tag{4.6}$$

[15] We write the kappa symmetry transformation in this way rather than the one in [1] because we want $P^{(0)}\mathrm{Ad}_f^{-1}\delta_\kappa\nu = 0$.

[16] A caveat is that the projections of $A_\pm$ in (4.5) with $P^{ij}_\mp$ do not vanish, while $P^{ij}_\mp A_{\pm j} = 0$. We trust that this will not create confusion, since the notation has clear advantages and those projections will never be needed.

JHEP10(2017)024

so that the total variation of the action under kappa symmetry transformations vanishes $(\delta_f+\delta_\nu+\delta_\gamma)S=0$. The kappa symmetry transformations for the fields may be also recast into the form

$$i_{\delta_\kappa z}E^{(2)}=0\,,\qquad i_{\delta_\kappa z}E^{(1)}=P_-^{ij}\{i\kappa_i^{(1)},E_j^{(2)}\}\,,\qquad i_{\delta_\kappa z}E^{(3)}=P_+^{ij}\{i\kappa_i^{(3)},E_j^{(2)}\}\,,\tag{4.7}$$

where $\kappa^{(1)}=\mathrm{Ad}_h\tilde\kappa^{(1)}$ and $\kappa^{(3)}=\tilde\kappa^{(3)}$ and where we made a choice for the bosonic and fermionic components of the supervielbeins

$$E^{(2)}=A_+^{(2)}=\mathrm{Ad}_hA_-^{(2)}\,,\qquad E^{(1)}=\mathrm{Ad}_hA_+^{(1)}\,,\qquad E^{(3)}=A_-^{(3)}\,.\tag{4.8}$$

The above transformations are the standard ones for kappa symmetry, and the action also takes the standard Green-Schwarz form

$$S=-\frac{T}{2}\int d^2\sigma\,\gamma^{ij}\mathrm{Str}(E_i^{(2)}E_j^{(2)})-T\int B\,,\tag{4.9}$$

where the B-field is

$$B=\frac{1}{4}\mathrm{Str}(J\wedge\hat d_fJ+(d\nu-\hat d_f^TJ)\wedge\widetilde{\mathcal{O}}^{-1}(d\nu+\hat d_fJ))\,.\tag{4.10}$$

As already noticed, $A_+^{(2)}$ and $A_-^{(2)}$ are related by a local Lorentz transformation, $A_+^{(2)}=\mathrm{Ad}_hA_-^{(2)}$ for some $h\in G^{(0)}$. For later convenience we can also relate other components of A_+ and A_- as follows[17]

$$\begin{aligned}A_-&=MA_+\,,\qquad\qquad P^{(2)}M=\mathrm{Ad}_h^{-1}P^{(2)}\,,\\M&=\mathrm{Ad}_f^{-1}[1-\tilde P-\widetilde{\mathcal{O}}^{-1}\widetilde{\mathcal{O}}^T-4\widetilde{\mathcal{O}}^{-1}\mathrm{Ad}_fP^{(2)}\mathrm{Ad}_f^{-1}(1-\tilde P)]\mathrm{Ad}_f\\&=1-4\mathrm{Ad}_f^{-1}\widetilde{\mathcal{O}}^{-1}\mathrm{Ad}_fP^{(2)}\,,\end{aligned}\tag{4.11}$$

while M^{-1} is given by the same expression as M but with $\widetilde{\mathcal{O}}$ replaced by its transpose $\widetilde{\mathcal{O}}^T=\tilde P^T(\hat d_f^T+\mathrm{ad}_\nu+\zeta\omega)\tilde P$. From this we can derive the useful relation

$$M^{-1}-1=-(M-1)\mathrm{Ad}_h\,.\tag{4.12}$$

5 Target space superfields

In this section we will derive the form of the target space supergravity superfields for the DTD model. The calculations are very similar to the ones performed in [13] for the η-model and λ-model. Once the action and kappa symmetry transformations are written in Green-Schwarz form as in (4.9) and (4.7), the easiest way to extract the background fields is by computing the torsion $T^a=dE^a+E^b\wedge\Omega_b{}^a$ and $T^\alpha=dE^\alpha-\frac{1}{4}(\Gamma_{ab}E)^\alpha\wedge\Omega^{ab}$ where

[17] As a consequence of this we have for example $A_+^{(3)}=E^{(3)}-P^{(3)}ME^{(2)}$.

Ω^{ab} is the spin connection superfield. It was shown in [9] that the constraints on the torsion implied by kappa symmetry take the form[18]

$$
\begin{aligned}
T^a &= -\frac{i}{2} E\gamma^a E\,, \\
T^{\alpha I} &= \frac{1}{2}E^{\alpha I}\, E\chi + \frac{1}{2}(\sigma^3 E)^{\alpha I}\, E\sigma^3\chi - \frac{1}{4}E\gamma_a E\,(\gamma^a\chi)^{\alpha I} - \frac{1}{4}E\gamma_a\sigma^3 E\,(\gamma^a\sigma^3\chi)^{\alpha I} \\
&\quad - \frac{1}{8}E^a\,(E\sigma^3\gamma^{bc})^{\alpha I} H_{abc} - \frac{1}{8}E^a\,(E\gamma_a\mathcal{S})^{\alpha I} + \frac{1}{2}E^b E^a\,\psi^{\alpha I}_{ab}\,,
\end{aligned} \tag{5.1}
$$

for the type IIB case.[19] The target space superfields contained here are the dilatino superfields $\chi_{\alpha I}$, the gravitino field strengths $\psi^{\alpha I}_{ab}$, where $I = 1, 2$ denotes the two Majorana-Weyl spinors of type IIB, as well as the NSNS three-form field strength $H = dB$ and "RR field strengths" encoded in the anti-symmetric 32×32 bispinor

$$
\mathcal{S} = -i\sigma^2\gamma^a\mathcal{F}_a - \frac{1}{3!}\sigma^1\gamma^{abc}\mathcal{F}_{abc} - \frac{1}{2\cdot 5!}i\sigma^2\gamma^{abcde}\mathcal{F}_{abcde}\,. \tag{5.2}
$$

Kappa symmetry implies that the target space is generically only a solution of the *generalised* type II supergravity equations defined in [9] and first written down, for the bosonic sector, in [10]. However, when the (Killing) vector

$$
K^a = -\frac{i}{16}(\gamma^a\sigma^3)^{\alpha I\beta J}\nabla_{\alpha I}\chi_{\beta J} \tag{5.3}
$$

vanishes one gets a solution of *standard* type II supergravity, and a one-loop Weyl invariant string sigma model. In that case there exists a dilaton superfield ϕ such that $\chi_{\alpha I} = \nabla_{\alpha I}\phi$ and the RR field strengths are defined in terms of potentials in the standard way $\mathcal{F} = e^{\phi}dC + \cdots$ [23, 24].

Given that the supervielbeins for the DTD model are defined in terms of $A_\pm$ as in (4.8) we need to compute the exterior derivative of $A_\pm$ defined in (4.5) to find the torsion. With a bit of work one finds the deformed "Maurer-Cartan" equations[20]

$$
dA_+ = \frac{1}{2}\{A_+, A_+\} - \frac{1}{2}\mathrm{Ad}_f^{-1}\widetilde{\mathcal{O}}^{-T}\mathrm{Ad}_f\big(\hat{d}^T\{A_+, A_+\} - 2\{A_+, \hat{d}^T A_+\}\big)\,, \tag{5.4}
$$

$$
dA_- = \frac{1}{2}\{A_-, A_-\} - \frac{1}{2}\mathrm{Ad}_f^{-1}\widetilde{\mathcal{O}}^{-1}\mathrm{Ad}_f\big(\hat{d}\{A_-, A_-\} - 2\{A_-, \hat{d}A_-\}\big)\,, \tag{5.5}
$$

where we have used the identity (A.1) and the fact that, due to the Jacobi identity and the 2-cocycle condition (2.5), both ad_ν and ω effectively act as derivations on the Lie bracket. Projecting the first equation with $P^{(2)}$ and using (4.8) and (4.11) we get

$$
\begin{aligned}
dE^{(2)} = \{A_+^{(0)}, E^{(2)}\} &+ \frac{1}{2}\{E^{(1)}, E^{(1)}\} + \frac{1}{2}\{E^{(3)}, E^{(3)}\} - \{E^{(3)}, P^{(3)}ME^{(2)}\} \\
&- P^{(2)}M^T\{E^{(2)}, E^{(3)}\} + \frac{1}{2}\{P^{(3)}ME^{(2)}, P^{(3)}ME^{(2)}\} \\
&+ P^{(2)}M^T\{E^{(2)}, P^{(3)}ME^{(2)}\} - \frac{1}{2}P^{(2)}M^T\{E^{(2)}, E^{(2)}\}\,.
\end{aligned} \tag{5.6}
$$

[18] This is valid only for a suitable choice of the spin connection, which can however be extracted from the same equations. We have dropped the $\wedge$'s for readability.

[19] Essentially identical expressions hold for type IIA, cf. [23].

[20] We use anti-commutators rather than commutators because the objects that appear are one-forms, and therefore naturally anti-commute.

Using $A_+^{(0)} = \frac{1}{2} A_+^{ab} J_{ab}$, $E^{(2)} = E^a P_a$ etc. and the algebra in appendix A of [13] this gives the form for the bosonic torsion T^a in (5.1) provided that we identify the spin connection with[21]

$$\Omega_{ab} = (A_+)_{ab} + 2i(E^2\gamma_{[a})_\beta M^{\beta 2}{}_{b]} + \frac{3i}{2} E^c M^{\alpha 2}{}_{[a}(\gamma_b)_{\alpha\beta} M^{\beta 2}{}_{c]} + \frac{1}{2} E^c (M_{ab,c} - 2M_{c[a,b]}) \,. \quad (5.7)$$

In a similar way, using (4.8) and (5.5) we find that

$$\begin{aligned} dE^{(3)} =& \{A_+^{(0)}, E^{(3)}\} + \{P^{(0)} M E^{(2)}, E^{(3)}\} \\ &+ \mathrm{Ad}_h^{-1}\{E^{(1)} + P^{(1)}\mathrm{Ad}_h M E^{(2)}, E^{(2)}\} + \frac{1}{2} P^{(3)} M \{E^{(3)}, E^{(3)}\} \\ &+ 2P^{(3)} \mathrm{Ad}_f^{-1} \widetilde{\mathcal{O}}^{-1} \mathrm{Ad}_f \Big(2\mathrm{Ad}_h^{-1}\{E^{(1)} + P^{(1)}\mathrm{Ad}_h M E^{(2)}, E^{(2)}\} + \mathrm{Ad}_h^{-1}\{E^{(2)}, E^{(2)}\} \Big), \end{aligned} \quad (5.8)$$

which leads to the torsion $T^{\alpha 2}$ taking the form in (5.1) with the background fields given by[22]

$$\begin{aligned} H_{abc} &= 3M_{[ab,c]} - 3iM^{\alpha 2}{}_{[a}(\gamma_b)_{\alpha\beta} M^{\beta 2}{}_{c]} \,, \\ \mathcal{S}^{\alpha 1\beta 2} &= -8i[\mathrm{Ad}_h(1 + 4\mathrm{Ad}_f^{-1}\widetilde{\mathcal{O}}^{-T}\mathrm{Ad}_f)]^{\alpha 1}{}_{\gamma 1} \widehat{\mathcal{K}}^{\gamma 1\beta 2}, \\ \chi^2_\alpha &= -\frac{i}{2} \gamma^a_{\alpha\beta} M^{\beta 2}{}_a \,, \\ \psi^{\alpha 2}_{ab} &= 2[\mathrm{Ad}_f^{-1}\widetilde{\mathcal{O}}^{-1}\mathrm{Ad}_f \mathrm{Ad}_h^{-1}]^{\alpha 2}{}_{cd} \widehat{\mathcal{K}}_{ab}{}^{cd} + \frac{1}{4}[\mathrm{Ad}_h M]^{\beta 1}{}_{[a}(\gamma_{b]}\mathcal{S}^{12})_\beta{}^\alpha \,. \end{aligned} \quad (5.9)$$

Here $\widehat{\mathcal{K}}^{AB}$ denotes the inverse of the metric defined by the supertrace $\mathrm{Str}(T_A T_B) = \mathcal{K}_{AB}$, see appendix A of [13] for more details on our conventions.

Since the DTD model contains NATD as a special case we obtain as a by-product the transformation rules for RR fields under NATD — starting from a supercoset model. As a check we can compare this to the formula conjectured in [14] based on analogy to the abelian case [25] — consistency of that formula was checked in some particular cases also in [8]. Setting $\zeta = 0$, which removes the deformation, and restricting to a bosonic $\tilde{\mathfrak{g}}$, so that $\tilde{P} = \tilde{P}(P^{(0)} + P^{(2)}) = (P^{(0)} + P^{(2)})\tilde{P}$, we find[23]

$$\mathcal{S}^{\alpha 1\beta 2} = -8i[\mathrm{Ad}_h|_{\theta=0}]^{\alpha 1}{}_{\gamma 1} \widehat{\mathcal{K}}^{\gamma 1\beta 2} + \text{fermions} \,, \quad (5.10)$$

which agrees with the transformations conjectured in [14]. Note that our result generalises this to the case where also fermionic T-dualities are involved.

Finally we must compute $T^{\alpha 1}$ to extract the other dilatino superfield χ^1. We find

$$\begin{aligned} dE^{(1)} =& \{\mathrm{Ad}_h A_+^{(0)} - dhh^{-1}, E^{(1)}\} + \mathrm{Ad}_h\{E^{(2)}, E^{(3)} - P^{(3)} M E^{(2)}\} \\ &+ \frac{1}{2} P^{(1)} \mathrm{Ad}_h M^{-1} \mathrm{Ad}_h^{-1}\{E^{(1)}, E^{(1)}\} \\ &+ 2P^{(1)} \mathrm{Ad}_h \mathrm{Ad}_f^{-1} \widetilde{\mathcal{O}}^{-T} \mathrm{Ad}_f \Big(2\{E^{(2)}, E^{(3)} - P^{(3)} M E^{(2)}\} + \{E^{(2)}, E^{(2)}\} \Big) \,. \end{aligned} \quad (5.11)$$

[21] The components of M are defined as $MT_A = T_B M^B{}_A$.

[22] These expressions have obvious close analogies with the ones found for the η-model in [13].

[23] Note that $(P^{(0)} + P^{(2)})\mathrm{Ad}_f P^{(1)} = 0$+fermions.

JHEP10(2017)024

Taking the exterior derivative of the equation $A_+^{(2)} = \mathrm{Ad}_h A_-^{(2)}$, cf. (4.11), we find the relation

$$[\mathrm{Ad}_h A_+^{(0)} - dhh^{-1}]_{ab} = \Omega_{ab} - \frac{1}{2} E^c H_{abc} + 2i(E^1\gamma_{[a})_\alpha [\mathrm{Ad}_h M]^{\alpha 1}{}_{b]}, \tag{5.12}$$

which can be used to show that the torsion again takes the form in (5.1), where the remaining components of the background fields are[24]

$$\chi^1_\alpha = \frac{i}{2}(\gamma^a)_{\alpha\beta}[\mathrm{Ad}_h M]^{\beta 1}{}_a\,, \quad \psi^{\alpha 1}_{ab} = 2[\mathrm{Ad}_h \mathrm{Ad}_f^{-1} \widetilde{\mathcal{O}}^{-T} \mathrm{Ad}_f]^{\alpha 1}{}_{cd} \widehat{\mathcal{K}}_{ab}{}^{cd} - \frac{1}{4}(\mathcal{S}^{12}\gamma_{[a})^\alpha{}_\beta M^{\beta 2}{}_{b]}\,. \tag{5.13}$$

It remains only to analyse the question of when this is a solution to the standard or the generalised type II supergravity equations, in other words to identify the conditions under which K^a defined in (5.3) vanishes. We do this in the next subsection.

5.1 Supergravity condition and dilaton

By analogy with the calculations performed in [13] there is a natural candidate for the dilaton superfield for the DTD model namely[25]

$$e^{-2\phi} = \mathrm{sdet}'\widetilde{\mathcal{O}}\,. \tag{5.14}$$

We will now show that this guess is indeed correct by verifying that its spinor derivatives reproduces the dilatini found above. Using the formula for the supertrace $\mathrm{Str}\mathcal{M} = \widehat{\mathcal{K}}^{AB}\mathrm{Str}(T_A \mathcal{M} T_B)$ we find

$$\begin{aligned}
d\phi = & -\frac{1}{2}\mathrm{Str}(d\widetilde{\mathcal{O}}\widetilde{\mathcal{O}}^{-1}) = -\frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{Str}\big\{([J,\hat{d}_f^T T_A] - \hat{d}_f^T[J,T_A] + [d\nu, T_A])\widetilde{\mathcal{O}}^{-1}T_B\big\} \\
= & -\frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{Str}\big\{\big([J,\hat{d}_f^T T_A] - \hat{d}_f^T[J,T_A] + [\mathrm{Ad}_f \hat{d}^T A_+, T_A] \\
& \qquad + [(\mathrm{ad}_\nu + \zeta\omega)(\mathrm{Ad}_f A_+ - J), T_A]\big)\widetilde{\mathcal{O}}^{-1} T_B\big\} \\
= & \frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{Str}\big\{T_A\big(\hat{d}[A_+, \mathrm{Ad}_f^{-1}\widetilde{\mathcal{O}}^{-1}\mathrm{Ad}_f T_B] \\
& \qquad + [\hat{d}^T A_+, \mathrm{Ad}_f^{-1}\widetilde{\mathcal{O}}^{-1}\mathrm{Ad}_f T_B] - [A_+, \hat{d}\mathrm{Ad}_f^{-1}\widetilde{\mathcal{O}}^{-1}\mathrm{Ad}_f T_B]\big)\big\} \\
& + \widehat{\mathcal{K}}^{AB}\mathrm{Str}\big\{[(\mathrm{Ad}_f A_+ - J), T_A]\tilde{P} T_B\big\}\,.
\end{aligned} \tag{5.15}$$

If the last term vanishes, then using (4.8), (5.13), (5.9) and (4.11) one may check that the $E^{(1,3)}$-terms are indeed equal to

$$E^{\alpha 1}\chi^1_\alpha + E^{\alpha 2}\chi^2_\alpha\,. \tag{5.16}$$

Therefore $\chi_{\alpha I} = \nabla_{\alpha I}\phi$ which implies that K^a in (5.3) vanishes and we have a solution to standard type II supergravity. Since $(\mathrm{Ad}_f A_+ - J) \in \tilde{\mathfrak{g}}$ can be regarded as an arbitrary

[24] Just as in [13], one finds a superficially different expression for H_{abc} namely

$$H_{abc} = 3[\mathrm{Ad}_h M]_{[ab,c]} + 3i[\mathrm{Ad}_h M]^{\alpha 1}{}_{[a}(\gamma_b)_{\alpha\beta}[\mathrm{Ad}_h M]^{\beta 1}{}_{c]}\,.$$

However consistency requires this to be the same as the expression in (5.9) and this can also be verified explicitly similarly to [13].

[25] The prime on the superdeterminant denotes the fact that we must restrict to the subspace where $\widetilde{\mathcal{O}}$ is defined, i.e. the subalgebra $\tilde{\mathfrak{g}}$.

element of the Lie algebra, the vanishing of the last term in (5.15) is equivalent to $f_{AB}{}^{A} = 0$ for the structure constants of $\tilde{\mathfrak{g}}$, i.e. $\tilde{\mathfrak{g}}$ must be unimodular. This condition is therefore sufficient to get a standard supergravity solution. Following a calculation similar to the one done in [13], computing K^a in (5.3) and requiring it to vanish one finds that this condition is also necessary.[26]

Our results imply that the DTD model gives a one-loop Weyl invariant string sigma model precisely[27] when the subalgebra $\tilde{\mathfrak{g}}$ is unimodular. This is in fact the same condition that was found long ago for NATD on bosonic sigma models by path integral considerations [11, 12]. Since the DTD model includes NATD as a special case, the analysis here coupled with the results of [9, 10], gives an alternative derivation of the Weyl anomaly for NATD of supercosets.

A nice fact is that we do not have to impose extra conditions on the cocycle ω used to construct the deformation. When ω is non-degenerate unimodularity of $\tilde{\mathfrak{g}}$ is equivalent to unimodularity of $R = \omega^{-1}$ as defined in [13], see the discussion there; this is consistent with the fact that the YB models are a special case of the DTD models.

6 Some explicit examples

Here we would like to collect some formulas that are useful when deriving the explicit background for a given DTD model, and then work out two examples in detail. We denote the generators of $\tilde{\mathfrak{g}} \subset \mathfrak{g}$ by T_i, $i = 1, \ldots, N = \dim(\tilde{\mathfrak{g}})$, and those of the dual $\tilde{\mathfrak{g}}^*$ by T^i. They satisfy $\mathrm{Str}(T^i T_j) = \delta^i_j$. The action of the projectors on a generic element $x \in \mathfrak{g}$ may be written as

$$\tilde{P}(x) = \mathrm{Str}(T^i x) T_i, \qquad \tilde{P}^T(x) = \mathrm{Str}(T_i x) T^i, \tag{6.1}$$

where summation of repeated indices is assumed. Given a cocycle $\omega = \frac{1}{2}\omega_{ij} T^i \wedge T^j$ with $\omega_{ji} = -\omega_{ij}$, its action on an element of the algebra is

$$\omega(x) = \omega_{ij} T^i \, \mathrm{Str}(T^j x), \tag{6.2}$$

and it must satisfy the cocycle condition, which may be written as

$$\mathrm{Str}\Big(T_k(\omega[T_i, T_j] - [T_i, \omega T_j] + [T_j, \omega T_i])\Big) = 0, \qquad \forall T_i, T_j, T_k \in \tilde{\mathfrak{g}}. \tag{6.3}$$

With the above definitions one may easily construct the operator $\widetilde{\mathcal{O}} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^*$ defined in (2.4), that can be encoded in an explicit $N \times N$ matrix

$$\tilde{O}_{ij} = \mathrm{Str}(\widetilde{\mathcal{O}}(T_i) T_j), \tag{6.4}$$

[26] In very special cases it is possible for K^a to decouple from the remaining generalized supergravity equations. One then obtains a background solving both the generalised and standard supergravity equations depending on if K^a is included or not. One such example is the pp-wave solution discussed in appendix B of [26]. We thank B. Hoare and S. van Tongeren for pointing this out.

[27] This is modulo possible subtleties with the special cases mentioned in the previous footnote. One should also note that this condition is true provided one only allows a *local* (Fradkin-Tseytlin) counter-term. If one relaxes this condition one can find a *non-local* counter-term also when K^a is non-zero, since solutions of the generalised supergravity equations are formally T-dual to solutions of the standard ones; see also [27]. This being said, cases where K^a is null may be subtle and deserve further study.

JHEP10(2017)024

so that $\widetilde{\mathcal{O}}(T_i) = \tilde{O}_{ij} T^j$. The matrix $\tilde{O}$ can be inverted with standard methods and used to construct the action of the inverse operator as $\widetilde{\mathcal{O}}^{-1}(x) = \mathrm{Str}(xT_i)(\tilde{O}^{-1})^{ij} T_j$, so that on the basis generators $\widetilde{\mathcal{O}}^{-1}(T^i) = (\tilde{O}^{-1})^{ij} T_j$. Obviously, when choosing a parametrisation for the group element f, one should make sure that the corresponding degrees of freedom cannot be gauged away by applying the local transformations discussed in section 2.

To obtain the background fields we use the results of section 5. The metric reads as $ds^2 = \eta^{ab} E_a E_b$, where the components of the bosonic supervielbein are obtained by $E_a = \mathrm{Str}(A_+ P_a)$, and the B-field is given by equation (4.10). From the superdeterminant of the matrix $\tilde{O}$ it is also straightforward to compute the (exponential of the) dilaton $e^\phi = (\mathrm{sdet}\, \tilde{O})^{-\frac{1}{2}}$. In order to determine the RR fields one first identifies the components of the matrix $M_{ab} = \mathrm{Str}((MP_a)P_b)$ and then one constructs the local Lorentz transformation on spinorial indices

$$(\mathrm{Ad}_h)^\beta{}_\alpha = \exp\left[-\frac{1}{4}(\log M)_{ab} \Gamma^{ab} \right]^\beta{}_\alpha \,, \tag{6.5}$$

so that $\mathrm{Ad}_h \Gamma_a \mathrm{Ad}_h^{-1} = M_a{}^b \Gamma_b$, where Γ_a are 32×32 Gamma-matrices.[28] From (5.2) and (5.9) one finds that the expression for RR fields is obtained by solving the equation

$$\left(\Gamma^a F_a + \frac{1}{3!} \Gamma^{abc} F_{abc} + \frac{1}{2 \cdot 5!} \Gamma^{abcde} F_{abcde} \right) \Pi = e^{-\phi} \, [\mathrm{Ad}_h (1 + 4 \mathrm{Ad}_f^{-1} \widetilde{\mathcal{O}}^{-T} \mathrm{Ad}_f)](4\Gamma_{01234})\Pi, \tag{6.6}$$

where $\Pi = \frac{1}{2}(1 - \Gamma_{11})$ is a projector[29] and $(-4\Gamma_{01234})\Pi$ corresponds to the 5-form flux of $\mathrm{AdS}_5 \times \mathrm{S}^5$. In order to find the component $F_{a_1 \ldots a_{2m+1}}$ it is then enough to multiply the above equation by $\Gamma_{a_1 \ldots a_{2m+1}}$ and take the trace. As already explained, when the subalgebra $\tilde{\mathfrak{g}}$ is bosonic the above result simplifies considerably, and only Ad_h remains inside square brackets. After obtaining the components in tangent indices we translate them into form language using $F^{(2m+1)} = \frac{1}{(2m+1)!} E^{a_{2m+1}} \wedge \ldots \wedge E^{a_1} F_{a_1 \ldots a_{2m+1}}$.

6.1 A TsT example

First we will work out a simple example where we dualise a two-dimensional abelian subalgebra of the isometry of the sphere $\mathfrak{so}(6)$, so that the deformation is equivalent to doing a TsT there [28–30]. This example was worked out already in [2] for the NSNS sector, and the RR fields were taken into account in [8] by following the T-duality rules of [14]. Here we will use the matrix realisation of the $\mathfrak{psu}(2,2|4)$ superalgebra used in [13], see also [31]. We take $\tilde{\mathfrak{g}}$ to be the abelian algebra spanned by two Cartans of $\mathfrak{so}(6)$, $T_1 \equiv J_{68}, T_2 \equiv J_{79}$, and for the dual generators we may just take $T^1 = J_{68}, T^2 = J_{79}$. We parametrise the bosonic fields as[30]

$$\nu = \tilde{\varphi}_i T^i, \qquad\qquad f = f_{\mathfrak{a}} \cdot \exp(\varphi P_5) \exp(-\xi J_{89}) \exp(-\arcsin r P_9), \tag{6.7}$$

where $f_{\mathfrak{a}}$ is a coset group element parametrised by fields in AdS_5. We take $\omega = T^1 \wedge T^2$ which obviously satisfies the cocycle condition. The matrix corresponding to $\widetilde{\mathcal{O}}$ is very

[28] Alternatively one can use the 16×16 gamma matrices used in the previous section.

[29] With these conventions the self-duality for the 5-form is $F^{(5)} = *F^{(5)}$.

[30] The group elements parametrised by φ, ξ and r coincide with those in (A.1) of [32].

simple

$$\tilde{O}_{ij} = \begin{pmatrix} 2r^2 \sin^2 \xi & \zeta \\ -\zeta & 2r^2 \cos^2 \xi \end{pmatrix}, \tag{6.8}$$

and it is easily inverted. Following the above discussion we immediately find the fields of the NSNS sector

$$\begin{aligned} ds^2 &= ds^2_{\mathfrak{a}} + \frac{r^2}{\zeta^2 + r^4 \sin^2(2\xi)}(\cos^2 \xi \, d\tilde{\varphi}_1^2 + \sin^2 \xi \, d\tilde{\varphi}_2^2) + (1 - r^2)d\varphi^2 + r^2 d\xi^2 + \frac{dr^2}{1 - r^2}, \\ e^\phi &= (\zeta^2 + r^4 \sin^2(2\xi))^{-\frac{1}{2}}, \qquad B = \frac{\zeta}{2} \frac{d\tilde{\varphi}_1 \wedge d\tilde{\varphi}_2}{\zeta^2 + r^4 \sin^2(2\xi)}, \end{aligned} \tag{6.9}$$

where $ds^2_{\mathfrak{a}}$ is the metric of AdS_5. After computing the matrix M_{ab} and the local Lorentz transformation[31] we get that only $F^{(3)}$ and $F^{(5)}$ are non-vanishing

$$\begin{aligned} F^{(3)} &= 4r^3 \sin(2\xi) d\varphi \wedge d\xi \wedge dr, \\ F^{(5)} &= -2\zeta(1 + *) \left(\frac{r^3 \sin(2\xi) \, d\tilde{\varphi}_1 \wedge d\tilde{\varphi}_2 \wedge d\varphi \wedge d\xi \wedge dr}{\zeta^2 + r^4 \sin^2(2\xi)} \right). \end{aligned} \tag{6.10}$$

Since ω is non-degenerate on $\tilde{\mathfrak{g}}$ we can relate the above background to a YB deformation of $AdS_5 \times S^5$, see also section 2.1. In this particularly simple example the R-matrix of the YB model is abelian, and therefore it corresponds just to a TsT transformation on the sphere, see also [21]. In fact, consider the following TsT transformation on $AdS_5 \times S^5$

$$\varphi_1 \to T(\varphi_1), \qquad \varphi_2 \to \varphi_2 - 2\eta T(\varphi_1), \qquad T(\varphi_1) \to \varphi_1, \tag{6.11}$$

which produces the following background[32]

$$\begin{aligned} ds^2 &= ds^2_{\mathfrak{a}} + \frac{r^2}{1 + \eta^2 r^4 \sin^2(2\xi)}(\cos^2 \xi \, d\varphi_2^2 + \sin^2 \xi \, d\varphi_1^2) + (1 - r^2)d\varphi^2 + r^2 d\xi^2 + \frac{dr^2}{1 - r^2}, \\ e^\phi &= (1 + \eta^2 r^4 \sin^2(2\xi))^{-\frac{1}{2}}, \qquad B = -\frac{\eta r^4 \sin^2(2\xi) d\varphi_1 \wedge d\varphi_2}{1 + \eta^2 r^4 \sin^2(2\xi)}, \end{aligned} \tag{6.12}$$

for the NSNS sector and

$$\begin{aligned} F^{(3)} &= 4\eta r^3 \sin(2\xi) d\varphi \wedge d\xi \wedge dr, \\ F^{(5)} &= -2(1 + *) \left(\frac{r^3 \sin(2\xi) \, d\varphi_1 \wedge d\varphi_2 \wedge d\varphi \wedge d\xi \wedge dr}{1 + \eta^2 r^4 \sin^2(2\xi)} \right), \end{aligned} \tag{6.13}$$

for the RR sector. To match with the above TsT background we need to implement the field redefinition (2.11) at the level of the DTD background, which in this case just reduces to $\tilde{\varphi}_1 = \eta^{-1}\varphi_2, \tilde{\varphi}_2 = -\eta^{-1}\varphi_1$ since $\tilde{\mathfrak{g}}$ is abelian. We find agreement only if we also use the gauge freedom for B to subtract the exact term $\frac{1}{2\eta} d\varphi_1 \wedge d\varphi_2$; moreover we also need to redefine the constant part of the dilaton to reabsorb a factor of η, which then appears in front of the RR fields.

[31] For 32×32 Gamma matrices we find convenient the basis used in [31].

[32] As a starting point we take the undeformed $AdS_5 \times S^5$ background as written in [31].

6.2 A new example

Let us now consider the example in (3.19)

$$\tilde{\mathfrak{g}} = \mathrm{span}\{p_1,\, p_2,\, p_3,\, J_{12}\}\,, \quad \tilde{\mathfrak{g}}^* = \mathrm{span}\Big\{ -\frac{1}{2}k_1,\, -\frac{1}{2}k_2,\, -\frac{1}{2}k_3,\, -J_{12}\Big\} \quad \omega = k_3 \wedge J_{12}\,. \tag{6.14}$$

In this case we have just one isometry of type 1 corresponding to p_0, and the isometries of type 2 are k_3 and J_{12}. Inspired by the parametrisation used in (6.19) of [13] we parametrise[33]

$$\nu = \tilde{\xi}\ J_{12} + \tilde{r}\ k_1 + \tilde{x}^3\ k_3\,, \qquad f = \exp(x^0 p_0)\exp(\log z D)\,. \tag{6.15}$$

The above is a good parametrisation because it is not possible to remove degrees of freedom by applying gauge transformations. This will be confirmed e.g. by the fact that we get a non-degenerate metric in target space. We find that the (matrix corresponding to the) operator $\widetilde{\mathcal{O}}$ is

$$\tilde{O}_{ij} = \begin{pmatrix} \frac{2}{z^2} & 0 & 0 & 0 \\ 0 & \frac{2}{z^2} & 0 & 2\tilde{r} \\ 0 & 0 & \frac{2}{z^2} & 2\zeta \\ 0 & -2\tilde{r} & -2\zeta & 0 \end{pmatrix}, \tag{6.16}$$

which is clearly invertible. We find the following NSNS sector fields

$$\begin{aligned} ds^2 &= \frac{-(dx^0)^2 + dz^2}{z^2} + d\tilde{r}^2 z^2 + \frac{d\tilde{\xi}^2}{4z^2\,(\zeta^2+\tilde{r}^2)} + \frac{\tilde{r}^2 z^2 (d\tilde{x}^3)^2}{\zeta^2+\tilde{r}^2} + ds^2_{\mathfrak{s}}\,, \\ e^{\phi} &= \left(\frac{16\,(\zeta^2+\tilde{r}^2)}{z^4}\right)^{-\frac{1}{2}}, \qquad B = -\frac{\zeta d\tilde{\xi}\wedge d\tilde{x}^3}{2\,(\zeta^2+\tilde{r}^2)}\,, \end{aligned} \tag{6.17}$$

where $ds^2_{\mathfrak{s}}$ is the metric on S^5. In the RR sector we have only three-form flux

$$F^{(3)} = -\frac{8(dx^0\wedge d\tilde{\xi}\wedge dz)}{z^5}\,. \tag{6.18}$$

According to the discussion in section 2.1 the above background is not related to a YB model by NATD.

7 Conclusions

We have argued that DTD models based on supercosets represent a large class of integrable string models which is closed under NATD as well as (certain) deformations. Besides being a useful tool to generate new integrable supergravity backgrounds it would be very interesting if these deformations could be understood on the dual field theory side. In the case when the 2-cocycle is invertible these models are equivalent to YB sigma models, which have been argued to correspond to non-commutative deformations, e.g. [33, 34],

[33] Even if present, one could remove k_2 in ν by means of a gauge transformation.

of the field theory [35–37] (see also [38]). This interpretation is consistent with the fact that TsT transformations are special cases of these models [21, 39] and this includes the so-called β and γ-deformations which have a known interpretation in $\mathcal{N}=4$ super Yang-Mills [28, 29, 40, 41]. Recently a certain limit of the γ-deformation has been used to construct a simplified integrable scalar field theory [42, 43] and it would be very interesting to explore similar limits of the more general class of deformations considered here to see whether one can learn more about the AdS/CFT duality for those cases.

Another important question is how the DTD model relates to the other known deformations of the $AdS_5 \times S^5$ string, i.e. the η-model with R-matrix solving the modified CYBE [44] and the λ-model [22]. These two deformations are related by Poisson-Lie T-duality and the fact that the latter is Weyl-invariant [13] while the former is not [10, 31] is explained by the fact that the obstruction to the duality at the quantum level again involves the trace of the structure constants [45].[34] The fact that NATD is used also in the construction of the λ-model suggests that there might be a bigger picture relating it to the DTD construction considered here. In fact this seems to be part of an even bigger picture of general integrable deformations of sigma models where T-duality and its generalizations play a central role, see for example the recent paper [46].

Acknowledgments

We thank B. Hoare, S. van Tongeren and A. Torrielli for interesting and useful discussions, and A. Tseytlin for illuminating discussions and comments on the manuscript. The work of R.B. was supported by the ERC advanced grant No 341222. We also thank the Galileo Galilei Institute for Theoretical Physics (GGI) for the hospitality and INFN for partial support during the completion of this work at the program *New Developments in* AdS_3/CFT_2 *Holography*.

A Useful identities

A useful identity is

$$\tilde{P}[\tilde{P}^T x, (1-\tilde{P})y] = 0\,, \qquad \forall x, y \in \mathfrak{g} \tag{A.1}$$

which is easily proven by taking the supertrace with an element of $\mathfrak{g}$. We will also need some relations related to the well-known formula for the derivative of the exponential map

$$de^x = e^x \frac{1-e^{-\mathrm{ad}_x}}{\mathrm{ad}_x} dx\,. \tag{A.2}$$

Let $x \in \tilde{\mathfrak{g}}$ and define a similar looking object $\mu = \tilde{P}^T e^{-x}\delta e^x$, where δ is the derivation acting as $\delta(x) = \omega(x)$ on $x \in \tilde{\mathfrak{g}}$. Note that this derivation is compatible with the Lie bracket due to the 2-cocycle condition (2.5), and following the same computations needed to prove the identity above, one may show that

$$\mu = \tilde{P}^T \frac{1-e^{-\mathrm{ad}_x}}{\mathrm{ad}_x} \omega x. \tag{A.3}$$

[34]We thank A. Tseytlin for this comment.

Taking $y \in \tilde{\mathfrak{g}}$, from the definition of μ we find $\tilde{P}^T \mathrm{ad}_\mu y = \tilde{P}^T \mathrm{Ad}_{e^x}^{-1} \delta(\mathrm{Ad}_{e^x} y) - \delta y$ which implies

$$\tilde{P}^T \mathrm{ad}_\mu \tilde{P} = \tilde{P}^T e^{-\mathrm{ad}_x} \omega e^{\mathrm{ad}_x} \tilde{P} - \omega. \tag{A.4}$$

Another useful identity valid for the derivative of μ is

$$d\mu = \mu e^{-x} de^x + \delta(e^{-x} de^x) + \tilde{P}^T de^{-x} e^x \mu = \tilde{P}^T (\mathrm{ad}_\mu + \omega)(e^{-x} de^x)\,. \tag{A.5}$$

Now, the identity (A.1) implies that

$$\tilde{P}^T \mathrm{ad}_{\tilde{P}^T \mathrm{Ad}_{\tilde{h}} \nu} \tilde{P} = \tilde{P}^T \mathrm{Ad}_{\tilde{h}} \mathrm{ad}_\nu \mathrm{Ad}_{\tilde{h}}^{-1} \tilde{P} = \tilde{P}^T \mathrm{Ad}_{\tilde{h}} \tilde{P}^T \mathrm{ad}_\nu \tilde{P} \mathrm{Ad}_{\tilde{h}}^{-1} \tilde{P} \tag{A.6}$$

and together with (A.4) it implies that if we redefine $\nu \to \tilde{P}^T \left(\mathrm{Ad}_{\tilde{h}} \nu + \zeta \mu\right)$ as in (2.7) then the operator in (2.4) transforms as

$$\widetilde{\mathcal{O}} \to \tilde{P}^T \mathrm{Ad}_{\tilde{h}} \widetilde{\mathcal{O}} \mathrm{Ad}_{\tilde{h}}^{-1} \tilde{P}. \tag{A.7}$$

Moreover, using (A.5) we also find

$$d\nu \to \tilde{P}^T \mathrm{Ad}_{\tilde{h}} (d\nu - (\mathrm{ad}_\nu + \zeta\omega)(\tilde{h}^{-1} d\tilde{h})). \tag{A.8}$$

B Derivation of the action

To derive the action of DTD models we start from the action of a supercoset sigma model, see e.g. [47], and we rewrite the group element as $g = \tilde{g} f$, where $\tilde{g} \in \tilde{G} \subset G$. We then gauge the $\tilde{G}$ symmetry and introduce the gauge fields $\tilde{A}_i$. If we fix the gauge $\tilde{g} = 1$ we essentially achieve $\tilde{g}^{-1} d\tilde{g} \to \tilde{A}$ when comparing to the initial supercoset action. At this point we add a Lagrange multiplier to impose the flatness of $\tilde{A}_i$, plus a ω-dependent term which deforms the model

$$\begin{aligned} S = -\frac{T}{2} \int d^2\sigma \Big[&\frac{\gamma^{ij} - \epsilon^{ij}}{2} \mathrm{Str}\left((\tilde{A}_i + J_i) \hat{d}_f (\tilde{A}_j + J_j)\right) \\ &- \epsilon^{ij} \mathrm{Str}\left(\nu(\partial_i \tilde{A}_j + \tilde{A}_i \tilde{A}_j) - \frac{\zeta}{2} \tilde{A}_i \omega \tilde{A}_j\right)\Big]. \end{aligned} \tag{B.1}$$

Instead of integrating out ν we integrate out $\tilde{A}$, so that we obtain the equations of motion

$$P_-^{ij} \left(\widetilde{\mathcal{O}} \tilde{A}_j + \partial_j \nu + \hat{d}_f J_j\right) + P_+^{ij} \left(\widetilde{\mathcal{O}}^T \tilde{A}_j - \partial_j \nu + \hat{d}_f^T J_j\right) = 0, \tag{B.2}$$

where

$$P_\pm^{ij} = \frac{\gamma^{ij} \pm \epsilon^{ij}}{2}, \tag{B.3}$$

are projectors

$$P_+^{ij} + P_-^{ij} = \gamma^{ij}, \qquad P_\pm^{il} P_{\pm l}{}^{j} = P_\pm^{ij}, \qquad P_\pm^{il} P_{\mp l}{}^{j} = 0. \tag{B.4}$$

Here we used also $\gamma^{ij} = \epsilon^{ik} \gamma_{kl} \epsilon^{lj}$. We also define $V_\pm^i \equiv P_\pm^{ij} V_j$, and it is useful to remember $P_\pm^{ij} A_i B_j = A_\mp^i \gamma_{ij} B_\pm^j$. We then solve for $\tilde{A}_\pm$

$$\tilde{A}_-^i = \widetilde{\mathcal{O}}^{-1} \left(-\partial_-^i \nu - \hat{d}_f J_-^i\right), \qquad \tilde{A}_+^i = \widetilde{\mathcal{O}}^{-T} \left(+\partial_+^i \nu - \hat{d}_f^T J_+^i\right). \tag{B.5}$$

JHEP10(2017)024

The action on the solutions to the equations of motion is

$$\begin{aligned} S &= -\frac{T}{2}\int d^2\sigma\, \mathrm{Str}\left[J_{+i}\hat{d}_f J_-{}^i + (\partial_{+i}\nu - \hat{d}_f^T J_{+i})\widetilde{\mathcal{O}}^{-1}(\partial_-^i\nu + \hat{d}_f J_-^i)\right] \\ &= -\frac{T}{2}\int d^2\sigma \frac{\gamma^{ij}-\epsilon^{ij}}{2}\,\mathrm{Str}\left[J_i\hat{d}_f J_j + (\partial_i\nu - \hat{d}_f^T J_i)\widetilde{\mathcal{O}}^{-1}(\partial_j\nu + \hat{d}_f J_j)\right]. \end{aligned} \tag{B.6}$$

C Classical integrability

Here we wish to be more explicit and show that the on-shell equations of DTD models can be recast into the flatness condition for a Lax connection. The argument follows the one presented in [1] in the case of DTD of Principal Chiral Models. First we compute the equations of motion for f and ν, which are obtained by the straightforward variations $\delta_f S$ and $\delta_\nu S$ of the action

$$\begin{aligned} \delta_f S &= +\frac{T}{2}\int d^2\sigma\, \mathrm{Str}\left(f^{-1}\delta f\ \mathcal{C}\right), \\ \delta_\nu S &= -\frac{T}{2}\int d^2\sigma\, \mathrm{Str}\left(\delta\nu\ \mathcal{F}^{\tilde{A}}\right) = -\frac{T}{2}\int d^2\sigma\, \mathrm{Str}\left((\mathrm{Ad}_f^{-1}\delta\nu)\mathcal{F}^A\right), \end{aligned} \tag{C.1}$$

where we defined

$$\begin{aligned} \mathcal{C} &\equiv \partial_{+i}(\hat{d}A_-^i) + \partial_{-i}(\hat{d}^T A_+^i) + [A_{+i}, \hat{d}A_-^i] + [A_{-i}, \hat{d}^T A_+^i], \\ \mathcal{F}^A &\equiv \partial_{+i}A_-^i - \partial_{-i}A_+^i + [A_{+i}, A_-^i] = -\epsilon^{ij}(\partial_i A_j + A_i A_j), \end{aligned} \tag{C.2}$$

and similarly for $\mathcal{F}^{\tilde{A}}$. Notice that $P^{(0)}\mathcal{C} = 0$. For convenience we also introduced the (projections of the) field $A_\pm^i \equiv \mathrm{Ad}_f^{-1}(\tilde{A}_\pm^i + J_\pm^i)$, where $\tilde{A}_\pm^i$ is given in (B.5). On the one hand, imposing the equations of motion $\delta_\nu S = 0$ is enough to get $\mathcal{F}^A = 0$. Notice that this equation is equivalent to imposing separately $\mathcal{F}^{\tilde{A}} = 0$ and $\mathcal{F}^J \equiv \partial_{+i}J_-^i - \partial_{-i}J_+^i - [J_{+i}, J_-^i] = 0$. On the other hand, the equations of motion $\delta_f S = 0$ imply that $\mathcal{C}$ vanishes only on a certain subspace of the superalgebra $\mathfrak{g}$. In fact, in the special case when the whole superalgebra is dualised $\tilde{\mathfrak{g}} = \mathfrak{g}$, there is no f for which we can compute the variation of the action, and we should find an independent argument to claim that the equation $\mathcal{C} = 0$ holds. We will now show that an appropriate (rotated) projection of $\mathcal{C}$ by $\tilde{P}^T$ indeed vanishes without appealing to the equations of motion for f. Consider the equations of motion for $\tilde{A}_\pm^i$ in (B.2) and let us rewrite them as $\mathcal{E}_\pm^i - M_\pm^{i\perp} = 0$ where

$$\begin{aligned} \mathcal{E}_+^i &\equiv +(\partial_+^i + \mathrm{ad}_{\tilde{A}_+^i})\nu - \hat{d}_f^T(J_+^i + \tilde{A}_+^i) - \zeta\omega\tilde{A}_+^i, \\ \mathcal{E}_-^i &\equiv -(\partial_-^i + \mathrm{ad}_{\tilde{A}_-^i})\nu - \hat{d}_f(J_-^i + \tilde{A}_-^i) + \zeta\omega\tilde{A}_-^i. \end{aligned} \tag{C.3}$$

Since we choose $M_\pm^{i\perp}$ to take values only in the complement of $\tilde{\mathfrak{g}}^*$, taking $\tilde{P}^T\mathcal{E}_\pm^i = 0$ gives indeed (B.2). Clearly $(\partial_{+i} + \mathrm{ad}_{\tilde{A}_{+i}})(\mathcal{E}_-^i - M_-^{i\perp}) + (\partial_{-i} + \mathrm{ad}_{\tilde{A}_{-i}})(\mathcal{E}_+^i - M_+^{i\perp}) = 0$ is identically true since it just follows from the above equations, and working out all the terms we find

$$\mathrm{Ad}_f\,\mathcal{C} = [\nu, \mathcal{F}^{\tilde{A}}] + \zeta\omega\mathcal{F}^{\tilde{A}} - (\partial_{-i} + \mathrm{ad}_{\tilde{A}_{-i}})M_+^{i\perp} - (\partial_{+i} + \mathrm{ad}_{\tilde{A}_{+i}})M_-^{i\perp}. \tag{C.4}$$

After projecting with $\tilde{P}^T$ all terms with $M_{\pm}^{i\perp}$ disappear. The remaining terms on the right-hand-side of the above equation vanish thanks to the flatness of $\tilde{A}$ ($\mathcal{F}^{\tilde{A}} = 0$) implied by the equations of motion for ν. To conclude, we obtain $\tilde{P}^T(\mathrm{Ad}_f\mathcal{C}) = 0$ as wanted, which together with the equations of motion for f is enough to claim $\mathcal{C} = 0$ on the whole superalgebra.

The on-shell equations $\mathcal{F}^A = 0$ and $\mathcal{C} = 0$ formally take the same form as those for a supercoset, where in that case A is the Maurer-Cartan form, see also [47, 48]. Therefore one may follow the derivation done in the case of the supercoset, and find that they are encoded in the flatness condition

$$\epsilon^{ij}(\partial_i\mathcal{L}_j + \mathcal{L}_i\mathcal{L}_j) = 0, \tag{C.5}$$

for the Lax connection

$$\mathcal{L}_i = A_i^{(0)} + zA_i^{(1)} + \frac{1}{2}\left(z^2 + z^{-2}\right)A_i^{(2)} + \frac{1}{2}\gamma_{ij}\epsilon^{jk}\left(z^{-2} - z^2\right)A_i^{(2)} + z^{-1}A_i^{(3)}, \tag{C.6}$$

where z is the spectral parameter. The existence of a Lax connection implies the presence of a tower of conserved charges, see e.g. [49] for a review. However, differently from the case of the supercoset, now fewer of them can be argued to be local. In fact, thanks to the gauge transformation it is always possible to define

$$\mathcal{L}'_i = h\mathcal{L}_ih^{-1} - \partial_ihh^{-1}, \tag{C.7}$$

so that $\mathcal{L}'_i$ is also flat. In the case of the supercoset, after noticing that $\mathcal{L}_i(z=1) = A_i = g^{-1}\partial_ig$, one may choose $h = g$ so that the new Lax connection vanishes at $z = 1$ $\mathcal{L}'_i(z=1) = 0$. Expanding around that point one finds

$$\mathcal{L}'_i(z = 1 + w) = w\ g\left(A_i^{(1)} - 2\gamma_{ij}\epsilon^{jk}A_k^{(2)} - A_i^{(3)}\right)g^{-1} + \mathcal{O}(w^2), \tag{C.8}$$

so that the flatness condition for $\mathcal{L}'_i$ at order w implies the conservation $\partial_i\mathcal{A}^i = 0$ for the current

$$\mathcal{A}^i = \epsilon^{ij}g\left(A_j^{(1)} - 2\gamma_{jk}\epsilon^{kl}A_l^{(2)} - A_j^{(3)}\right)g^{-1}. \tag{C.9}$$

This is how in the supercoset case one can argue from the Lax connection that the isometries corresponding to the superalgebra $\mathfrak{g}$ correspond to *local* charges. In the case of DTD models A is not of the Maurer-Cartan form, and in general it is not possible to find a group element h for which a gauge-equivalent Lax connection vanishes at $z = 1$. With the exception of the isometries discussed in section 3, we therefore expect that the initial symmetries of the undeformed model are traded for non-local charges.

References

[1] R. Borsato and L. Wulff, *Integrable deformations of T-dual σ models*, *Phys. Rev. Lett.* **117** (2016) 251602 [arXiv:1609.09834] [INSPIRE].

[2] B. Hoare and A.A. Tseytlin, *Homogeneous Yang-Baxter deformations as non-abelian duals of the AdS_5 σ-model*, *J. Phys.* **A 49** (2016) 494001 [arXiv:1609.02550] [INSPIRE].

[3] D. Orlando, S. Reffert, J.-i. Sakamoto and K. Yoshida, *Generalized type IIB supergravity equations and non-Abelian classical r-matrices*, *J. Phys.* **A 49** (2016) 445403 [arXiv:1607.00795] [INSPIRE].

[4] C. Klimčík, *Yang-Baxter σ-models and dS/AdS T duality*, *JHEP* **12** (2002) 051 [hep-th/0210095] [INSPIRE].

[5] C. Klimčík, *On integrability of the Yang-Baxter σ-model*, *J. Math. Phys.* **50** (2009) 043508 [arXiv:0802.3518] [INSPIRE].

[6] F. Delduc, M. Magro and B. Vicedo, *An integrable deformation of the $AdS_5 \times S^5$ superstring action*, *Phys. Rev. Lett.* **112** (2014) 051601 [arXiv:1309.5850] [INSPIRE].

[7] I. Kawaguchi, T. Matsumoto and K. Yoshida, *Jordanian deformations of the $AdS_5 x S^5$ superstring*, *JHEP* **04** (2014) 153 [arXiv:1401.4855] [INSPIRE].

[8] B. Hoare and D.C. Thompson, *Marginal and non-commutative deformations via non-abelian T-duality*, *JHEP* **02** (2017) 059 [arXiv:1611.08020] [INSPIRE].

[9] L. Wulff and A.A. Tseytlin, *κ-symmetry of superstring σ-model and generalized* 10*d* *supergravity equations*, *JHEP* **06** (2016) 174 [arXiv:1605.04884] [INSPIRE].

[10] G. Arutyunov, S. Frolov, B. Hoare, R. Roiban and A.A. Tseytlin, *Scale invariance of the η-deformed $AdS_5 \times S^5$ superstring, T-duality and modified type-II equations*, *Nucl. Phys.* **B 903** (2016) 262 [arXiv:1511.05795] [INSPIRE].

[11] E. Alvarez, L. Álvarez-Gaumé and Y. Lozano, *On non-Abelian duality*, *Nucl. Phys.* **B 424** (1994) 155 [hep-th/9403155] [INSPIRE].

[12] S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer and G. Veneziano, *Remarks on non-Abelian duality*, *Nucl. Phys.* **B 435** (1995) 147 [hep-th/9409011] [INSPIRE].

[13] R. Borsato and L. Wulff, *Target space supergeometry of η and λ-deformed strings*, *JHEP* **10** (2016) 045 [arXiv:1608.03570] [INSPIRE].

[14] K. Sfetsos and D.C. Thompson, *On non-abelian T-dual geometries with Ramond fluxes*, *Nucl. Phys.* **B 846** (2011) 21 [arXiv:1012.1320] [INSPIRE].

[15] Y. Lozano, E. O Colgain, K. Sfetsos and D.C. Thompson, *Non-Abelian T-duality, Ramond fields and coset geometries*, *JHEP* **06** (2011) 106 [arXiv:1104.5196] [INSPIRE].

[16] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B 533** (1998) 109 [hep-th/9805028] [INSPIRE].

[17] K. Zarembo, *Strings on semisymmetric superspaces*, *JHEP* **05** (2010) 002 [arXiv:1003.0465] [INSPIRE].

[18] L. Wulff, *Superisometries and integrability of superstrings*, *JHEP* **05** (2014) 115 [arXiv:1402.3122] [INSPIRE].

[19] K. Sfetsos, *Integrable interpolations: from exact CFTs to non-Abelian T-duals*, *Nucl. Phys.* **B 880** (2014) 225 [arXiv:1312.4560] [INSPIRE].

JHEP10(2017)024

[20] A.A. Belavin and V.G. Drinfel'd, *Solutions of the classical Yang-Baxter equation for simple Lie algebras*, *Funktsional. Anal. i Prilozhen.* **16** (1982) 1.

[21] D. Osten and S.J. van Tongeren, *Abelian Yang-Baxter deformations and TsT transformations*, *Nucl. Phys.* **B 915** (2017) 184 [arXiv:1608.08504] [INSPIRE].

[22] T.J. Hollowood, J.L. Miramontes and D.M. Schmidtt, *An integrable deformation of the $AdS_5 \times S^5$ superstring*, *J. Phys.* **A 47** (2014) 495402 [arXiv:1409.1538] [INSPIRE].

[23] L. Wulff, *The type-II superstring to order θ^4*, *JHEP* **07** (2013) 123 [arXiv:1304.6422] [INSPIRE].

[24] A.A. Tseytlin, *On dilaton dependence of type-II superstring action*, *Class. Quant. Grav.* **13** (1996) L81 [hep-th/9601109] [INSPIRE].

[25] S.F. Hassan, *T duality, space-time spinors and RR fields in curved backgrounds*, *Nucl. Phys.* **B 568** (2000) 145 [hep-th/9907152] [INSPIRE].

[26] B. Hoare and S.J. van Tongeren, *On Jordanian deformations of AdS_5 and supergravity*, *J. Phys.* **A 49** (2016) 434006 [arXiv:1605.03554] [INSPIRE].

[27] J.-i. Sakamoto, Y. Sakatani and K. Yoshida, *Weyl invariance for generalized supergravity backgrounds from the doubled formalism*, *PTEP* **2017** (2017) 053B07 [arXiv:1703.09213] [INSPIRE].

[28] O. Lunin and J.M. Maldacena, *Deforming field theories with* $\mathrm{U}(1) \times \mathrm{U}(1)$ *global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [hep-th/0502086] [INSPIRE].

[29] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for superconformal deformations of $N = 4$ super Yang-Mills theory*, *JHEP* **07** (2005) 045 [hep-th/0503192] [INSPIRE].

[30] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, *JHEP* **05** (2005) 069 [hep-th/0503201] [INSPIRE].

[31] G. Arutyunov, R. Borsato and S. Frolov, *Puzzles of η-deformed $AdS_5 \times S^5$*, *JHEP* **12** (2015) 049 [arXiv:1507.04239] [INSPIRE].

[32] G. Arutyunov, R. Borsato and S. Frolov, *S-matrix for strings on η-deformed $AdS_5 \times S^5$*, *JHEP* **04** (2014) 002 [arXiv:1312.3542] [INSPIRE].

[33] A. Hashimoto and N. Itzhaki, *Noncommutative Yang-Mills and the AdS/CFT correspondence*, *Phys. Lett.* **B 465** (1999) 142 [hep-th/9907166] [INSPIRE].

[34] J.M. Maldacena and J.G. Russo, *Large-N limit of noncommutative gauge theories*, *JHEP* **09** (1999) 025 [hep-th/9908134] [INSPIRE].

[35] T. Matsumoto and K. Yoshida, *Integrability of classical strings dual for noncommutative gauge theories*, *JHEP* **06** (2014) 163 [arXiv:1404.3657] [INSPIRE].

[36] S.J. van Tongeren, *Yang-Baxter deformations, AdS/CFT and twist-noncommutative gauge theory*, *Nucl. Phys.* **B 904** (2016) 148 [arXiv:1506.01023] [INSPIRE].

[37] S.J. van Tongeren, *Almost abelian twists and AdS/CFT*, *Phys. Lett.* **B 765** (2017) 344 [arXiv:1610.05677] [INSPIRE].

[38] T. Araujo et al., *Yang-Baxter σ-models, conformal twists and noncommutative Yang-Mills theory*, *Phys. Rev.* **D 95** (2017) 105006 [arXiv:1702.02861] [INSPIRE].

JHEP10(2017)024

[39] T. Matsumoto and K. Yoshida, *Lunin-Maldacena backgrounds from the classical Yang-Baxter equation — Towards the gravity/CYBE correspondence*, JHEP **06** (2014) 135 [arXiv:1404.1838] [INSPIRE].

[40] R.G. Leigh and M.J. Strassler, *Exactly marginal operators and duality in four-dimensional $N = 1$ supersymmetric gauge theory*, Nucl. Phys. **B 447** (1995) 95 [hep-th/9503121] [INSPIRE].

[41] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for (non)supersymmetric deformations of $N = 4$ super Yang-Mills theory*, Nucl. Phys. **B 731** (2005) 1 [hep-th/0507021] [INSPIRE].

[42] O. Gurdogan and V. Kazakov, *New integrable 4D quantum field theories from strongly deformed planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory*, Phys. Rev. Lett. **117** (2016) 201602 [arXiv:1512.06704] [INSPIRE].

[43] N. Gromov, V. Kazakov, G. Korchemsky, S. Negro and G. Sizov, *Integrability of conformal fishnet theory*, arXiv:1706.04167 [INSPIRE].

[44] F. Delduc, M. Magro and B. Vicedo, *On classical q-deformations of integrable σ-models*, JHEP **11** (2013) 192 [arXiv:1308.3581] [INSPIRE].

[45] E. Tyurin and R. von Unge, *Poisson-lie T duality: the path integral derivation*, Phys. Lett. **B 382** (1996) 233 [hep-th/9512025] [INSPIRE].

[46] C. Klimčík, *Yang-Baxter σ-model with WZNW term as $\mathcal{E}$-model*, Phys. Lett. **B 772** (2017) 725 [arXiv:1706.08912] [INSPIRE].

[47] G. Arutyunov and S. Frolov, *Foundations of the $AdS_5 \times S^5$ superstring. Part I*, J. Phys. **A 42** (2009) 254003 [arXiv:0901.4937] [INSPIRE].

[48] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, Phys. Rev. **D 69** (2004) 046002 [hep-th/0305116] [INSPIRE].

[49] A. Torrielli, *Lectures on classical Integrability*, J. Phys. **A 49** (2016) 323001 [arXiv:1606.02946] [INSPIRE].

Integrable Deformations of T-Dual σ Models

Riccardo Borsato* and Linus Wulff†
The Blackett Laboratory, Imperial College, London SW7 2AZ, United Kingdom
(Received 21 October 2016; published 16 December 2016)

We present a method to deform (generically non-Abelian) T duals of two-dimensional σ models, which preserves classical integrability. The deformed models are identified by a linear operator ω on the dualized subalgebra, which satisfies the 2-cocycle condition. We prove that the so-called homogeneous Yang-Baxter deformations are equivalent, via a field redefinition, to our deformed models when ω is invertible. We explain the details for deformations of T duals of principal chiral models, and present the corresponding generalization to the case of supercoset models.

DOI: 10.1103/PhysRevLett.117.251602

Introduction.—Integrable models in two dimensions have played a pivotal role in the understanding of (quantum) field theory, have numerous applications in condensed matter theory, and have recently attracted attention also in the context of the AdS/CFT correspondence [1], which relates certain string theories on $(d+1)$-dimensional anti–de Sitter (AdS) backgrounds to conformal field theories in d dimensions. The most studied example that exhibits integrable structures is that of the superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ [2] and its dual $\mathcal{N}=4$ super Yang-Mills theory in four dimensions [3], see Refs. [4,5] for reviews. On the string side the two-dimensional world sheet theory is classically integrable; i.e., there is a Lax pair whose flatness condition is equivalent to the equations of motion of the σ model. The Lax pair depends on an auxiliary spectral parameter z, and its expansion around a fixed z_0 yields an infinite set of conserved charges, see Ref. [6] for a review. Integrability has provided the most stringent tests of AdS/CFT, culminating with the possibility of computing the spectrum of the quantum theory in the large N limit exactly [7–10].

Given this tremendous success it is natural to ask whether other theories that are not maximally (super)symmetric are still integrable. Integrability could then also be a guiding principle to discover new models that are interesting in their own right. The β deformation [11–13] or certain gravity duals of noncommutative gauge theories [14,15] are examples that are integrable but reduce to the maximally symmetric case only when a deformation parameter is sent to zero. These instances actually fall into a larger class that goes under the name of Yang-Baxter (YB) models [16–19], sometimes also called η deformations after the deformation parameter. A YB model is identified by an R matrix solving the classical Yang-Baxter equation (CYBE), which in general has a rich set of solutions. Each R generates a background that reduces to the undeformed model (e.g., $\mathrm{AdS}_5 \times \mathrm{S}^5$) in the $\eta \to 0$ limit. Here, we will not consider the case of the "modified" CYBE.

In this Letter we explore another possibility; we deform the original σ model by adding a topological term (a closed B field) and then apply non-Abelian T duality (NATD) [20] with respect to a subgroup $\tilde{G}$ of the isometry group G. The special case when $\tilde{G}$ is Abelian gives so-called TsT transformations [11–13]. We refer to the resulting actions as deformed T dual (DTD) models, since sending the deformation parameter $\zeta \to 0$ they reduce to NATD. DTD models are in one-to-one correspondence with the 2-cocycles ω of the Lie algebra of $\tilde{G}$. The cocycle condition (3) guarantees that integrability is preserved, and plays the same role as the CYBE for YB models.

The analogy goes even further. When ω is invertible its inverse $R=\omega^{-1}$ solves the CYBE, and each solution of the CYBE corresponds to an invertible 2-cocycle [21]. We use this identification to show that the action of YB models can be recast in the form of DTD models, where the two deformation parameters are simply related by $\eta = \zeta^{-1}$. As explained later, this translates into our language a recent conjecture by Hoare and Tseytlin [22]. We prove it by providing the explicit field redefinition that relates YB to DTD models. The field redefinition is local, albeit in general nonlinear, and it allows us to interpolate between a certain σ model ($\zeta \to \infty$) and its NATD ($\zeta \to 0$). In the case when ω is degenerate, the DTD model is equivalent to a combination of YB deformation and NATD.

We first construct the DTD of the principal chiral model (PCM), since it provides a simpler setup where all the essential features already appear. Later, we generalize it to the case of supercosets, which is more relevant to the study of deformations of superstrings. The supercoset case will be described in more detail elsewhere [23].

DTD of the PCM.—We start from a PCM parametrized by a group element $g \in G$, with the familiar action $S[g] = -\frac{1}{2}\int \mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g)$. Since we want to dualize a $\tilde{G}$ subgroup of the left copy of G [24] we rewrite [25]

$$S[f,\tilde{A},\nu] = -\frac{1}{2}\int \mathrm{Tr}((\tilde{A}_+ + J_+)(\tilde{A}_- + J_-) + \nu \tilde{F}_{+-}). \quad (1)$$

Here, $J = dff^{-1}$ is a right-invariant Maurer-Cartan form for $f \in G$, depending on fields that remain spectators under NATD. At the same time $\tilde{A} \in \tilde{\mathfrak{g}}$ and $\nu \in \tilde{\mathfrak{g}}^*$ identify each of the two T-dual frames. If T_i are generators for $\tilde{\mathfrak{g}}$, a basis for the dual algebra $\tilde{\mathfrak{g}}^*$ is given by T^i, where $\mathrm{Tr}(T_i T^j) = \delta_i^j$. The curvature of $\tilde{A}$ is $\tilde{F}_{+-} = \partial_+ \tilde{A}_- - \partial_- \tilde{A}_+ + [\tilde{A}_+, \tilde{A}_-]$. The original PCM is recovered upon integrating out ν since $\tilde{F}_{+-} = 0$ implies that $\tilde{A}$ is pure gauge, i.e., $\tilde{A} = \bar{g}^{-1} d\bar{g}$ for a $\bar{g} \in \tilde{G}$, and we get the desired action with $g = \bar{g} f$. The NATD with respect to $\tilde{G}$, on the other hand, is obtained by integrating out $\tilde{A}$.

We now add a deformation with parameter ζ given by

$$S'[f, \tilde{A}, \nu] = S[f, \tilde{A}, \nu] + \frac{\zeta}{2} \int \mathrm{Tr}(\tilde{A}_+ \omega \tilde{A}_-). \tag{2}$$

Here, ω: $\tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^*$ is a linear antisymmetric [i.e., $\mathrm{Tr}(x\omega y) = -\mathrm{Tr}(\omega x y)$] map satisfying the cocycle condition [26]

$$\omega \mathrm{ad}_x y = \mathrm{ad}_x \omega y - \mathrm{ad}_y \omega x, \quad \forall\ x, y \in \tilde{\mathfrak{g}}. \tag{3}$$

This property is needed to have local $\tilde{G}$ invariance also for $\zeta \neq 0$, which ensures that # d.o.f. $= \dim(G)$ [27]. Equations of motion for $\tilde{A}$ give $\int \mathrm{Tr}(\delta \tilde{A}_\mp \mathcal{E}_\pm) = 0$, where

$$\mathcal{E}_\pm \equiv (1 \pm \mathrm{ad}_\nu \pm \zeta\omega) \tilde{A}_\pm \mp \partial_\pm \nu + J_\pm. \tag{4}$$

This implies $\tilde{P}^T \mathcal{E}_\pm = 0$, where $\tilde{P}$ projects onto $\tilde{\mathfrak{g}}$, $\tilde{P}^T$ onto $\tilde{\mathfrak{g}}^*$. We solve these equations by defining the linear operator $\tilde{\mathcal{O}} = \tilde{P}^T (1 - \mathrm{ad}_\nu - \zeta\omega) \tilde{P}$, which is a map $\tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^*$

$$\tilde{A}_- = \tilde{\mathcal{O}}^{-1}(-\partial_- \nu - J_-), \qquad \tilde{A}_+ = \tilde{\mathcal{O}}^{-T}(\partial_+ \nu - J_+) \tag{5}$$

and $\tilde{\mathcal{O}}^{-T}$ is the inverse of its transpose. Note that $\tilde{\mathcal{O}}^{-1} \tilde{\mathcal{O}} = \tilde{P}$ as the lhs is defined only on $\tilde{\mathfrak{g}}$. Evaluating S' on the solution we get the DTD action

$$S'[f, \nu] = -\frac{1}{2} \int \mathrm{Tr}(J_+ J_- + (\partial_+ \nu - J_+) \tilde{\mathcal{O}}^{-1} (\partial_- \nu + J_-)). \tag{6}$$

A second interpretation of DTD comes from integrating out ν rather than $\tilde{A}$ from Eq. (2), which gives again $\tilde{A} = \bar{g}^{-1} d\bar{g}$. The resulting action is a topological deformation of the PCM, since the cocycle condition implies that $B = \zeta\omega(\bar{g}^{-1} d\bar{g}, \bar{g}^{-1} d\bar{g})$ is closed. At the classical level adding this term has no effect, and in fact this picture of a deformation that is trivial in the dual frame is reminiscent of YB models: in some cases they correspond to TsT transformations [22,28–30], which are just field redefinitions in a T-dual frame. Since DTD is a NATD of a topological deformation of the PCM, it is classically integrable, where NATD can be applied thanks to closure of B. In fact, the equation $\tilde{A} = \bar{g}^{-1} d\bar{g}$ with $\tilde{A}$ given in Eq. (5) allows us to relate the variables of the deformed model to those of the original PCM. In the special case of Abelian subalgebra $\tilde{\mathfrak{g}}$ the relation simplifies and the deformed model becomes equivalent to the PCM with twisted boundary conditions, consistent with the TsT interpretation [12].

A third interpretation of DTD comes from the possibility of applying NATD to a centrally extended subalgebra. This idea first appeared in Ref. [22] and was the original motivation for considering the deformation (2). One can indeed replace $\tilde{A}$ in Eq. (1) with $\tilde{A}' \in \tilde{\mathfrak{g}}_{\mathrm{c.e.}} = \tilde{\mathfrak{g}} \oplus \mathfrak{c}$ and $\mathfrak{c}$ central; similarly $\nu' \in \tilde{\mathfrak{g}}^*_{\mathrm{c.e.}}$. We decompose $\tilde{A}' = \tilde{A} + \tilde{A}^{\mathfrak{c}}$, $\nu' = \nu + \nu^{\mathfrak{c}}$ with obvious notation, and extend the definition of the trace $\mathrm{Tr}(\mathfrak{c}^2) = 1$, $\mathrm{Tr}(\mathfrak{c}\mathfrak{g}) = 0$. Equations for $\tilde{A}^{\mathfrak{c}}$ imply that $\nu^{\mathfrak{c}}$ is constant, $\nu^{\mathfrak{c}} = \zeta\mathfrak{c}$. At this point $\mathrm{Tr}(\nu' \tilde{F}'_{+-}) = \mathrm{Tr}(\nu \tilde{F}_{+-}) + \zeta \mathbf{f}_{ab} \tilde{A}^a_+ \tilde{A}^b_-$, where $\mathbf{f}_{ab}$ are the structure constants introduced by the central extension $[T_a, T_b] = f^c_{ab} T_c + \mathbf{f}_{ab}\mathfrak{c}$. Introducing a map ω whose components are $\omega_{ab} = -\mathbf{f}_{ab}$ we just notice that it is antisymmetric and satisfies the cocycle condition, a consequence of the Jacobi identity in $\tilde{\mathfrak{g}}_{\mathrm{c.e.}}$ projected on $\mathfrak{c}$.

For some ω's DTD reduces to just NATD; i.e., the deformation parameter can be removed by a field redefinition. This happens when ω is a coboundary, i.e., $\omega(x, y) = f([x, y])$ for some function f. Therefore, nontrivial deformations are in one-to-one correspondence with 2-cocycles modulo coboundaries, i.e., with elements of the second cohomology group $H^2(\tilde{\mathfrak{g}})$. The same holds also for nontrivial central extensions. In particular, there are none for semisimple $\tilde{\mathfrak{g}}$. Trivial deformations are equivalently described as adding an exact B field to the PCM.

An example.—Before continuing our general discussion, let us provide an explicit example: a PCM on $U(2)$. We use generators $T_j = i\sigma_j \in \mathfrak{su}(2)$ and $T_4 = i\mathbf{1}$, with duals $T^j = -(i/2)\sigma_j$ and $T^4 = -(i/2)\mathbf{1}$. We parametrize the group element by $g = \exp(i\theta\mathbf{1}) \exp(i\phi_+ \sigma_1) \check{g}(\xi) \exp(i\phi_- \sigma_2)$, where $\phi_\pm = (\phi_1 \pm \phi_2)/2$ and $\check{g}(\xi) = \mathrm{diag}(i^{-1/2} e^{i\xi}, i^{1/2} e^{-i\xi})$. The PCM action yields the metric of $\mathrm{S}^3 \times \mathrm{S}^1$

$$ds^2 = d\xi^2 + \sin^2\xi d\phi_1^2 + \cos^2\xi d\phi_2^2 + d\theta^2. \tag{7}$$

Suppose we want to dualize the coordinates ϕ_+ in S^3 and θ in S^1, corresponding to the Abelian subalgebra $\tilde{\mathfrak{g}} = \mathrm{span}\{T_1, T_4\}$. We take $f = \check{g}(\xi) \exp(i\phi_- \sigma_2)$ and $\nu = 2(\tilde{\phi}_+ T^1 + \tilde{\theta} T^4)$, where $\tilde{\phi}_+$, $\tilde{\theta}$ are dual coordinates. We deform the dual theory by taking $\omega = 2T^1 \wedge T^4$; namely, $\omega T_1 = -2T^4$, $\omega T_4 = 2T^1$. From Eq. (6) we find the action of DTD $S' = \int \partial_+ X^i (G_{ij} - B_{ij}) \partial_- X^j$, with the metric and B field

$$\begin{aligned} ds^2 &= d\xi^2 + (1+\zeta^2)^{-1}(d\tilde{\phi}_+^2 + (\zeta^2 + \sin^2 2\xi)d\phi_-^2 \\ &\quad + d\tilde{\theta}^2 + 2\zeta\cos 2\xi d\tilde{\theta} d\phi_-), \\ B &= (1+\zeta^2)^{-1}(\cos 2\xi d\phi_- - \zeta d\tilde{\theta}) \wedge d\tilde{\phi}_+. \end{aligned} \tag{8}$$

The $\zeta \to 0$ limit yields the T-dual model of $S^3 \times S^1$ with respect to $\tilde{\mathfrak{g}}$. To relate this simple example to a YB model it is enough to take $\nu = \eta^{-1}R(\vartheta T^4 + \varphi_+ T^1)$ with $R = \frac{1}{2}(T_4 \wedge T_1)$. However, when $\tilde{\mathfrak{g}}$ is non-Abelian, the field redefinition is more complicated, see Eq. (13).

Integrability.—Above we argued that DTD models must be integrable; however, it is instructive to show this explicitly to see how the cocycle condition enters and write a Lax connection. We will show that the equations of motion formally resemble those of the PCM, for which a Lax pair is known. Suppose we consider a PCM with group element $g = \bar{g}f$, with $\bar{g} \in \tilde{G}$, $f \in G$. We prefer to rewrite its on-shell equations in terms of the left and right currents $\tilde{A} = \bar{g}^{-1}d\bar{g}$ and $J = dff^{-1}$. To start, the flatness condition for $A = g^{-1}dg$ is equivalent to $\mathcal{F}^J = 0$, $\mathcal{F}^{\tilde{A}} = 0$

$$\begin{aligned} \mathcal{F}^J &\equiv \partial_+ J_- - \partial_- J_+ - [J_+, J_-], \\ \mathcal{F}^{\tilde{A}} &\equiv \partial_+ \tilde{A}_- - \partial_- \tilde{A}_+ + [\tilde{A}_+, \tilde{A}_-]. \end{aligned} \tag{9}$$

Moreover, the equations of motion for the PCM, i.e., conservation of A, become $\mathcal{C} = 0$,

$$\begin{aligned} \mathcal{C} &\equiv \partial_+(J_- + \tilde{A}_-) + \partial_-(J_+ + \tilde{A}_+) \\ &\quad + [\tilde{A}_+, J_-] + [\tilde{A}_-, J_+]. \end{aligned} \tag{10}$$

Let us now rederive the above equations for DTD models, where now importantly $\tilde{A}$ is identified as in Eq. (5). To start, the flatness condition $\mathcal{F}^J = 0$ still follows from the definition of J. Flatness for $\tilde{A}$, instead, now arises as the equations of motion for ν, which are $\delta_\nu S'[f,\nu] = -\frac{1}{2}\int \mathrm{Tr}(\delta\nu \mathcal{F}^{\tilde{A}}) = 0$. It is nice that the known mechanism familiar from T duality of trading flatness for an equation of motion still holds for DTD models.

The equations of motion for f are $\delta_f S'[f,\nu] = +\frac{1}{2}\int \mathrm{Tr}(\delta f f^{-1}\mathcal{C}) = 0$, essentially as in the previous example of the PCM. However, in that case it is only thanks to the equations of motion for $\bar{g}$ [i.e., $\int \mathrm{Tr}(\bar{g}^{-1}\delta\bar{g}\mathcal{C}) = 0$] that one can claim $\mathcal{C} = 0$. In analogy to the PCM, it is then clear that our task is to show that $\tilde{P}^T\mathcal{C} = 0$ also for DTD models. We generalize the argument of Ref. [31] for NATD of the PCM, and consider the equations $\mathcal{E}_\pm = M_\pm^\perp$, for some $M_\pm^\perp$ for which $\tilde{P}^T M_\pm^\perp = 0$. They imply $\tilde{P}^T\mathcal{E}_\pm = 0$; i.e., they are equivalent to the solutions for $\tilde{A}$ as in Eq. (5). They obviously imply also the equation $(\partial_+ + \mathrm{ad}_{\tilde{A}_+})(\mathcal{E}_- - M_-^\perp) + (\partial_- + \mathrm{ad}_{\tilde{A}_-})(\mathcal{E}_+ - M_+^\perp) = 0$, which reads as

$$\begin{aligned} \mathcal{C} &= [\partial_- + \mathrm{ad}_{\tilde{A}_-}, \partial_+ + \mathrm{ad}_{\tilde{A}_+}]\nu \\ &\quad - (\partial_- + \mathrm{ad}_{\tilde{A}_-})M_+^\perp - (\partial_+ + \mathrm{ad}_{\tilde{A}_+})M_-^\perp \\ &\quad + \zeta[\omega(\partial_+\tilde{A}_- - \partial_-\tilde{A}_+) + \mathrm{ad}_{\tilde{A}_+}\omega\tilde{A}_- - \mathrm{ad}_{\tilde{A}_-}\omega\tilde{A}_+]. \end{aligned}$$

The first line on the right-hand side is rewritten as $[\nu, \tilde{F}_{+-}]$, and hence vanishes thanks to the flatness of $\tilde{A}$. The second line vanishes upon projecting with $\tilde{P}^T$ [32]. Finally, the last line vanishes thanks to the cocycle condition: using Eq. (3) it is rewritten as $-\zeta\omega(\tilde{F}_{+-})$, which is again zero. Since also $\tilde{P}^T\mathcal{C} = 0$ holds, we conclude that the whole set of on-shell equations for the DTD models is formally equivalent to those of a PCM, provided the proper $\tilde{A}$ is used. We can furthermore write the Lax pair as

$$L_\pm = \frac{1}{2}(1 + z^{\mp 2})\mathrm{Ad}_f^{-1}(\tilde{A}_\pm + J_\pm) \tag{11}$$

with z a spectral parameter. In fact, the flatness condition $\partial_+ L_- - \partial_- L_+ + [L_+, L_-] = 0$ is equivalent to the on-shell equations just derived.

Relation to Yang-Baxter models.—We now prove that YB deformations for the PCM on the group G are equivalent to DTD. This was checked for many particular examples in Ref. [22]. YB models are identified by an R matrix solving the CYBE on the Lie algebra $\mathfrak{g}$. If $g \in G$

$$S_{\mathrm{YB}}[g] = -\frac{1}{2}\int \mathrm{Tr}\left(g^{-1}\partial_+ g \frac{1}{1 - \eta R_g} g^{-1}\partial_- g\right). \tag{12}$$

R is invertible on a certain subalgebra and its inverse is a 2-cocycle [21]. As anticipated, we identify $R = \omega^{-1}$, where ω is the operator defining the DTD model. Then, R: $\tilde{\mathfrak{g}}^* \to \tilde{\mathfrak{g}}$. The two deformation parameters will be related by $\eta = \zeta^{-1}$.

We first split the group element parametrizing the YB model as $g = \tilde{g}f$, where $\tilde{g} \in \tilde{G}$ and $f \in G$. We identify f with the homonym appearing on the DTD side. Our proof of equivalence of the two actions will then consist in giving the field redefinition relating $\tilde{g}$ and ν. Since R is invertible, we can always take $\tilde{g} = \exp(RX)$ for some $X \in \tilde{\mathfrak{g}}^*$. One can check that taking $X = \eta\nu + (\eta^2/2)\tilde{P}^T[R\nu, \nu] + \mathcal{O}(\eta^3)$ the two actions are equivalent up to terms that are at least cubic in η. The generalization to all orders can be obtained by requiring that the $dfdf$ terms in the two actions match. This leads to the condition $(1 - \eta R_{\tilde{g}})^{-1} = 1 - \tilde{\mathcal{O}}^{-1}$ whose solution can be shown to be

$$\nu = \frac{1}{\eta}\tilde{P}^T \frac{1 - e^{-\mathrm{ad}_{RX}}}{\mathrm{ad}_{RX}} X = \frac{1}{\eta}\tilde{P}^T \frac{1 - \mathrm{Ad}_{\tilde{g}}^{-1}}{\log \mathrm{Ad}_{\tilde{g}}}\omega \log \tilde{g}. \tag{13}$$

It follows that $d\nu = (\tilde{P}^T - \tilde{\mathcal{O}})\tilde{g}^{-1}d\tilde{g}$ or, equivalently,

$$\mathbf{A}_\pm = \mathrm{Ad}_f^{-1}(J_\pm + \tilde{A}_\pm), \tag{14}$$

where we defined $\mathbf{A}_\pm = (1 \pm \eta R_g)^{-1}(g^{-1}\partial_\pm g)$ on the YB side. Using these relations it is not hard to check that the two actions are the same up to the topological term $\zeta\omega(\tilde{g}^{-1}d\tilde{g}, \tilde{g}^{-1}d\tilde{g})$, which has no effect in the classical theory as remarked earlier.

We have proven the equivalence of the DTD and YB models when ω is nondegenerate. In the case of a degenerate ω it is always possible to choose it in such a way that it is nondegenerate on a subalgebra $\hat{\mathfrak{g}} \subset \tilde{\mathfrak{g}}$ [33] and acts trivially on its complement $\check{\mathfrak{g}}$ in $\tilde{\mathfrak{g}}$, also an algebra thanks to Eq. (3). We interpret it as NATD on $\check{\mathfrak{g}}$ of the YB model corresponding to restricting ω to $\hat{\mathfrak{g}}$.

DTD of supercosets.—The construction of DTD models for supercosets follows the steps explained in the simpler case of the PCM. Here, we only present the main results, whose derivation will be collected in Ref. [23].

We still denote by G the group of superisometries, e.g., $PSU(2,2|4)$ for superstrings on $\mathrm{AdS}_5 \times \mathrm{S}^5$, see Ref. [34] for a review. Its Lie superalgebra $\mathfrak{g}$ admits a $\mathbb{Z}_4$ decomposition, and we denote by $P^{(j)}$ the projectors onto the four subspaces. They typically appear in the combination $\hat{d} = P^{(1)} + 2P^{(2)} - P^{(3)}$ or its transpose $\hat{d}^T$. The absence of $P^{(0)}$ in $\hat{d}$ is necessary for the local $\mathfrak{g}^{(0)}$ invariance of the action, i.e., local Lorentz transformations. The action for DTD models of supercosets is [35]

$$S'[f,\nu] = -\frac{T}{2}\int \mathrm{Str}(J_+\hat{d}_f J_- + (\partial_+\nu - \hat{d}_f^T J_+)\tilde{\mathcal{O}}^{-1}(\partial_-\nu + \hat{d}_f J_-)), \tag{15}$$

where $\hat{d}_f \equiv \mathrm{Ad}_f \hat{d} \mathrm{Ad}_f^{-1}$. We keep the same definitions for J, ν, which however now take values in superalgebras. Moreover, now $\tilde{\mathcal{O}} = \tilde{P}^T(\hat{d}_f - \mathrm{ad}_\nu - \zeta\omega)\tilde{P}$.

The model is integrable since we can write down a Lax pair. This is more conveniently expressed in terms of $A = \mathrm{Ad}_f^{-1}(\tilde{A} + J)$, where

$$\tilde{A}_+ = \tilde{\mathcal{O}}^{-T}(+\partial_+\nu - \hat{d}_f^T J_+),$$
$$\tilde{A}_- = \tilde{\mathcal{O}}^{-1}(-\partial_-\nu - \hat{d}_f J_-). \tag{16}$$

The flatness condition $\partial_+\mathcal{L}_- - \partial_-\mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-] = 0$ for

$$\mathcal{L}_\pm = A_\pm^{(0)} + zA_\pm^{(1)} + z^{\mp 2}A_\pm^{(2)} + z^{-1}A_\pm^{(3)} \tag{17}$$

is equivalent to the on-shell equations of the DTD model.

DTD models of supercosets possess kappa symmetry, and therefore correspond to solutions of the generalized supergravity equations of Refs. [36,37]. Kappa symmetry transformations are $\delta f f^{-1} = \hat{d}_f^T(\delta\nu) = \rho_{1,-} + \rho_{3,+}$, where

$$\rho_{j,\pm} = \{i\mathrm{Ad}_f\kappa^{(j)}, J_\pm^{(2)} + \tilde{A}_\pm^{(2)}\} \tag{18}$$

and $\kappa^{(j)}$, $j = 1, 3$ are two local parameters of grading j. The action (15) is invariant under these transformations upon using the Virasoro constraints. If we were not fixing conformal gauge, the variation of the action would be compensated by the variation of the world sheet metric. From these kappa symmetry transformations it is possible to extract the background fields of DTD models [23].

The equivalence to YB models for invertible ω's holds also in the case of DTD models of supercosets. Remarkably, the field redefinition is still given by Eq. (13) as for the PCM. We have further verified that kappa symmetry transformations of YB models [18] take the above form under this field redefinition, when we fix the $\tilde{G}$ gauge to get $\delta f f^{-1} = \hat{d}_f^T(\delta\nu)$.

Conclusions.—We provided a unified picture of (non-Abelian) T duality and homogeneous YB deformations as DTD of σ models. As pointed out in Ref. [22], an advantage of this formulation is that it can be realized at the path integral level, giving a better handle on the quantum theory. In fact, it also explains why the condition for one-loop Weyl invariance, i.e., unimodularity of $\tilde{\mathfrak{g}}$, is the same for both the YB model and NATD [30,38,39].

Despite the close relation, it is still worth viewing the DTD models as a distinct class of deformations. In fact, the field redefinition that relates it to the YB model is singular in the two undeformed limits; the YB model becomes degenerate when taking the undeformed (i.e., $\zeta \to 0$) limit of DTD models, and vice versa. Therefore, the interpretation as deformation applies to just one of the two models in the T-dual pair. It would be interesting to understand if there is any connection to the λ model of Refs. [31,40,41], which is also a deformation of NATD and is related to the inhomogeneous YB deformation [16–18].

Although our motivation was integrability, such deformations can be applied also to nonintegrable models, which provides an interesting and potentially useful way to generate new supergravity solutions.

We thank Ben Hoare, Stijn van Tongeren, and Arkady Tseytlin for interesting discussions and comments on the Letter. R. B. thanks also Bogdan Stefański for related discussions. R. B. thanks also Wim Hennink and his group in Utrecht for the kind hospitality during part of this project. This work was supported by ERC Advanced Grant No. 290456. The work of L. W. was also supported by STFC Consolidated Grant No. ST/L00044X/1.

*riccardo.borsato@su.se
†l.wulff@imperial.ac.uk

[1] J. M. Maldacena, Int. J. Theor. Phys. **38**, 1113 (1999); Adv. Theor. Math. Phys. **2**, 231 (1998).
[2] I. Bena, J. Polchinski, and R. Roiban, Phys. Rev. D **69**, 046002 (2004).
[3] J. Minahan and K. Zarembo, J. High Energy Phys. 03 (2003) 013.

[4] N. Beisert *et al.*, Lett. Math. Phys. **99**, 3 (2012).
[5] D. Bombardelli, A. Cagnazzo, R. Frassek, F. Levkovich-Maslyuk, F. Loebbert, S. Negro, I. M Szécsényi, A. Sfondrini, S. J van Tongeren, and A. Torrielli, J. Phys. A **49**, 320301 (2016).
[6] A. Torrielli, J. Phys. A **49**, 323001 (2016).
[7] D. Bombardelli, D. Fioravanti, and R. Tateo, J. Phys. A **42**, 375401 (2009).
[8] N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, Lett. Math. Phys. **91**, 265 (2010).
[9] G. Arutyunov and S. Frolov, J. High Energy Phys. 05 (2009) 068 (2009).
[10] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, Phys. Rev. Lett. **112**, 011602 (2014).
[11] O. Lunin and J. M. Maldacena, J. High Energy Phys. 05 (2005) 033.
[12] S. A. Frolov, R. Roiban, and A. A. Tseytlin, J. High Energy Phys. 07 (2005) 045.
[13] S. Frolov, J. High Energy Phys. 05 (2005) 069.
[14] A. Hashimoto and N. Itzhaki, Phys. Lett. B **465**, 142 (1999).
[15] J. M. Maldacena and J. G. Russo, J. High Energy Phys. 09 (1999) 025.
[16] C. Klimcik, J. High Energy Phys. 12 (2002) 051.
[17] C. Klimcik, J. Math. Phys. (N.Y.) **50**, 043508 (2009).
[18] F. Delduc, M. Magro, and B. Vicedo, Phys. Rev. Lett. **112**, 051601 (2014).
[19] I. Kawaguchi, T. Matsumoto, and K. Yoshida, J. High Energy Phys. 04 (2014) 153.
[20] X. C. de la Ossa and F. Quevedo, Nucl. Phys. **B403**, 377 (1993).
[21] A. Stolin, J. Pure Appl. Algebra **137**, 285 (1999).
[22] B. Hoare and A. A. Tseytlin, J. Phys. A **49**, 494001 (2016).
[23] R. Borsato and L. Wulff (to be published).
[24] The construction could be generalized to include the right copy. That DTD model should then be related to the bi-YB model of Ref. [17].
[25] We omit the integration measure $d\sigma^+ d\sigma^-$, where $\sigma^\pm = \tau \pm \sigma$.
[26] We use standard notation $\mathrm{Ad}_g M = gMg^{-1}$ and $\mathrm{ad}_x M = [x, M]$. Equivalently, Eq. (3) takes the form $\omega(x,[y,z]) + \omega(y,[z,x]) + \omega(z,[x,y]) = 0$ for ω: $\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}} \to \mathbb{R}$.
[27] Local invariance is found by including also a shift proportional to ζ in the transformation for ν. We thank A. Tseytlin for pointing this out.
[28] D. Orlando, S. Reffert, J.-i. Sakamoto, and K. Yoshida, J. Phys. A **49**, 445403 (2016).
[29] D. Osten and S. J. van Tongeren, arXiv:1608.08504.
[30] R. Borsato and L. Wulff, J. High Energy Phys. 10 (2016) 045.
[31] K. Sfetsos, Nucl. Phys. **B880**, 225 (2014).
[32] If $\tilde{P}^T(M^\perp_\pm) = 0$ then also $\tilde{P}^T(\mathrm{ad}_x M^\perp_\pm) = 0$ with $x \in \tilde{\mathfrak{g}}$.
[33] V. G. Drinfeld, Sov. Math. Dokl. **27**, 68 (1983).
[34] G. Arutyunov and S. Frolov, J. Phys. A **42**, 254003 (2009).
[35] We have fixed conformal gauge, $\gamma^{+-} = \gamma^{-+} = \epsilon^{-+} = -\epsilon^{+-} = 2$.
[36] G. Arutyunov, S. Frolov, B. Hoare, R. Roiban, and A. A. Tseytlin, Nucl. Phys. **B903**, 262 (2016).
[37] L. Wulff and A. A. Tseytlin, J. High Energy Phys. 06 (2016) 174.
[38] S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer, and G. Veneziano, Nucl. Phys. **B435**, 147 (1995).
[39] E. Alvarez, L. Alvarez-Gaume, and Y. Lozano, Nucl. Phys. **B424**, 155 (1994).
[40] T. J. Hollowood, J. L. Miramontes, and D. M. Schmidtt, J. High Energy Phys. 11 (2014) 009.
[41] T. J. Hollowood, J. L. Miramontes, and D. M. Schmidtt, J. Phys. A **47**, 495402 (2014).

Published for SISSA by Springer

Received: *September 1, 2016*
Accepted: *October 7, 2016*
Published: *October 10, 2016*

Target space supergeometry of η and λ-deformed strings

Riccardo Borsato and Linus Wulff

Blackett Laboratory, Imperial College,
London SW7 2AZ, U.K.

E-mail: r.borsato@imperial.ac.uk, l.wulff@imperial.ac.uk

Abstract: We study the integrable η and λ-deformations of supercoset string sigma models, the basic example being the deformation of the $AdS_5 \times S^5$ superstring. We prove that the kappa symmetry variations for these models are of the standard Green-Schwarz form, and we determine the target space supergeometry by computing the superspace torsion. We check that the λ-deformation gives rise to a standard (generically type II*) supergravity background; for the η-model the requirement that the target space is a supergravity solution translates into a simple condition on the R-matrix which enters the definition of the deformation. We further construct all such non-abelian R-matrices of rank four which solve the homogeneous classical Yang-Baxter equation for the algebra $\mathfrak{so}(2,4)$. We argue that most of the corresponding backgrounds are equivalent to sequences of non-commuting TsT-transformations, and verify this explicitly for some of the examples.

Keywords: Integrable Field Theories, Supergravity Models, Superstrings and Heterotic Strings, Sigma Models

ArXiv ePrint: 1608.03570

doi:10.1007/JHEP10(2016)045

Contents

JHEP10(2016)045

1 Introduction and summary of results

A remarkable property of the $AdS_5 \times S^5$ superstring sigma model is its classical integrability [1], see [2] for a review. In fact, this property extends to several other symmetric space string backgrounds [3, 4]. Recently two interesting deformations of the $AdS_5 \times S^5$ superstring sigma model[1] were proposed which preserve the integrability. The η-model [5] and λ-model [6], named after the corresponding deformation parameters. The former is based on the Yang-Baxter deformation of [7–9], it generalises the case of bosonic coset models [10], and its essential ingredient is an R-matrix which satisfies the modified classical Yang-Baxter equation (MCYBE). The λ-model was originally proposed by [11] and

[1]These deformations extend to any $\mathbb{Z}_4$-symmetric supercoset sigma model, i.e. symmetric space RR string background preserving supersymmetry.

it extends the case of bosonic cosets [12] (see also [13]). The construction is based on a G/G gauged Wess-Zumino-Witten (WZW) model, and it is more naturally interpreted as a deformation of the non-abelian T-dual of the original string. The two deformations are closely related; in fact, in both cases the original symmetry algebra gets q-deformed [14, 15] (with q real and root of unity respectively), and the two models are related, at least at the classical level, by the Poisson-Lie T-duality of [16, 17], see [18–20].

The attempt of interpreting these deformations as string theories has raised interesting questions. In fact, both models possess a local fermionic symmetry believed to be the standard kappa symmetry — which was expected to guarantee a string theory interpretation. However, the target space of the η-model derived in [21, 22][2] does not solve the type IIB supergravity equations [22], but rather a generalisation of them as suggested in [25]. These generalised equations ensure scale invariance for the sigma model, but are not enough to have the full Weyl invariance, which is present only when the target space satisfies the standard equations of supergravity. For the λ-model, on the other hand, it was shown that the target space does solve the supergravity equations, at least in the case of λ-deformed $AdS_2 \times S^2 \times T^6$ [26] and $AdS_3 \times S^3 \times T^4$ [27] string sigma models.[3]

A possible resolution for the puzzle posed by the η-model could have been that, after all, the possessed local fermionic symmetry was not the standard kappa symmetry of Green-Schwarz. However this state of affairs was clarified recently in [30] where it was shown that, contrary to what was commonly believed, kappa symmetry of the type II Green-Schwarz superstring does not imply the full equations of motion of type II supergravity.[4] Rather it implies a weaker (generalized) version of these equations, whose bosonic subsector coincides with the equations written down in [25]. These generalized supergravity equations involve a Killing vector field K^a, and reduce to the standard type II supergravity equations when this vector field is set to zero. This fact implies that kappa-symmetric backgrounds whose metric does not allow for isometries must in fact solve the standard type II equations. The λ-model falls into this class, which is consistent with the fact that the corresponding target spaces were found to be supergravity backgrounds.[5] On the other hand, the η-model typically leads to a target-space metric which possesses isometries, so that a priori it is not possible to exclude the possibility that it solves only the generalized supergravity equations. It should be mentioned that, given a solution of the generalized supergravity equations and provided that K^a is space-like, it is possible to find a genuine supergravity solution which is formally T-dual to it [25, 32] (i.e. only at the classical level of the sigma model, ignoring the fact that the dilaton is linear in the coordinate along which T-duality is implemented). We will not consider this possibility here.

[2] See [23] for lower dimensional examples of bosonic truncations and [24] for a review.

[3] These results differ from the ones proposed in [28]. The metric in target space of the λ-deformed $AdS_5 \times S^5$ was obtained in [29].

[4] Earlier indications of this was seen in the pure spinor string in [31].

[5] We will actually see that the kappa symmetry transformations of the λ-model take the standard form only after inserting proper factors of i (see section 2.2). This leads to a target space geometry which is a solution of type II* rather than type II supergravity. In the case of $AdS_2 \times S^2 \times T^6$ [26] it was shown how one can get a standard (and real) type IIB background by analytic continuation, or equivalently by picking a different coordinate patch. The same should be true for the deformation of $AdS_3 \times S^3 \times T^4$ [27], and probably $AdS_5 \times S^5$.

JHEP10(2016)045

Target space supergeometry. The procedure for the η-deformation can be generalised[6] also to the case when the R-matrix satisfies the classical Yang-Baxter equation (CYBE) [33–35]. Therefore several solutions exist and the question is which choices lead to a string theory, i.e. a target space that solves the standard type II supergravity equations. Here we will answer this question and find a simple (necessary and sufficient) condition on R. We will also determine the form of the target space (super) fields for both the η and the λ-model in terms of the ingredients that define them (see section 2 for their definition); we check that the models can be written in Green-Schwarz form and we work out the superspace torsion. The target space fields can then be read off by comparing to the expressions in [30, 36]. This gives a simple way of extracting the target space backgrounds, much simpler than previous methods. The metric and B-field are easily read off directly from the sigma model Lagrangian, see (2.7). The NSNS three-form and RR fluxes are found to be given by the expressions[7]

$$H_{abc} = 3M_{[ab,c]} - 3i \begin{Bmatrix} \hat{\eta}^2 \\ -\lambda^2 \end{Bmatrix} M^{\hat{\alpha}2}{}_{[a}(\gamma_b)_{\hat{\alpha}\hat{\beta}}M^{\hat{\beta}2}{}_{c]}\,, \tag{1.1}$$

$$\mathcal{S}^{\hat{\alpha}1\hat{\beta}2} = 8i \begin{Bmatrix} [\mathrm{Ad}_h(1+2\hat{\eta}^{-2} - 4\mathcal{O}_+^{-1})]^{\hat{\alpha}1}{}_{\hat{\gamma}1} \\ i\lambda[\mathrm{Ad}_h(1+\lambda(1-\lambda^{-4})\mathcal{O}_+^{-1})]^{\hat{\alpha}1}{}_{\hat{\gamma}1} \end{Bmatrix} \widehat{\mathcal{K}}^{\hat{\gamma}1\hat{\beta}2}\,, \tag{1.2}$$

where the upper (lower) expression in curly brackets refers to the η (λ) model and $\hat{\eta} = \sqrt{1-c\eta^2}$. The RR field strengths are encoded in the bispinor defined as [30, 36]

$$\mathcal{S} = -i\sigma^2\gamma^a\mathcal{F}_a - \frac{1}{3!}\sigma^1\gamma^{abc}\mathcal{F}_{abc} - \frac{1}{2\cdot 5!}i\sigma^2\gamma^{abcde}\mathcal{F}_{abcde}\,, \tag{1.3}$$

where for standard supergravity backgrounds $\mathcal{F} = e^{\phi}F$ contains the exponential of the dilaton. The remaining ingredients in these equations are defined in section 2, in particular the operators $\mathcal{O}_+$, M and the group element h are defined in (2.5), (2.2), (B.2) and (2.12). From our computation we obtain also the Killing vector of the generalised type II equations

$$K^a = -\frac{i}{16}(\gamma^a)^{\hat{\alpha}\hat{\beta}}(\nabla_{\hat{\alpha}1}\chi_{\hat{\beta}1} - \nabla_{\hat{\alpha}2}\chi_{\hat{\beta}2})\,, \tag{1.4}$$

where χ^I $(I=1,2)$ are the would be dilatino superfields

$$\chi^1_{\hat{\alpha}} = \frac{i}{2}\begin{Bmatrix} \hat{\eta} \\ -1 \end{Bmatrix}\gamma^b_{\hat{\alpha}\hat{\beta}}[\mathrm{Ad}_h M]^{\hat{\beta}1}{}_b\,, \qquad \chi^2_{\hat{\alpha}} = -\frac{i}{2}\begin{Bmatrix} \hat{\eta} \\ i\lambda \end{Bmatrix}\gamma^a_{\hat{\alpha}\hat{\beta}}M^{\hat{\beta}2}{}_a\,. \tag{1.5}$$

When K^a vanishes we have a standard supergravity solution and the dilaton is given by[8]

$$e^{-2\phi} = \mathrm{sdet}(\mathcal{O}_+)\,. \tag{1.6}$$

[6]We will use the names "η-deformation" and "Yang-Baxter deformation" for both the homogeneous (CYBE) and inhomogeneous (MCYBE) cases, as we can treat them both at the same time.

[7]Note that here we write the λ-model as a solution of type IIB supergravity, and the corresponding RR flux is imaginary. The background is real when written as a solution of type IIB*. The reason for this is a non-standard sign in the kappa symmetry transformations of the lambda model, see sec 2.2.

[8]For the λ-model this formula was argued in [6]. It is also consistent with the form of the bosonic dilaton suggested in [37] for the η-model based on bosonic R-matrices.

JHEP10(2016)045

For the λ-model K^a automatically vanishes and the target space is always a supergravity solution, consistently with the observation of [30] and the previous findings [26, 27].

The η-model as a string. For the η-model the situation is more subtle. Let us review some details at this point and recall that the η-deformation is defined by an antisymmetric R-matrix on the algebra $R: \mathfrak{g} \to \mathfrak{g}$, $R^T = -R$, satisfying the (M)CYBE

$$[R(x), R(y)] - R([R(x), y] + [x, R(y)]) = c[x, y]\,, \quad \forall x, y \in \mathfrak{g}\,, \quad \begin{cases} c = 0 & \text{CYBE} \\ c = \pm 1 & \text{MCYBE} \end{cases}. \tag{1.7}$$

In section 4.1 we prove that the condition $K^a = 0$ for the η-model is equivalent to the following algebraic condition on the R-matrix[9]

$$\mathrm{STr}(R\mathrm{ad}_x) = 0\,, \qquad \forall x \in \mathfrak{g} \qquad (\text{i.e.} \qquad R^B{}_A f^A{}_{BC} = 0)\,. \tag{1.8}$$

We will refer to R-matrices satisfying this condition[10] as "unimodular", for reasons that will be clear in section 5. Therefore the η-model has an interpretation as a string sigma model precisely for the unimodular R-matrices.

Let us consider the η-deformation based on an R-matrix which is a non-split[11] ($c = 1$ in (1.7)) solution of the MCYBE for the supercoset on $AdS_5 \times S^5$ with superalgebra $\mathfrak{psu}(2,2|4)$, as in [5]. A standard choice is to take R that multiplies by $-i$ $(+i)$ positive (negative) roots of the complexified algebra, and annihilates Cartan elements. Choices of different real forms of the superalgebra correspond to inequivalent R-matrices, but one can check that none of the examples considered so far [5, 14, 22, 39] are unimodular, which is consistent with the findings of [22, 39]. We are not aware of a complete classification of solutions of the MCYBE for $\mathfrak{psu}(2,2|4)$, which leaves open the possibility of having unimodular non-split R-matrices that would lead to genuine string deformations. We will not analyze this question further here.

As first pointed out in [33], there is a rich set of solutions to the CYBE ($c = 0$ in (1.7)) which can be used to define an η-deformation of the supercoset. These R-matrices can be divided into two classes: abelian and non-abelian. Writing the R-matrix as (sums over repeated indices are understood)

$$R = \frac{1}{2} r^{ij} b_i \wedge b_j\,, \qquad (R(x) = r^{ij} b_i \mathrm{Str}(b_j x),\ x \in \mathfrak{g}), \tag{1.9}$$

abelian R-matrices are the ones for which $[b_i, b_j] = 0 \quad \forall i, j$ while non-abelian ones have $[b_i, b_j] \neq 0$ for some i, j. The unimodularity condition (1.8) takes the form

$$r^{ij} [b_i, b_j] = 0\,. \tag{1.10}$$

[9] Essentially the same condition was argued to appear from the analysis of vertex operators of the β-deformed $AdS_5 \times S^5$ superstring in [38], see equation (87) there. That discussion would correspond to the truncation of our deformed action at order $\mathcal{O}(\eta^2)$. We thank Arkady Tseytlin for pointing this reference out to us.

[10] It is easy to see that this condition is compatible with the (M)CYBE.

[11] For the split case ($c = -1$) there exist no solution for the compact subalgebra $\mathfrak{su}(4) \subset \mathfrak{psu}(2,2|4)$. It seems then not possible to have a split solution for the full superalgebra.

JHEP10(2016)045

This is trivially satisfied by any abelian R-matrix, which is consistent with observations in the literature, see e.g [37, 40, 41]. This is also in line with the expectation that abelian R-matrices always have an interpretation in terms of (commuting) TsT-transformations[12] [35]. For non-abelian R-matrices the unimodularity condition (1.10) is non-trivial, and it is interesting to find all the compatible ones. In fact, as explained in section 5 it rules out most of the R-matrices of the so-called Jordanian type, which is the only class considered in the literature so far [33, 35, 37, 40, 41].

Here we will focus on the problem of classifying all R-matrices which satisfy the CYBE on the bosonic subalgebra $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6) \subset \mathfrak{psu}(2,2|4)$ and are unimodular. The question is non-trivial only for non-abelian R-matrices, which we classify by the rank. From (1.10), any unimodular R-matrix of rank two $R = a \wedge b$ must be abelian, i.e. $[a,b] = 0$, so non-abelian unimodular R-matrices have at least rank four. In tables 1 and 2 we write down all non-abelian rank four R-matrices for $\mathfrak{so}(2,4)$ (the second table gives the inequivalent ones from the point of view of the string sigma model), and in table 3 we provide the bosonic isometries and the number of supersymmetries that they preserve. These R-matrices are constructed in section 5, where we also show that the only other possibility is rank six. The extension to $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6)$ is essentially trivial as it turns out that they must be abelian.[13] in $\mathfrak{so}(6)$. Therefore there are no new marginal deformations of the dual CFT.[14] Notice that R_6, R_{13} and R_{15} can be embedded in $\mathfrak{so}(2,3)$ and can therefore be used to define deformations of AdS_4. To have non-abelian deformations of AdS_3, instead, one must involve also generators from the sphere.

Because abelian R-matrices seem to generate backgrounds which can be equivalently obtained by doing (commuting) TsT-transformations on the undeformed model, one might suspect that η-deformed strings always correspond to TsT-transformations. With the exception of the last three R-matrices our results appear to be consistent with this expectation, see section 5 for a discussion.

The outline of the rest of the paper is as follows. In section 2 we first review the definitions of the deformed models, we introduce a notation that highlights their similarities, and prove that the local fermionic symmetries of both deformed models are of the standard Green-Schwarz form. In section 3 we derive the target space supergeometry for the λ-model, and by comparing to the results of [30] we extract the corresponding background fields. Section 4 achieves the same goal for the η-model. Here we also show how the unimodularity condition for the R-matrix is derived. In section 5 we study this condition in detail. We discuss its compatibility with Jordanian R-matrices, and derive all rank-four non-abelian unimodular R-matrices for $\mathfrak{so}(2,4)$ which solve the CYBE. In section 6 we consider the case of backgrounds generated by R-matrices which act only on the bosonic

[12] TsT stands for T-duality — shift — T-duality [42–44]. Here we use it in the most general possible sense, e.g. including non-compact and fermionic T-dualities.

[13] This includes R-matrices mixing generators of AdS and S, e.g. as in the so-called dipole deformations of [45].

[14] This statement remains to be true also if we further allow the R-matrix to act non-trivially on supercharges: after imposing unimodularity, preservation of the $\mathfrak{so}(2,4)$ isometry, reality and CYBE, we find that the only possible R-matrices are abelian and they act just on $\mathfrak{so}(6)$.

$R_1 = p_1 \wedge p_2 + (p_0 + p_3) \wedge (J_{01} - J_{13})$
$R_2 = p_1 \wedge p_2 + (p_0 + p_3) \wedge (p_3 + J_{01} - J_{13})$
$R_3 = p_1 \wedge (J_{02} - J_{23}) + (p_0 + p_3) \wedge (p_2 + J_{01} - J_{13})$
$R_4 = (p_1 - J_{02} + J_{23}) \wedge (k_0 + k_3 + 2p_3 - 2J_{12}) + 2(p_0 + p_3) \wedge (p_2 + J_{01} - J_{13})$
$R_5 = p_1 \wedge (J_{02} - J_{23}) + (p_0 + p_3) \wedge (D + J_{03})$
$R_6 = p_1 \wedge J_{03} + 2p_0 \wedge p_3$
$R_7 = J_{03} \wedge J_{12} + 2p_0 \wedge p_3$
$R_8 = p_1 \wedge p_2 + (p_0 + p_3) \wedge J_{12}$
$R_9 = p_1 \wedge p_2 + (p_0 + p_3) \wedge (p_3 + J_{12})$
$R_{10} = p_1 \wedge p_2 + p_3 \wedge (p_0 + J_{12})$
$R_{11} = p_1 \wedge p_2 + p_3 \wedge J_{12}$
$R_{12} = p_1 \wedge p_2 + p_0 \wedge (p_3 + J_{12})$
$R_{13} = p_1 \wedge p_2 + p_0 \wedge J_{12}$
$R_{14} = p_1 \wedge p_2 + J_{12} \wedge J_{03}$
$R_{15} = p_1 \wedge p_3 + (J_{01} - J_{13}) \wedge (p_0 + p_3)$
$R_{16} = p_1 \wedge p_3 + (p_2 + J_{01} - J_{13}) \wedge (p_0 + p_3)$
$R_{17} = p_1 \wedge (p_3 + J_{02} - J_{23}) + (p_0 + p_3) \wedge (p_2 + J_{01} - J_{13})$

Table 1. All non-abelian unimodular rank-four R-matrices (CYBE) of $\mathfrak{so}(2,4)$ up to automorphisms of the corresponding subalgebras (see section 5).

subalgebra. We work out certain examples generated by the R-matrices previously derived, and we check in some cases that they are equivalent to sequences of TsT transformations on the original undeformed model.

2 η and λ-deformed string sigma models

The η and λ deformations are deformations of supercoset sigma models that preserve the classical integrability of the original models. In the string theory context the most studied example is the deformation of the $AdS_5 \times S^5$ string[15] described by a $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)}$ supercoset sigma model [47]. However, there are many other backgrounds where at least a subsector of the string worldsheet theory is described by a supercoset sigma model, e.g. $AdS_4 \times \mathbb{CP}^3$ [48–50], $AdS_3 \times S^3 \times T^4$ [51], $AdS_2 \times S^2 \times T^6$ [52] and several others [3].

We start by reviewing the definitions of the deformed models. The relevant superalgebra conventions are collected in appendix A.

[15] Another supercoset closely related to this is the pp-wave background of [46].

$R_1 = (p_1 + a(J_{01} - J_{13})) \wedge p_2 + (p_0 + p_3) \wedge (J_{01} - J_{13})$
$R_2 = (p_1 + a(p_3 + J_{01} - J_{13}) + b(p_0 + p_3)) \wedge p_2 + (p_0 + p_3) \wedge (p_3 + J_{01} - J_{13})$
$R_3 = (p_1 + a(p_2 + J_{01} - J_{13})) \wedge (p_1 + J_{02} - J_{23}) + (p_0 + p_3) \wedge (p_2 + J_{01} - J_{13})$
$R_4 = ((p_1 - J_{02} + J_{23}) + 2a(p_2 + J_{01} - J_{13}) + 2b(p_0 + p_3)) \wedge (k_0 + k_3 + 2p_3 - 2J_{12} + c(p_0 + p_3))$ $+2d(p_0 + p_3) \wedge (p_2 + J_{01} - J_{13})$
$R_5 = p_1 \wedge (J_{02} - J_{23}) + a(p_0 + p_3) \wedge (D + J_{03})$
$R_6 = p_1 \wedge J_{03} + 2p_0 \wedge p_3$
$R_7 = J_{03} \wedge J_{12} + 2p_0 \wedge p_3$
$R_8 = p_1 \wedge p_2 + (p_0 + p_3) \wedge J_{12}$
$R_9 = p_1 \wedge p_2 + a\,(p_0 + p_3) \wedge (p_3 + J_{12})$
$R_{10} = p_1 \wedge p_2 + a\, p_3 \wedge (p_0 + J_{12})$
$R_{11} = p_1 \wedge p_2 + p_3 \wedge J_{12}$
$R_{12} = p_1 \wedge p_2 + a\, p_0 \wedge (p_3 + J_{12})$
$R_{13} = p_1 \wedge p_2 + p_0 \wedge J_{12}$
$R_{14} = p_1 \wedge p_2 + J_{12} \wedge J_{03}$
$R_{15} = (p_1 + a(p_0 + p_3)) \wedge p_3 + (J_{01} - J_{13}) \wedge (p_0 + p_3)$
$R_{16} = (p_1 + a(p_0 + p_3)) \wedge p_3 + (p_2 + J_{01} - J_{13}) \wedge (p_0 + p_3)$
$R_{17} = (p_1 + a(p_0 + p_3)) \wedge (p_1 + p_3 + J_{02} - J_{23}) + (p_0 + p_3) \wedge (p_2 + J_{01} - J_{13})$

Table 2. All non-abelian unimodular rank four R-matrices (CYBE) of $\mathfrak{so}(2,4)$ up to inner automorphisms.

2.1 Lagrangians of the deformed models

The η-model Lagrangian takes the form [5, 33]

$$\mathcal{L} = -\frac{(1+c\eta^2)^2}{4(1-c\eta^2)}(\gamma^{ij} - \varepsilon^{ij})\mathrm{Str}(g^{-1}\partial_i g\; \hat{d}\,\mathcal{O}_-^{-1}(g^{-1}\partial_j g))\,, \tag{2.1}$$

where g is a group element of G, i, j are worldsheet indices, γ^{ij} is the (Weyl-invariant) worldsheet metric and $\varepsilon^{01} = +1$. Here η is the deformation parameter, and setting $\eta = 0$ yields the Lagrangian of the undeformed supercoset sigma model. The deformation involves the Lie algebra operators

$$\mathcal{O}_+ = 1 + \eta R_g \hat{d}^T\,, \qquad \mathcal{O}_- = 1 - \eta R_g \hat{d}\,, \tag{2.2}$$

where $R_g = \mathrm{Ad}_g^{-1} R \mathrm{Ad}_g$, $R^T = -R$ and R satisfies the (M)CYBE (1.7). Our derivation is general and we will not need to pick a particular solution of (1.7): we only need to assume the above properties for R, and we will treat the homogeneous ($c = 0$, CYBE) and the inhomogeneous ($c = 1$, MCYBE) cases at the same time. In the Lagrangian the following

	supercharges	bosonic isometries
R_1	8	$p_0+p_3,\ p_1,\ p_2,\ p_0-p_3-2(J_{02}-J_{23}), \quad (a=0)$
	8	$p_0+p_3,\ p_1+a(J_{01}-J_{13}),\ p_2, \quad (a\neq 0)$
R_2	8	$p_0+p_3,\ p_1,\ p_2,\ p_0-p_3-J_{01}-J_{02}+J_{13}+J_{23}, \quad (a=0)$
	8	$p_0+p_3,\ p_1+a(J_{01}-J_{13}),\ p_2, \quad (a\neq 0)$
R_3	8	$p_0+p_3,\ p_1,\ J_{02}-J_{23}, \quad (a=0)$
	8	$p_0+p_3,\ p_1+(J_{02}-J_{23}),\ J_{02}-J_{23}-a(J_{01}-J_{13}+p_2), \quad (a\neq 0)$
R_4	0	$-J_{02}+J_{23}+p_1+2a(J_{01}-J_{13}+p_2),\ p_0+p_3,\ 2J_{12}-2p_3-k_0-k_3,$
R_5	8	$D+J_{03},\ p_0+p_3,$
R_6	0	$J_{03},\ p_1,\ p_2,$
R_7	0	$J_{03},\ J_{12},$
R_8	0	$p_0,\ p_3,\ J_{12},$
R_9	0	$p_0,\ p_3,\ J_{12},$
R_{10}	0	$p_0,\ p_3,\ J_{12},$
R_{11}	0	$p_0,\ p_3,\ J_{12},$
R_{12}	0	$p_0,\ p_3,\ J_{12},$
R_{13}	0	$p_0,\ p_3,\ J_{12},$
R_{14}	0	$J_{03},\ J_{12},$
R_{15}	8	$p_0+p_3,\ p_1,\ p_2,$
R_{16}	8	$p_0+p_3,\ p_1,\ p_2,$
R_{17}	8	$p_0+p_3,\ p_1,\ J_{02}-J_{23}-p_2+p_3,$

Table 3. For each R-matrix of table 2 we indicate the number of unbroken supercharges and we list the unbroken bosonic isometries.

combinations of projection operators appear

$$\begin{aligned} \hat{d} &= P^{(1)} + 2\hat{\eta}^{-2}P^{(2)} - P^{(3)}\,, & \hat{\eta} &= \sqrt{1-c\eta^2}\,. \\ \hat{d}^T &= -P^{(1)} + 2\hat{\eta}^{-2}P^{(2)} + P^{(3)}\,, & \text{where } \hat{d}+\hat{d}^T &= 4\hat{\eta}^{-2}P^{(2)}\,. \end{aligned} \tag{2.3}$$

The λ-model is defined as a deformation of the G/G gauged WZW model. To get a standard string sigma model one integrates out the gauge-field which leads to a Lagrangian[16] somewhat similar to that of the η-model, namely [6]

$$\mathcal{L} = -\frac{k}{2\pi}(\gamma^{ij}-\varepsilon^{ij})\mathrm{Str}(g^{-1}\partial_i g(1+\widehat{B}_0-2\mathcal{O}_-^{-1})(g^{-1}\partial_j g))\,. \tag{2.4}$$

[16]This is the classical Lagrangian. At the quantum level there is also a Fradkin-Tseytlin term $R^{(2)}\phi$ present, where ϕ is the dilaton superfield, generated by integrating out the gauge-field, whose form will be discussed in section 3.

Here k is the level of the WZW model,[17] and the Lie algebra operators $\mathcal{O}_\pm$ are now defined as

$$\mathcal{O}_+ = \mathrm{Ad}_g^{-1} - \Omega^T , \qquad \mathcal{O}_- = 1 - \mathrm{Ad}_g^{-1}\Omega . \tag{2.5}$$

In this case things are written in terms of the combinations of projectors

$$\begin{aligned} \Omega &= P^{(0)}+\lambda^{-1}P^{(1)}+\lambda^{-2}P^{(2)}+\lambda P^{(3)} , \\ \Omega^T &= P^{(0)}+\lambda P^{(1)}+\lambda^{-2}P^{(2)}+\lambda^{-1}P^{(3)} , \quad 1-\Omega\Omega^T = 1-\Omega^T\Omega = (1-\lambda^{-4})P^{(2)} . \end{aligned} \tag{2.6}$$

Both the Lagrangian (2.1) of the η and (2.4) of the λ-model can be formally written in the same way[18]

$$\mathcal{L} = -\frac{T}{2}\gamma^{ij}\mathrm{Str}(A^{(2)}_{-i}A^{(2)}_{-j}) + \frac{T}{2}\varepsilon^{ij}\mathrm{Str}(A_{-i}\widehat{B}A_{-j}) , \tag{2.7}$$

in terms of the one-forms

$$A_\pm = \mathcal{O}_\pm^{-1}(g^{-1}dg) , \tag{2.8}$$

where the string tension T and the operator $\widehat{B}$ (responsible for the B-field) in the two cases are

$$\begin{aligned} &\eta-\textbf{model}: \quad T = \left(\frac{1+c\eta^2}{1-c\eta^2}\right)^2 , \quad \widehat{B} = \frac{\hat{\eta}^2}{2}(P^{(1)}-P^{(3)}+\eta\hat{d}^T R_g \hat{d}) , \\ &\lambda-\textbf{model}: \quad T = \frac{k}{\pi}(\lambda^{-4}-1) , \quad \widehat{B} = (\lambda^{-4}-1)^{-1}(\mathcal{O}_-^T\widehat{B}_0\mathcal{O}_- + \Omega^T\mathrm{Ad}_g - \mathrm{Ad}_g^{-1}\Omega) . \end{aligned} \tag{2.9}$$

An important role is played by the operator

$$M = \mathcal{O}_-^{-1}\mathcal{O}_+ \tag{2.10}$$

which relates A_- to A_+ as $A_- = MA_+$. Using the expressions in (B.2) it is not hard to show that

$$M^T P^{(2)} M = P^{(2)} , \tag{2.11}$$

which implies that the operator $P^{(2)}MP^{(2)}$ implements a Lorentz transformation on the subspace with grading-2 of the superisometry algebra. This implies that there exists an element $h \in H = G^{(0)} \subset G$ such that

$$P^{(2)}MP^{(2)} = \mathrm{Ad}_h^{-1}P^{(2)} = P^{(2)}\mathrm{Ad}_h^{-1} . \tag{2.12}$$

The fact that Ad_h is a Lorentz transformation implies the basic relation between the action on vectors and spinors

$$[\mathrm{Ad}_h]^{\hat{\gamma}}{}_{\hat{\alpha}}\gamma^a_{\hat{\gamma}\hat{\delta}}[\mathrm{Ad}_h]^{\hat{\delta}}{}_{\hat{\beta}} = [\mathrm{Ad}_h]^a{}_b\gamma^b_{\hat{\alpha}\hat{\beta}} . \tag{2.13}$$

We refer to appendix B for some useful identities satisfied by the operators entering the deformed models.

[17] $\widehat{B}_0 = -\widehat{B}_0^T$ is related to the original WZ-term, see section 3.
[18] We have used (2.3), (2.6), $\mathrm{Ad}_g^T = \mathrm{Ad}_g^{-1}$ and $R_g^T = -R_g$.

2.2 Kappa symmetry transformations in Green-Schwarz form

Both the η and λ model have a local fermionic symmetry which removes 16 of the 32 fermions, and here we show that it takes the form of the standard kappa symmetry of the GS superstring. The transformations for the local fermionic symmetry take the form [5, 6, 33]

$$\mathcal{O}_{+}^{-1}(g^{-1}\delta_\kappa g) = P_{-}^{ij}\{i\tilde{\kappa}_i^{(1)}, A_{-j}^{(2)}\} + \zeta^s P_{+}^{ij}\{i\tilde{\kappa}_i^{(3)}, A_{+j}^{(2)}\}\,, \tag{2.14}$$

where we denote the parameter by $\tilde{\kappa}$, which is related to the kappa symmetry parameter κ of the GS string as explained below. The above transformations are accompanied by the variation of the worldsheet metric

$$\delta_\kappa \gamma^{ij} = \frac{\zeta^2}{2}\Big(\mathrm{Str}(W[(P_+ i\tilde{\kappa}^{(1)})^i, (P_+ A_+^{(1)})^j]) + \mathrm{Str}(W[(P_- i\tilde{\kappa}^{(3)})^i, (P_- A_-^{(3)})^j])\Big)\,, \tag{2.15}$$

where we have defined

$$P_{\pm}^{ij} = \frac{1}{2}(\gamma^{ij} \pm \varepsilon^{ij})\,, \qquad \zeta = \begin{cases} \hat{\eta} \\ \lambda \end{cases}, \qquad s = \begin{cases} 0 & \eta-\textbf{model} \\ 1 & \lambda-\textbf{model} \end{cases}. \tag{2.16}$$

Using the fact that $A_-^{(2)}$ is related to $A_+^{(2)}$ by a gauge transformation, i.e.

$$A_-^{(2)} = P^{(2)} M A_+^{(2)} = \mathrm{Ad}_h^{-1} A_+^{(2)}\,, \tag{2.17}$$

we can write the kappa transformations as[19]

$$\begin{gathered} i_{\delta_\kappa} E^{(2)} = 0\,, \qquad i_{\delta_\kappa} E^{(1)} = P_-^{ij}\{i\kappa_i^{(1)}, E_j^{(2)}\}\,, \qquad i_{\delta_\kappa} E^{(3)} = P_+^{ij}\{i\kappa_i^{(3)}, E_j^{(2)}\} \\ \delta_\kappa \gamma^{ij} = \frac{1}{2}\mathrm{Str}(W[(P_+ i\kappa^{(1)})^i, (P_+ E^{(1)})^j]) + \frac{1}{2}\mathrm{Str}(W[(P_- i\kappa^{(3)})^i, (P_- E^{(3)})^j])\,, \end{gathered} \tag{2.18}$$

where $\kappa^{(1)} = \zeta \mathrm{Ad}_h \tilde{\kappa}^{(1)}$ and $\kappa^{(3)} = (-i)^s \zeta \tilde{\kappa}^{(3)}$. This shows that the kappa symmetry variations have the standard GS form, and at the same time it allows us to identify the supervielbeins with projections of $A_\pm$ as[20]

$$E^{(2)} \equiv E^a P_a = A_+^{(2)}\,, \quad E^{(1)} \equiv E^{\hat{\alpha}1} Q_{\hat{\alpha}}^1 = \zeta \mathrm{Ad}_h A_+^{(1)}\,, \quad E^{(3)} \equiv E^{\hat{\alpha}2} Q_{\hat{\alpha}}^2 = i^s \zeta A_-^{(3)}\,. \tag{2.19}$$

In terms of these the Lagrangian (2.7) takes the standard form

$$L = -\frac{T}{2}\gamma^{ij}\mathrm{Str}(E_i^{(2)} E_j^{(2)}) + \frac{T}{2}\varepsilon^{ij} B_{ij}\,, \tag{2.20}$$

where the B-field can be read off from (2.9).

[19] In writing the transformations in this form we used (B.4).

[20] The explicit i in $E^{(3)}$ and $\kappa^{(3)}$ in the case of the λ-model is needed to put the transformations in the standard type IIB form. The reason for having i can be traced to the relative sign between $P^{(1)}$ and $P^{(3)}$ in (2.6) compared to (2.3). Alternatively, insisting on manifest reality of the model, the kappa symmetry transformations and superspace constraints become those of type IIB* rather than type IIB. This is rather natural since the λ-model is a deformation of the non-abelian T-dual of the $AdS_5 \times S^5$ string, which involves also a T-duality in the time direction.

Since the action and kappa symmetry transformations take the standard GS form, it follows from the analysis of [30] that the target superspace of these models solves the generalized type II supergravity equations derived there. If the Killing vector K^a appearing in these equations vanishes, they reduce to the standard supergravity equations. In the next sections we will derive the form of the target space supergeometry for the η and λ-deformed strings. Having identified the supervielbeins of the background superspace we can find the supergeometry by calculating the torsion[21]

$$T^a = dE^a + E^b \wedge \Omega_b{}^a\,, \qquad T^{\hat{\alpha}I} = dE^{\hat{\alpha}I} - \frac{1}{4}(\gamma_{ab}E^I)^{\hat{\alpha}} \wedge \Omega^{ab} \qquad (I=1,2)\,, \tag{2.21}$$

and reading off the background superfields by comparing to the general expressions derived in [30]. These are valid for a generalized type II supergravity background and reduce to those of a standard supergravity background (see e.g. [36]) only when $K^a = 0$. We will see that the λ-model background is a solution to standard (type II*) supergravity. For the η-model background we will derive the condition on the R-matrix of the η-model for it to give rise to a standard type II background.

3 Target superspace for the λ-model

In this section we present the derivation for the λ-model. We refer to appendix B.1 for more details. The supervielbeins are defined in terms of projections of $A_\pm$ by (2.19). To calculate the torsion we therefore need to calculate the exterior derivative of $A_\pm$. Using $A_+ = \mathcal{O}_+^{-1}(g^{-1}dg)$ where $\mathcal{O}_\pm$ are defined in (2.5) we find

$$\begin{aligned} dA_+ &= \mathcal{O}_+^{-1}(d\mathcal{O}_+ \wedge A_+) + \mathcal{O}_+^{-1}(g^{-1}dg \wedge g^{-1}dg) \\ &= -\mathcal{O}_+^{-1}\{g^{-1}dg, \mathrm{Ad}_g^{-1}A_+\} + \frac{1}{2}\mathcal{O}_+^{-1}\{g^{-1}dg, g^{-1}dg\} \\ &= -\frac{1}{2}\mathcal{O}_+^{-1}\{\mathrm{Ad}_g^{-1}A_+, \mathrm{Ad}_g^{-1}A_+\} + \frac{1}{2}\mathcal{O}_+^{-1}\{\Omega^T A_+, \Omega^T A_+\} \\ &= -\frac{1}{2}\{A_+, A_+\} - \frac{1}{2}\mathcal{O}_+^{-1}(\Omega^T\{A_+, A_+\} - \{\Omega^T A_+, \Omega^T A_+\})\,, \end{aligned} \tag{3.1}$$

where we used the fact that $g^{-1}dg = \mathcal{O}_+ A_+ = (\mathrm{Ad}_g^{-1} - \Omega^T)A_+$ to write everything in terms of A_+. An almost identical calculation gives

$$dA_- = \frac{1}{2}\{A_-, A_-\} + \frac{1}{2}\mathcal{O}_-^{-1}\mathrm{Ad}_g^{-1}(\Omega\{A_-, A_-\} - \{\Omega A_-, \Omega A_-\})\,. \tag{3.2}$$

In the above equations it is useful to expand out the expressions inside parenthesis, see (B.5), (B.6). Projecting equation (B.5) with $P^{(2)}$ we find

$$\begin{aligned} dE^{(2)} = {} & \frac{1}{2}\{E^{(1)}, E^{(1)}\} + \frac{1}{2}\{E^{(3)}, E^{(3)}\} - \{A_+^{(0)}, E^{(2)}\} - i\lambda\{E^{(3)}, P^{(3)}ME^{(2)}\} \\ & - i\lambda P^{(2)}M^T\{E^{(2)}, E^{(3)}\} - \frac{1}{2}\lambda^2\{P^{(3)}ME^{(2)}, P^{(3)}ME^{(2)}\} - \frac{1}{2}P^{(2)}M^T\{E^{(2)}, E^{(2)}\} \\ & - \lambda^2 P^{(2)}M^T\{E^{(2)}, P^{(3)}ME^{(2)}\}\,. \end{aligned} \tag{3.3}$$

[21]Our conventions are the same as those of [30]. In particular d acts from the right and components of superforms are defined as $\omega_n = \frac{1}{n!}E^{A_n} \wedge \cdots \wedge E^{A_1}\omega_{A_1\cdots A_n}$.

where the result has been rewritten in terms of the supervielbeins (2.19), and we have used (B.4) and (2.12). Using the explicit form of the commutators in (A.1) and (A.2) we find that the component T^a of the torsion takes the standard form (here and in the following we drop the $\wedge$'s for readability)

$$T^a = dE^a + E^b \Omega_b{}^a = -\frac{i}{2} E^1 \gamma^a E^1 - \frac{i}{2} E^2 \gamma^a E^2 \,, \tag{3.4}$$

if we identify the spin connection as[22]

$$\Omega_{ab} = -(A_+)_{ab} - 2\lambda (E^2 \gamma_{[a})_{\hat\alpha} M^{\hat\alpha 2}{}_{b]} - \frac{3i}{2} \lambda^2 E^c M^{\hat\alpha 2}{}_{[a} (\gamma_b)_{\hat\alpha\hat\beta} M^{\hat\beta 2}{}_{c]} + \frac{1}{2} E^c (M_{ab,c} - 2M_{c[a,b]}) \,. \tag{3.5}$$

To derive the other components of the torsion we first need to compute the exterior derivative of the fermionic supervielbeins. Using (B.6) and (2.19) we find

$$\begin{aligned} dE^{(3)} &= \frac{i}{2} \lambda P^{(3)} M \{E^{(3)}, E^{(3)}\} - \{A_+^{(0)}, E^{(3)}\} + \{P^{(0)} M E^{(2)}, E^{(3)}\} \\ &\quad - i\lambda \big[1 + \lambda(1 - \lambda^{-4}) P^{(3)} (\mathcal{O}_+^T)^{-1}\big] \mathrm{Ad}_h^{-1} \big(\{E^{(2)}, E^{(1)}\} - \{E^{(2)}, \mathrm{Ad}_h P^{(1)} M E^{(2)}\}\big) \\ &\quad + \frac{i}{2} \lambda (1 - \lambda^{-4}) P^{(3)} (\mathcal{O}_+^T)^{-1} \mathrm{Ad}_h^{-1} \{E^{(2)}, E^{(2)}\} \,. \end{aligned} \tag{3.6}$$

Since we have already identified the form of the spin connection (3.5) from the previous computation, we can now find the corresponding component of the torsion (2.21) and compare it to the standard form given in [30], i.e.

$$T^{\hat\alpha 2} = E^{\hat\alpha 2}\, E^2 \chi^2 - \frac{1}{2} E^2 \gamma^a E^2 (\gamma_a \chi^2)^{\hat\alpha} + \frac{1}{8} E^a (E^2 \gamma^{bc})^{\hat\alpha} H_{abc} - \frac{1}{8} E^a (E^1 \gamma_a \mathcal{S}^{12})^{\hat\alpha} + \frac{1}{2} E^b E^a \psi_{ab}^{\hat\alpha 2} \,, \tag{3.7}$$

where H is the NSNS three-form, $\mathcal{S}$ the RR bispinor, $\chi_{\hat\alpha}^I$ the dilatino and $\psi_{ab}^{\hat\alpha I}$ the gravitino field strength superfields. We find that $T^{\hat\alpha 2}$ takes the above form if we identify

$$H_{abc} = 3M_{[ab,c]} + 3i\lambda^2 M^{\hat\alpha 2}{}_{[a} (\gamma_b)_{\hat\alpha\hat\beta} M^{\hat\beta 2}{}_{c]} \,, \tag{3.8}$$

$$\mathcal{S}^{\hat\alpha 1 \hat\beta 2} = -8\lambda \big[\mathrm{Ad}_h (1 + \lambda(1 - \lambda^{-4}) \mathcal{O}_+^{-1})\big]^{\hat\alpha 1}{}_{\hat\gamma 1} \widehat{\mathcal{K}}^{\hat\gamma 1 \hat\beta 2} \,, \tag{3.9}$$

$$\chi_{\hat\alpha}^2 = \frac{1}{2} \lambda \gamma^a_{\hat\alpha\hat\beta} M^{\hat\beta 2}{}_a \,, \tag{3.10}$$

$$\psi_{ab}^{\hat\alpha 2} = \frac{i}{4} \lambda (1 - \lambda^{-4}) [(\mathcal{O}_+^T)^{-1} \mathrm{Ad}_h^{-1}]^{\hat\alpha 2}{}_{cd} \widehat{\mathcal{K}}_{ab}{}^{cd} - \frac{1}{4} [\mathrm{Ad}_h M]^{\hat\beta 1}{}_{[a} (\gamma_{b]})_{\hat\beta\hat\gamma} \mathcal{S}^{\hat\gamma 1 \hat\alpha 2} \,. \tag{3.11}$$

As already remarked, the RR bispinor superfield is imaginary if we interpret the λ-model target space as a solution of type II supergravity, as here, rather than type II* supergravity.[23] This determines the bosonic target space fields, with the exception of the dilaton which we will determine shortly. First, let us calculate also the remaining components of the femionic superfields, which we will extract from the corresponding component of the

[22] Here we rewrote $A_\pm^{(0)} = \frac{1}{2} A_\pm^{ab} J_{ab}$ and used the relation between components of M and M^T in (A.11).

[23] Let us recall that at least in some cases it is possible to define a real type II background, after analytic continuation or proper choice of coordinate patch [26, 27].

torsion, $T^{\hat\alpha 1}$. From (B.5) and using (2.19) we find

$$\begin{aligned}
dE^{(1)} &= -\{\mathrm{Ad}_h A_+^{(0)} + dhh^{-1}, E^{(1)}\} + \frac{1}{2}\lambda(1-\lambda^{-4})P^{(1)}\mathrm{Ad}_h\mathcal{O}_+^{-1}\mathrm{Ad}_h^{-1}\{E^{(1)},E^{(1)}\} \\
&\quad - i\lambda\mathrm{Ad}_h\{E^{(2)},E^{(3)}\} - \lambda^2\mathrm{Ad}_h\{E^{(2)},P^{(3)}ME^{(2)}\} - i\lambda^2(1-\lambda^{-4})P^{(1)}\mathrm{Ad}_h\mathcal{O}_+^{-1}\{E^{(2)},E^{(3)}\} \\
&\quad - \frac{1}{2}\lambda(1-\lambda^{-4})P^{(1)}\mathrm{Ad}_h\mathcal{O}_+^{-1}\big(\{E^{(2)},E^{(2)}\} + 2\lambda^2\{E^{(2)},P^{(3)}ME^{(2)}\}\big)\,.
\end{aligned} \tag{3.12}$$

Using this expression we find[24]

$$T^{\hat\alpha 1} = E^{\hat\alpha 1}\,E^1\chi^1 - \frac{1}{2}E^1\gamma^a E^1(\gamma_a\chi^1)^{\hat\alpha} - \frac{1}{8}E^a(E^1\gamma^{bc})^{\hat\alpha}H_{abc} - \frac{1}{8}E^a(E^2\gamma_a\mathcal{S}^{21})^{\hat\alpha} + \frac{1}{2}E^bE^a\psi_{ab}^{\hat\alpha 1}\,, \tag{3.13}$$

is again of the standard form given in [30], where $\mathcal{S}^{\hat\beta 2\hat\alpha 1} = -\mathcal{S}^{\hat\alpha 1\hat\beta 2}$ and

$$\chi^1_{\hat\alpha} = -\frac{i}{2}\gamma^b_{\hat\alpha\hat\beta}[\mathrm{Ad}_hM]^{\hat\beta 1}{}_b\,,\quad \psi_{ab}^{\hat\alpha 1} = -\frac{1}{2}\lambda(1-\lambda^{-4})[\mathrm{Ad}_h\mathcal{O}_+^{-1}]^{\hat\alpha 1}{}_{cd}\widehat{\mathcal{K}}_{ab}{}^{cd} - \frac{i}{4}\lambda(\mathcal{S}^{12}\gamma_{[a})^{\hat\alpha}{}_{\hat\beta}M^{\hat\beta 2}{}_{b]}\,. \tag{3.14}$$

We complete the set of background superfields for the λ-model by noting that the B-field can be written in the two equivalent forms

$$\begin{aligned}
B &= (\lambda^{-4}-1)^{-1}\big[B_0 + \mathrm{Str}(g^{-1}dg\wedge A_-)\big]\,, \qquad dB_0 = \frac{1}{3}\mathrm{Str}(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg)\,, \\
&= (\lambda^{-4}-1)^{-1}\big[B_0 - \mathrm{Str}(g^{-1}dg\wedge\Omega^TA_+)\big]\,,
\end{aligned} \tag{3.15}$$

and that the dilaton is given by

$$e^{-2\phi} = \mathrm{sdet}(\mathcal{O}_+) = \mathrm{sdet}(\mathrm{Ad}_g - \Omega)\,. \tag{3.16}$$

This result for the dilaton arises from integrating out the gauge-fields in the deformed gauged WZW model [6]. To verify that the λ-model gives rise to a standard supergravity background[25] it is enough to verify that the dilatino's found in (3.10) and (3.14) are indeed the spinor derivatives of ϕ

$$\begin{aligned}
\nabla_{\hat\alpha 2}\phi &= \frac{i}{2}\lambda\widehat{\mathcal{K}}^{\hat\beta 1\hat\gamma 2}\mathrm{STr}(Q^1_{\hat\beta}M[Q^2_{\hat\alpha},Q^2_{\hat\gamma}]) = \chi^2_{\hat\alpha}\,, \\
\nabla_{\hat\alpha 1}\phi &= \frac{1}{2}(1-\lambda^{-4})[\mathrm{Ad}_h^{-1}]^{\hat\beta}{}_{\hat\alpha}\mathrm{STr}(P^a\mathcal{O}_-^{-1}[Q^1_{\hat\beta},P_a]) = \chi^1_{\hat\alpha}\,.
\end{aligned} \tag{3.17}$$

[24]To calculate this component of the torsion we must first find the Lorentz-transformed spin connection $\mathrm{Ad}_hA_+^{(0)} + dhh^{-1}$ appearing in the first term, see equation (B.9) and the corresponding derivation.

[25]As pointed out in [30] this was clear from the fact that the metric of the λ-model does not admit any isometries, so that the Killing vector K^a of the generalized supergravity equations vanishes.

4 Target superspace for the η-model

The calculations for the η-model proceed along the same lines as those for the λ-model with only minor differences. We begin by calculating the derivative of A_+

$$\begin{aligned}
dA_+ &= \mathcal{O}_+^{-1}(d\mathcal{O}_+ \wedge A_+) + \mathcal{O}_+^{-1}(g^{-1}dg \wedge g^{-1}dg) \\
&= \eta\mathcal{O}_+^{-1}R_g\{g^{-1}dg, \hat{d}^T A_+\} - \eta\mathcal{O}_+^{-1}\{g^{-1}dg, R_g\hat{d}^T A_+\} + \frac{1}{2}\mathcal{O}_+^{-1}\{g^{-1}dg, g^{-1}dg\} \\
&= \frac{1}{2}\mathcal{O}_+^{-1}\{A_+, A_+\} + \eta\mathcal{O}_+^{-1}R_g\{A_+, \hat{d}^T A_+\} + \eta^2\mathcal{O}_+^{-1}R_g\{R_g\hat{d}^T A_+, \hat{d}^T A_+\} \\
&\quad - \frac{1}{2}\eta^2\mathcal{O}_+^{-1}\{R_g\hat{d}^T A_+, R_g\hat{d}^T A_+\} \\
&= \frac{1}{2}\mathcal{O}_+^{-1}\{A_+, A_+\} - \frac{1}{2}c\eta^2\mathcal{O}_+^{-1}\{\hat{d}^T A_+, \hat{d}^T A_+\} + \eta\mathcal{O}_+^{-1}R_g\{A_+, \hat{d}^T A_+\}\,,
\end{aligned} \tag{4.1}$$

where we used the fact that $g^{-1}dg = \mathcal{O}_+A_+$ and in the last step we used the fact that R (as well as R_g) satisfies the (M)CYBE equation, so that

$$\{R_g\hat{d}^T A_+, R_g\hat{d}^T A_+\} - 2R_g\{R_g\hat{d}^T A_+, \hat{d}^T A_+\} - c\{\hat{d}^T A_+, \hat{d}^T A_+\} = 0\,. \tag{4.2}$$

The result for dA_- is simply obtained by changing the sign of η and replacing $\hat{d}^T \to \hat{d}$ in the above expression

$$dA_- = \frac{1}{2}\mathcal{O}_-^{-1}\{A_-, A_-\} - \frac{1}{2}c\eta^2\mathcal{O}_-^{-1}\{\hat{d}A_-, \hat{d}A_-\} - \eta\mathcal{O}_-^{-1}R_g\{A_-, \hat{d}A_-\}\,. \tag{4.3}$$

After rewriting dA_+ as in (B.13) and projecting with $P^{(2)}$ we find

$$\begin{aligned}
dE^{(2)} &= \{A_+^{(0)}, E^{(2)}\} + \frac{1}{2}\{E^{(1)}, E^{(1)}\} + \frac{1}{2}\{E^{(3)}, E^{(3)}\} - 2\hat{\eta}\{E^{(3)}, P^{(3)}\mathcal{O}_-^{-1}E^{(2)}\} \\
&\quad + 4\hat{\eta}^{-1}P^{(2)}\mathcal{O}_+^{-1}\{E^{(2)}, E^{(3)}\} - 8P^{(2)}\mathcal{O}_+^{-1}\{E^{(2)}, P^{(3)}\mathcal{O}_-^{-1}E^{(2)}\} \\
&\quad + 2\hat{\eta}^2\{P^{(3)}\mathcal{O}_-^{-1}E^{(2)}, P^{(3)}\mathcal{O}_-^{-1}E^{(2)}\} + 2\eta\hat{\eta}^{-2}P^{(2)}\mathcal{O}_+^{-1}R_g\{E^{(2)}, E^{(2)}\}\,,
\end{aligned} \tag{4.4}$$

where we have used (2.19) to write the result in terms of the supervielbeins, together with (B.4) and (2.12). We check again that the bosonic torsion T^a takes the standard form (3.4), where we can now identify the spin connection for the η-model background as

$$\Omega_{ab} = (A_+)_{ab} + 2i\hat{\eta}(\gamma_{[a}E^2)_{\hat{\alpha}}M^{\hat{\alpha}2}{}_{b]} + \frac{3i}{2}\hat{\eta}^2E^c\,M^{\hat{\alpha}2}{}_{[a}(\gamma_b)_{\hat{\alpha}\hat{\beta}}M^{\hat{\beta}2}{}_{c]} - \frac{1}{2}E^c(2M_{c[a,b]} - M_{ab,c})\,. \tag{4.5}$$

As before, we continue by computing the remaining components of the torsion. First, from (B.14) we get

$$\begin{aligned}
dE^{(3)} &= \{A_+^{(0)}, E^{(3)}\} + \hat{\eta}P^{(3)}\mathcal{O}_-^{-1}\{E^{(3)}, E^{(3)}\} + 2\{P^{(0)}\mathcal{O}_-^{-1}E^{(2)}, E^{(3)}\} \\
&\quad + P^{(3)}(4\mathcal{O}_-^{-1} - 1 - 2\hat{\eta}^{-2})\mathrm{Ad}_h^{-1}\{E^{(2)}, E^{(1)}\} - 2\eta\hat{\eta}^{-1}P^{(3)}\mathcal{O}_-^{-1}R_g\mathrm{Ad}_h^{-1}\{E^{(2)}, E^{(2)}\} \\
&\quad + 2\hat{\eta}P^{(3)}(4\mathcal{O}_-^{-1} - 1 - 2\hat{\eta}^{-2})\{\mathrm{Ad}_h^{-1}E^{(2)}, P^1\mathcal{O}_-^{-1}E^{(2)}\}\,,
\end{aligned} \tag{4.6}$$

which we use to check that also $T^{\hat\alpha 2}$ is of the standard form (3.7). To do this we make use of the spin connection (4.5) and we identify the following superfields for the η-model

$$\begin{aligned} H_{abc} &= 3M_{[ab,c]} - 3i\hat\eta^2 M^{\hat\alpha 2}{}_{[a}(\gamma_b)_{\hat\alpha\hat\beta}M^{\hat\beta 2}{}_{c]}\,, && (4.7)\\ \mathcal{S}^{\hat\alpha 1\hat\beta 2} &= 8i[\mathrm{Ad}_h(1+2\hat\eta^{-2}-4\mathcal{O}_+^{-1})]^{\hat\alpha 1}{}_{\hat\gamma 1}\widehat{\mathcal{K}}^{\hat\gamma 1\hat\beta 2}\,, && (4.8)\\ \chi^2_{\hat\alpha} &= -\frac{i}{2}\hat\eta\gamma^a_{\hat\alpha\hat\beta}M^{\hat\beta 2}{}_a\,, && (4.9)\\ \psi^{\hat\alpha 2}_{ab} &= -2\eta\hat\eta^{-1}[\mathcal{O}_-^{-1}R_g\mathrm{Ad}_h^{-1}]^{\hat\alpha 2}{}_{cd}\widehat{\mathcal{K}}_{ab}{}^{cd} + \frac{1}{4}\hat\eta[\mathrm{Ad}_h M]^{\hat\beta 1}{}_{[a}(\gamma_{b]}\mathcal{S}^{12})_{\hat\beta}{}^{\hat\alpha}\,. && (4.10)\end{aligned}$$

To identify the last component of the spinor superfields we must compute torsion $T^{\hat\alpha 1}$. Starting from (B.13) we find

$$\begin{aligned} dE^{(1)} &= \{\mathrm{Ad}_h A_+^{(0)} - dhh^{-1}, E^{(1)}\} + \hat\eta P^{(1)}\mathrm{Ad}_h\mathcal{O}_+^{-1}\mathrm{Ad}_h^{-1}\{E^{(1)},E^{(1)}\}\\ &\quad + P^{(1)}\mathrm{Ad}_h(4\mathcal{O}_+^{-1} - 1 - 2\hat\eta^{-2})\{E^{(2)},E^{(3)}\} + 2\eta\hat\eta^{-1}P^{(1)}\mathrm{Ad}_h\mathcal{O}_+^{-1}R_g\{E^{(2)},E^{(2)}\}\\ &\quad - 2\hat\eta P^{(1)}\mathrm{Ad}_h(4\mathcal{O}_+^{-1} - 1 - 2\hat\eta^{-2})\{E^{(2)},P^{(3)}\mathcal{O}_-^{-1}E^{(2)}\}\,. \end{aligned} \qquad (4.11)$$

Using this expression we can check[26] that $T^{\hat\alpha 1}$ is standard, see (3.13), where $\mathcal{S}^{\hat\beta 2\hat\alpha 1} = -\mathcal{S}^{\hat\alpha 1\hat\beta 2}$ and

$$\chi^1_{\hat\alpha} = \frac{i}{2}\hat\eta\gamma^b_{\hat\alpha\hat\beta}[\mathrm{Ad}_h M]^{\hat\beta 1}{}_b\,, \quad \psi^{\hat\alpha 1}_{ab} = 2\eta\hat\eta^{-1}[\mathrm{Ad}_h\mathcal{O}_+^{-1}R_g]^{\hat\alpha 1}{}_{cd}\widehat{\mathcal{K}}_{ab}{}^{cd} - \frac{1}{4}\hat\eta(\mathcal{S}^{12}\gamma_{[a})^{\hat\alpha}{}_{\hat\beta}M^{\hat\beta 2}{}_{b]}\,. \qquad (4.12)$$

Let us also note that in the case of the η-model the B-field can be written in the two ways

$$B = \frac{\hat\eta^2}{4}\mathrm{Str}(g^{-1}dg\wedge\hat d^T A_+) = -\frac{\hat\eta^2}{4}\mathrm{Str}(g^{-1}dg\wedge\hat d A_-)\,, \qquad (4.13)$$

which are equivalent thanks to the properties of $\mathcal{O}_\pm$ under transposition.

4.1 Dilaton and supergravity condition

Unlike in the case of the λ-model, the η-model does not come with a natural candidate dilaton. Indeed, in general the target space geometry of the η-model is a solution of the generalized type II supergravity equations of [25, 30] rather than the standard ones, and a dilaton does not exist. One of our goals is to determine precisely when a dilaton exists for the η-model. To do this, let us define a would-be dilaton in the same way as the dilaton is defined in the λ-model

$$e^{-2\phi} = \mathrm{sdet}(\mathcal{O}_+) = \mathrm{sdet}(1+\eta R_g\hat d^T)\,. \qquad (4.14)$$

For this to be the actual dilaton of the η-model its spinor derivatives must coincide with the dilatinos in (4.9) and (4.12). In (B.18) we write down the result for $d\phi$. In particular

[26] As in the previous section, we need to first find an expression for $\mathrm{Ad}_h A_+^{(0)} - dhh^{-1}$, see (B.16).

we find[27]

$$\begin{aligned}\nabla_{\hat\alpha 2}\phi &= -2\hat\eta^{-1}\mathrm{STr}(P^a\mathcal{O}_+^{-1}[Q^2_{\hat\alpha},P_a]) - \frac{\eta}{2}\hat\eta^{-1}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A R_g[T_B,Q^2_{\hat\alpha}])\\ &= \chi^2_{\hat\alpha} - \frac{\eta}{2}\hat\eta^{-1}\widehat{\mathcal{K}}^{AB}\mathrm{STr}([T_A,RT_B]gQ^2_{\hat\alpha}g^{-1})\,, \end{aligned} \tag{4.15}$$

$$\begin{aligned}\nabla_{\hat\alpha 1}\phi &= -\hat\eta[\mathrm{Ad}_h^{-1}]^{\hat\beta}{}_{\hat\alpha}(\widehat{\mathcal{K}}^{\hat\gamma 1\hat\delta 2}\mathrm{STr}(Q^2_{\hat\delta}\mathcal{O}_+^{-1}[Q^1_{\hat\beta},Q^1_{\hat\gamma}]) - \frac{\eta}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A R_g[T_B,Q^1_{\hat\beta}]))\\ &= \chi^1_{\hat\alpha} + \frac{\eta}{2}\hat\eta[\mathrm{Ad}_h^{-1}]^{\hat\beta}{}_{\hat\alpha}\widehat{\mathcal{K}}^{AB}\mathrm{STr}([T_A,RT_B]gQ^1_{\hat\beta}g^{-1})\,. \end{aligned} \tag{4.16}$$

Therefore a sufficient condition for the η-model to lead to a standard supergravity background is that

$$\widehat{\mathcal{K}}^{AB}\mathrm{STr}([T_A,RT_B]gQ^I_{\hat\alpha}g^{-1}) = 0\,, \tag{4.17}$$

or, since g is an arbitrary group element (modulo gauge-transformations),

$$\mathrm{STr}(R\mathrm{ad}_x) = 0\,, \qquad \forall x\in\mathfrak{g} \qquad (\text{i.e. } R^B{}_A f^A{}_{BC}=0, \text{ or } R^{BC}f^A{}_{BC}=0)\,. \tag{4.18}$$

To see that this condition is also necessary we calculate the Killing vector superfield K^a appearing in the generalized supergravity equations of [30], which in general is given by

$$K^a = -\frac{i}{16}(\gamma^a)^{\hat\alpha\hat\beta}(\nabla_{\hat\alpha 1}\chi_{\hat\beta 1} - \nabla_{\hat\alpha 2}\chi_{\hat\beta 2})\,, \tag{4.19}$$

and whose result is collected in (B.19). The η-model has a standard type II supergravity solution as target space if $K^a=0$. In fact, it must be that it vanishes order by order in the deformation parameter η. At linear order we find the equation

$$\widehat{\mathcal{K}}^{AB}\mathrm{STr}([T_A,RT_B]gP_ag^{-1}) = 0\,, \tag{4.20}$$

which, since $g\in G$ is arbitrary implies (4.18). Therefore the condition (4.18) is both necessary and sufficient, and also the higher order terms in η in (B.19) vanish when this condition is fulfilled.

5 Non-abelian R-matrices and the unimodularity condition

In this section we study the unimodularity condition (1.8) for the R-matrix. First we analyse its compatibility with a class of non-abelian R-matrices — the Jordanian ones — and then we explain how to classify all unimodular R-matrices solving the CYBE on the bosonic subalgebra of the superisometry algebra.

Following [53] we define an "extended Jordanian" R-matrix for a Lie superalgebra $\mathfrak{g}$ as follows: we fix a Cartan element h ($\deg(h)=0$) and a positive root e as well as a collection of roots $e_{\gamma_{\pm i}}$ with $i\in\{1,2,\dots,N\}$ such that $\deg(e)=\deg(e_{\gamma_i})+\deg(e_{\gamma_{-i}})$ (mod 2) and satisfying

$$\begin{aligned} &[h,e]=e\,, \qquad [h,e_{\gamma_i}]=(1-t_{\gamma_i})e_{\gamma_i}\,, \quad [h,e_{\gamma_{-i}}]=t_{\gamma_i}e_{\gamma_{-i}}\,, \qquad\qquad (t_{\gamma_i}\in\mathbb{C})\\ &[e_{\gamma_{\pm i}},e]=0\,, \quad [e_{\gamma_k},e_{\gamma_l}]=\delta_{k,-l}e\,, \qquad\qquad (k>l\in\{\pm1,\pm2,\dots,\pm N\})\,. \end{aligned} \tag{5.1}$$

[27]Here we used the fact that $\mathcal{O}_\pm^{-1}P^{(0)}=P^{(0)}$.

The extended Jordanian R-matrix is then defined as

$$R = h \wedge e + \sum_{i=1}^{N} (-1)^{\deg(e_{\gamma_i})\deg(e_{\gamma_{-i}})} e_{\gamma_i} \wedge e_{\gamma_{-i}} \,. \tag{5.2}$$

It is now easy to see that for a bosonic deformation, i.e. $\deg(e) = 0$, we have

$$r^{ij}[b_i, b_j] = (N_0 - N_1 + 1)e \,, \tag{5.3}$$

with $N = N_0 + N_1$, N_0 (N_1) being the number of bosonic (fermionic) roots e_{γ_i}. For this to vanish we need precisely one more fermionic e_{γ_i} than bosonic. This is clearly a very strong restriction on the allowed Jordanian R-matrices. Let us note that this result is compatible with the findings of [37, 40, 41], where Jordanian R-matrices acting only on bosonic generators were found to produce backgrounds which do not solve the standard supergravity equations. We have considered certain examples of bosonic Jordanian R-matrices (namely $R = J_{01} \wedge (P_0 - P_1)$, $R = J_{03} \wedge (J_{01} - J_{13})$ and $R = D \wedge p_i,\ i = 0, \ldots, 3$) and we have checked that it is not possible to find a positive and a negative fermionic root satisfying (5.1) without spoiling the reality of the extended R-matrix. If possible, it would be interesting to find extended Jordanian unimodular R-matrices for $\mathfrak{psu}(2,2|4)$, but we will not analyze this question further here.

From now on we will restrict to the bosonic subalgebra $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6) \subset \mathfrak{psu}(2,2|4)$. Let us recall some known facts about solutions to the CYBE, (1.7) with $c = 0$, for ordinary Lie algebras. The first important fact, due to Stolin [54, 55], is that there is a one-to-one correspondence between constant solutions of the CYBE for a Lie algebra $\mathfrak{g}$ and quasi-Frobenius (or symplectic) subalgebras $\mathfrak{f} \subset \mathfrak{g}$ (see also [56]). Notice that we do not need to assume anything about the Lie algebra $\mathfrak{g}$, in particular it does not need to be simple. A Lie algebra is quasi-Frobenius if it has a non-degenerate 2-cocycle ω, i.e.

$$\omega(x,y) = -\omega(y,x)\,, \qquad \omega([x,y],z) + \omega([z,x],y) + \omega([y,z],x) = 0\,, \qquad \forall x,y,z \in \mathfrak{f}\,. \tag{5.4}$$

It is Frobenius if ω is a coboundary, i.e. $\omega(x,y) = f([x,y])$ for some linear function f. If R is a solution to the CYBE for $\mathfrak{g}$, then there is a subalgebra $\mathfrak{f}$ on which R is non-degenerate. This subalgebra is necessarily quasi-Frobenius, and writing R in the form (1.9) the 2-cocycle is the inverse of the R-matrix, i.e. $\omega(b_i, b_j) = (r^{-1})_{ij}$. The converse is also true, i.e. if $\mathfrak{f} \subset \mathfrak{g}$ is quasi-Frobenius then the inverse of the 2-cocycle ω gives a solution to the CYBE, as is easily verified. Therefore, finding solutions to the CYBE for a given $\mathfrak{g}$ reduces to finding all quasi-Frobenius subalgebras[28] of $\mathfrak{g}$. A fact with important consequences for our analysis is that if $\mathfrak{g}$ is compact then $\mathfrak{f}$ must be abelian [58]. This leads to the conclusion that deformations involving only S^5 (i.e. marginal deformations of the dual CFT) must necessarily have abelian R-matrices.

We now show that the unimodularity condition (1.8) for the R-matrix adds a further property to the quasi-Frobenius subalgebra $\mathfrak{f}$. If we write the structure constants as $f^i{}_{jk}$ in some basis, the 2-cocycle condition is

$$(r^{-1})_{i[j} f^i{}_{kl]} = 0\,. \tag{5.5}$$

[28]This was done for $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ in [57].

JHEP10(2016)045

$\mathfrak{f}$	Defining Lie brackets
$\mathbb{R}^4$	—
$\mathfrak{h}_3 \oplus \mathbb{R}$	$[e_1, e_2] = e_3$
$\mathfrak{r}_{3,-1} \oplus \mathbb{R}$	$[e_1, e_2] = e_2,\ [e_1, e_3] = -e_3$
$\mathfrak{r}'_{3,0} \oplus \mathbb{R}$	$[e_1, e_2] = -e_3,\ [e_1, e_3] = e_2$
$\mathfrak{n}_4$	$[e_1, e_2] = -e_4,\ [e_4, e_2] = e_3$

Table 4. The four-dimensional real unimodular quasi-Frobenius Lie algebras. In all cases the 2-cocycle can be taken as $\omega = e^1 \wedge e^4 + e^2 \wedge e^3$, where e^i denotes the dual basis of $\mathfrak{f}^*$.

Contracting this equation with r^{jk} we get $(r^{-1})_{il} f^i{}_{jk} r^{jk} = -2 f^i{}_{il}$, which together with the unimodularity condition for the R-matrix written as (1.10), i.e. $f^i{}_{jk} r^{jk} = 0$, implies

$$f^i{}_{il} = 0 \quad \Leftrightarrow \quad \mathrm{tr}(ad_x) = 0 \quad \forall x \in \mathfrak{f}\,. \tag{5.6}$$

Therefore $\mathfrak{f}$ is a *unimodular* Lie algebra. Clearly the converse is also true and we have established that *solutions of the CYBE for a Lie algebra $\mathfrak{g}$ which satisfy the condition (1.8) are in one-to-one correspondence with unimodular quasi-Frobenius subalgebras of $\mathfrak{g}$.*

For this reason we refer also to the R-matrices which satisfy (1.8) as unimodular. A quasi-Frobenius Lie algebra must clearly have even dimension, and if the dimension is two the algebra must be abelian to respect unimodularity. To find a non-abelian R-matrix we must therefore consider at least the case of rank four. Luckily the real quasi-Frobenius Lie algebras of dimension four were classified in [59], and the five unimodular ones (Corollary 2.5 in [59]) are listed in table 4. The task of finding all R-matrices of rank four which solve the CYBE and lead to a deformation of the $AdS_5 \times S^5$ string with a proper supergravity background is therefore reduced to finding all inequivalent embeddings of these subalgebras in $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6)$. The most interesting problem is to find the embedding of the non-abelian algebras[29] in $\mathfrak{so}(2,4)$. This is still quite challenging, but it becomes simpler by the following observation. A unimodular quasi-Frobenius Lie algebra is solvable [58], and solvable subalgebras of $\mathfrak{so}(2,4)$ must be embeddable in one of the maximal solvable subalgebras of $\mathfrak{so}(2,4)$, see [60] for a proof of this. Besides the Cartan subalgebra which is not relevant for our purposes, Patera, Winternitz and Zassenhaus in [61] showed that there are two maximal solvable subalgebras of $\mathfrak{so}(2,4)$, $\mathfrak{s}_1$ and $\mathfrak{s}_2$ of dimension 9 and 8 respectively. It is most convenient to write them using the conformal form of the $\mathfrak{so}(2,4)$ algebra, with dilatation generator D, translations and special conformal generators p_i, k_i $(i = 0, \ldots 3)$ and Lorentz transformations and rotations J_{ij}. They are related to the form of $\mathfrak{so}(2,4)$ in (A.1) with $\widehat{\mathcal{K}}_{ij}{}^{kl} = -2\delta^k_{[i}\delta^l_{j]}$ by

$$p_i = P_i + J_{i4}\,, \qquad k_i = -P_i + J_{i4}\,, \qquad D = P_4\,, \tag{5.7}$$

[29]The extension to $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6)$ is essentially trivial and amounts to adding in commuting generators from $\mathfrak{so}(6)$ in such a way that the commutation relations of the algebra are preserved.

and the non-vanishing commutators are

$$[D,p_i]=p_i\,,\qquad [D,k_i]=-k_i\,,\qquad [p_i,k_j]=-2\eta_{ij}D+2J_{ij}\,, \tag{5.8}$$
$$[J_{ij},p_k]=2\eta_{k[i}p_{j]}\,,\quad [J_{ij},k_k]=2\eta_{k[i}k_{j]}\,,\quad [J_{ij},J_{kl}]=\eta_{ik}J_{jl}-\eta_{jk}J_{il}-\eta_{il}J_{jk}+\eta_{jl}J_{ik}\,.$$

The metric on the Lie algebra is given by $\mathrm{tr}(DD)=1$, $\mathrm{tr}(p_ik_j)=-2\eta_{ij}$, $\mathrm{tr}(J_{ij}J_{kl})=-2\eta_{i[k}\eta_{l]j}$. The two non-abelian maximal solvable subalgebras of $\mathfrak{so}(2,4)$ then take the form

$$\begin{aligned}&\mathfrak{s}_1=\mathrm{span}(p_i,\,J_{01}-J_{13},\,J_{02}-J_{23},\,J_{03},\,J_{12},\,D)\,,\\&\mathfrak{s}_2=\mathrm{span}(p_0+p_3,\,p_1,\,p_2,\,J_{01}-J_{13},\,J_{02}-J_{23},\,J_{12},\,J_{03}-D,\,k_0+k_3+2p_3)\,,\end{aligned}\tag{5.9}$$

up to automorphisms. Our task is reduced to finding all embeddings of the non-abelian algebras in table 4 in $\mathfrak{s}_1$ and $\mathfrak{s}_2$. To simplify this problem further we will single out the element e_3 in this table[30] and use automorphisms generated by elements of $\mathfrak{s}_1$ ($\mathfrak{s}_2$) to simplify it as much as possible. Using this freedom we can bring e_3 to one of the following forms

$$\begin{aligned}\mathfrak{s}_1:\quad &(1)\quad e_3=p_1\,,\quad (2)\quad e_3=J_{02}-J_{23}\,,\quad (3)\quad e_3=p_1+J_{02}-J_{23}\,,\qquad (4)\quad e_3=p_0\,,\\&(5)\quad e_3=p_3\,,\quad (6)\quad e_3=p_0+p_3\,,\quad (7)\quad e_3=p_0-p_3+J_{01}-J_{13}\,,\end{aligned}\tag{5.10}$$
$$\mathfrak{s}_2:\quad (1)\quad e_3=p_1\,,\quad (2)\quad e_3=p_0+p_3\,,\quad (3)\quad e_3=ap_1+bp_2+J_{01}-J_{13}\,.\tag{5.11}$$

The rest is a straightforward if slightly tedious calculation. The results are summarized in tables 5–8. Note that in writing these embeddings we have used automorphisms of the four-dimensional subalgebras which are not always inner automorphisms of $\mathfrak{so}(2,4)$. This must be accounted for when constructing the list of inequivalent R-matrices. In table 1 in the introduction we write the corresponding R-matrices, $R=e_1\wedge e_4+e_2\wedge e_3$ up to automorphisms. In table 2 instead we list the inequivalent, modulo inner automorphisms of $\mathfrak{so}(2,4)$, R-matrices. This is the result which is interesting from the string sigma model perspective, since inner automorphisms correspond to field redefinitions in the sigma model, i.e. coordinate transformations in target space. In table 3 we write down the bosonic isometries and the number of supercharges that each R-matrix preserves. Given a generator t of the superalgebra $\mathfrak{g}$, the condition that it is preserved by the R-matrix is given by

$$[t,R(x)]=R([t,x])\,,\qquad \forall x\in\mathfrak{g}\,.\tag{5.12}$$

Most of these R-matrices all have a form which suggests that they should correspond to non-commuting TsT-transformations,[31] in the sense that they involve sequences of T-dualities along non-commuting directions. All but the last three R-matrices in table 1 have the form

$$R=a\wedge b+c\wedge d\,,\tag{5.13}$$

[30] The reason for picking e_3 is that it always arises as a commutator of two other elements. Since the last three generators in $\mathfrak{s}_1$ or $\mathfrak{s}_2$ are never generated in commutators, they do not appear in e_3.

[31] Here we use TsT in a generalized sense, where we can involve also non-compact directions.

where $[a, b] = [c, d] = 0$ and c, d generate isometries of the corresponding background. It is natural to conjecture that such R-matrices correspond to two successive TsT-transformations, the first using isometries a, b and the second using isometries c, d. Note that unlike in standard applications of TsT-tranformations, e.g. [62], the pairs of isometries a, b and c, d do not commute with each other. This means that after the first TsT is implemented, it is necessary to make a change of coordinates in order to realize the isometries of the second TsT transformation as shift isometries. We will confirm this in section 6, when we will check in some examples that the deformed backgrounds are indeed equivalent to such sequences of TsT-transformations. These considerations suggest a very simple picture for how TsT-transformations are interpreted at the level of the R-matrix: the TsT-transformation involving isometries a, b should be simply implemented by adding a term $a \wedge b$ to the R-matrix.[32] Notice that the number of free parameters entering the definitions of the R-matrices (plus the overall deformation parameter) does not need to be equal to the number of TsT-transformations implemented. In fact, the number of parameters could be reduced in some cases, if they can be reabsorbed by means of field redefinitions. In other cases one might have more parameters than expected, which suggests the possibility of applying TsT-transformations on linear combinations of the isometric coordinates.

The structure of the last three R-matrices in table 1 is different, and one observes that now a, c generate isometries. However, one can check explicitly that the background corresponding to R_{15}, for example, is self-dual (up to field redefinitions) under a TsT-transformation involving a, c.[33] This example is particularly instructive because it can be embedded in $\mathfrak{so}(2, 3)$: in this algebra, the deformed background does not preserve other bosonic isometries than a, c, which suggests that backgrounds corresponding to the algebra $\mathfrak{n}_4$ are not of TsT-type. Note that $\mathfrak{n}_4$ is the only algebra considered which is not the direct sum of a three-dimensional algebra and a commuting generator. One possibility is that *non-abelian* T-duality of the corresponding subalgebra should instead play a role in the interpretation of these backgrounds. A hint towards this direction comes from the results of [63], where it was shown that a conformal anomaly is encountered when implementing non-abelian T-duality on a subalgebra, unless all generators have vanishing trace.[34] In the case of the adjoint representation this condition is precisely that of unimodularity of the corresponding subalgebra.

Let us now consider the case of higher ranks, which can only be six or eight. We have not done a systematic study for the case of rank six R-matrices. One would first need to identify all 6-dimensional subalgebras of $\mathfrak{s}_1$ and $\mathfrak{s}_2$, and check which of them are unimodular and quasi-Frobenius. We have found that the subalgebra of $\mathfrak{s}_1$ generated by $\{p_i, J_{03}, J_{12}\}$ has both properties. It is straightforward to find the 2-form ω that solves the cocycle condition (5.4), and invert it to find the corresponding R-matrix. For particular choices of the free parameters this can be written e.g. as $R = p_0 \wedge p_1 + p_2 \wedge p_3 + J_{01} \wedge J_{23}$.

[32] It is easy to check that this is compatible with the CYBE, since a, b are isometries and satisfy (5.12).

[33] Note that this is consistent with our above proposal on how to interpret the action of TsT at the level of the R-matrix; in fact, in this case the addition of the term $a \wedge c$ to R_{15} can be removed by an inner automorphism of $\mathfrak{so}(2, 4)$. Here a, c can be chosen to be $p_1, p_0 + p_3$.

[34] We thank Arkady Tseytlin for pointing this reference out to us.

JHEP10(2016)045

$\mathfrak{h}_3 \oplus \mathbb{R}$	e_1	e_2	e_3	e_4
1.	p_1	$J_{01} - J_{13}$	$p_0 + p_3$	p_2
2.	p_1	$p_3 + J_{01} - J_{13}$	$p_0 + p_3$	p_2
3.	p_1	$p_2 + J_{01} - J_{13}$	$p_0 + p_3$	$p_1 + J_{02} - J_{23}$
4.	$\frac{1}{2}p_1 - \frac{1}{2}(J_{02} - J_{23})$	$p_2 + J_{01} - J_{13}$	$p_0 + p_3$	$k_0 + k_3 + 2p_3 - 2J_{12}$

Table 5. Embeddings of $\mathfrak{h}_3 \oplus \mathbb{R}$ in $\mathfrak{so}(2,4)$ up to automorphism.

$\mathfrak{r}_{3,-1} \oplus \mathbb{R}$	e_1	e_2	e_3	e_4
1.	$-D - J_{03}$	$J_{02} - J_{23}$	p_1	$p_0 + p_3$
2.	J_{03}	$p_0 - p_3$	$p_0 + p_3$	p_1
3.	J_{03}	$p_0 - p_3$	$p_0 + p_3$	J_{12}
(4.)	$D + 2J_{03}$	p_1	$p_0 + p_3$	$-$

Table 6. Embeddings of $\mathfrak{r}_{3,-1} \oplus \mathbb{R}$ in $\mathfrak{so}(2,4)$ up to automorphism. The last case is an embedding of $\mathfrak{r}_{3,-1}$ which does not extend to an embedding of $\mathfrak{r}_{3,-1} \oplus \mathbb{R}$. It is the only case where this happens and included only since it is relevant for constructing all non-abelian R-matrices of $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6)$.

$\mathfrak{r}'_{3,0} \oplus \mathbb{R}$	e_1	e_2	e_3	e_4
1.	J_{12}	p_2	p_1	$p_0 + p_3$
2.	$p_3 + J_{12}$	p_2	p_1	$p_0 + p_3$
3.	$p_0 + J_{12}$	p_2	p_1	p_3
4.	J_{12}	p_2	p_1	p_3
5.	$p_3 + J_{12}$	p_2	p_1	p_0
6.	J_{12}	p_2	p_1	p_0
7.	J_{12}	p_2	p_1	J_{03}

Table 7. Embeddings of $\mathfrak{r}'_{3,0} \oplus \mathbb{R}$ in $\mathfrak{so}(2,4)$ up to automorphism.

$\mathfrak{n}_4$	e_1	e_2	e_3	e_4
1.	p_3	$J_{01} - J_{13}$	$p_0 + p_3$	p_1
2.	p_3	$p_2 + J_{01} - J_{13}$	$p_0 + p_3$	p_1
3.	$p_1 + p_3 + J_{02} - J_{23}$	$p_2 + J_{01} - J_{13}$	$p_0 + p_3$	p_1

Table 8. Embeddings of $\mathfrak{n}_4$ in $\mathfrak{so}(2,4)$ up to automorphism.

JHEP10(2016)045

We have also checked that there is no 8-dimensional subalgebra which is at the same time unimodular and quasi-Frobenius. Therefore there is no rank eight R-matrix which produces a background that solves the supergravity equations of motion. It is in fact easy to check that $\mathfrak{s}_2$ (which is 8-dimensional) is quasi-Frobenius but not unimodular. To identify all 8-dimensional subalgebras of $\mathfrak{s}_1$ (which is 9-dimensional), we first define $e = \sum_{j=1}^{9} \lambda_j e_j$ to be the generator which we want to remove, where e_j are the generators of $\mathfrak{s}_1$. Then for a generic element $X \in \mathfrak{s}_1$ we define its component perpendicular to e as $X^\perp = X - P(X)$, where P projects[35] along e. Then the condition to have a subalgebra is $P[X^\perp, Y^\perp] = 0,\ \forall X, Y \in \mathfrak{s}_1$. These equations give two possible solutions, depending on some unconstrained parameters

$$
\begin{aligned}
&(a) \qquad e = \lambda_7 J_{12} + \lambda_8 J_{03} + \lambda_9 D\,,\\
&(b) \qquad e = \lambda_1 (p_0 - p_3) + \lambda_8 (J_{03} - D)\,.
\end{aligned} \tag{5.14}
$$

In the case (a) we find[36] that the subalgebra is unimodular if $\lambda_7 = 0$ and $\lambda_9 = 2\lambda_8$. However, for this choice it is not quasi-Frobenius — the cocycle condition gives a 2-form of rank six. In the case (b) the subalgebra is not unimodular for any choice of $\lambda_1,\ \lambda_8$.

6 Some examples of supergravity backgrounds

In this section we give a brief discussion on the η-model backgrounds generated by solutions of the CYBE ($c = 0$), when we restrict R to act only on the bosonic subalgebra. In most cases a convenient parameterisation of the group element $g = g_a \cdot g_s \in \mathrm{SO}(2,4) \times \mathrm{SO}(6)$ is

$$
g_a = \exp\left(x^i p_i\right) \cdot \exp\left(\log z\, D\right)\,, \tag{6.1}
$$

where p_i, D are the generators defined in (5.7). Here we will be interested only on deformations of AdS, so we will not need to specify the parameterisation that we use for g_s on the sphere. In this coordinate system the undeformed metric takes the familiar form

$$
ds^2_{\eta=0} = \frac{\eta_{ij} dx^i\, dx^j + dz^2}{z^2} + ds^2_s\,. \tag{6.2}
$$

Because of our restriction on R, it is enough to look at the action of the operators $\mathcal{O}_\pm$ on the bosonic subalgebra. They take a block form

$$
\left(\begin{array}{c|c} \mathbf{1} & (\mathcal{O}_\pm)^{bc}{}_a \\ \hline \mathbf{0} & (\mathcal{O}_\pm)^b{}_a \end{array}\right)\,, \tag{6.3}
$$

[35] We define $P(X) = e\,\mathrm{STr}(Xe^*)$, where e^* is a dual to e, $\mathrm{STr}(ee^*) = 1$. We can take it as $e^* = \sum_{j=1}^{9} \frac{\lambda_j}{||\lambda||^2} e^j$, where $||\lambda||^2 = \sum_{j=1}^{9} \lambda_j^2$ and e^j are the duals of the generators in the basis such that $\mathrm{STr}(e_i e^j) = \delta_i^j$.

[36] In both cases (a) and (b) one needs to choose carefully a basis for the 8-dimensional subalgebra, in such a way that the generators are linearly independent and non-degenerate for generic choices of the remaining λ_j. A way to do it is to pick an orthogonal basis, and normalise the vectors such that they can be degenerate only if $\lambda_j = 0\ \forall j$.

JHEP10(2016)045

since[37] $\mathcal{O}_\pm P^{(0)} = P^{(0)}$. All the information about background fields of the deformed model can be extracted by studying just the block ${(\mathcal{O}_+)^b}_a$ — or in other words $P^{(2)}\mathcal{O}_+ P^{(2)}$. Notice that the results for ${(\mathcal{O}_-)^b}_a$ are simply obtained by changing the sign of the deformation parameter η. The dilaton of the deformed model is easily obtained by computing the determinant of ${(\mathcal{O}_+)^b}_a$

$$e^\phi = (\det \mathcal{O}_+)^{-1/2} . \tag{6.4}$$

The rest of the background fields are written in terms of ${(\mathcal{O}_+^{-1})^b}_a$ — the inverse of the block ${(\mathcal{O}_+)^b}_a$. The vielbein components for the deformed model are

$$E^a = {(\mathcal{O}_+^{-1})^a}_b e^b , \tag{6.5}$$

where e^a is the bosonic vielbein of the *undeformed* background, related to the Maurer-Cartan form as

$$g^{-1}dg = e^a P_a + \frac{1}{2}\omega^{ab} J_{ab} . \tag{6.6}$$

The spacetime metric of the deformed background is then straightforwardly obtained, $ds^2 = \eta_{ab} E^a E^b$. The B-field can be extracted immediately from the action of the bosonic σ-model, and it reads as

$$B = \frac{1}{2} dX^n \wedge dX^m B_{mn} = \frac{1}{2}(\mathcal{O}_-^{-1})_{ab}\, e^a \wedge e^b , \tag{6.7}$$

where it is assumed that indices are raised and lowered with η_{ab}. To get the Ramond-Ramond fields we first need to consider the local Lorentz transformation given by M in (2.10) and write its action on spinors

$$(\mathrm{Ad}_h)^{\hat\beta}{}_{\hat\alpha} = \exp\left[-\frac{1}{4}(\log M)_{ab}\Gamma^{ab}\right]^{\hat\beta}{}_{\hat\alpha} , \tag{6.8}$$

where here we have introduced a basis for 32×32 Gamma-matrices.[38] The RR fields are obtained by solving the equation (note that (1.2) simplifies considerably for R-matrices of the bosonic subalgebra)

$$\left(\Gamma^a F_a + \frac{1}{3!}\Gamma^{abc}F_{abc} + \frac{1}{2\cdot 5!}\Gamma^{abcde}F_{abcde}\right)\Pi = e^{-\phi}\ \mathrm{Ad}_h(-4\Gamma_{01234})\Pi \tag{6.9}$$

where $\Pi = \frac{1}{2}(1-\Gamma_{11})$ is a projector and $(-4\Gamma_{01234})\Pi$ encodes the 5-form flux of the undeformed model. The various components of F's are found by multiplying the above equation by the relevant Gamma-matrix $\Gamma_{a_1\ldots a_{2m+1}}$ and then taking the trace. This computation[39] yields the F's expressed with tangent indices, which are translated into form language by $F^{(2m+1)} = \frac{1}{(2m+1)!}E^{a_{2m+1}} \wedge \ldots \wedge E^{a_1} F_{a_1\ldots a_{2m+1}}$.

[37] We recall that $P^{(0)}$ and $P^{(2)}$ are projectors on the subspaces spanned by the generators J_{ab} and P_a respectively. A useful matrix realisation of the algebra generators can be found in [22]. Here we identify $P_a = \mathbf{P}_a$, and $J_{ab} = -\mathbf{J}_{ab}$, where $\mathbf{P}_a$, $\mathbf{J}_{ab}$ are the generators used in [22].

[38] For a convenient basis see [22].

[39] For $F^{(5)}$ it is enough to look at half of the components, e.g. F_{0bcde}, and construct the corresponding form $f^{(5)}$. Then $F^{(5)} = (1+*)f^{(5)}$, such that $F^{(5)} = *F^{(5)}$.

JHEP10(2016)045

In the rest of this section we present some backgrounds solving the standard supergravity equations which we have derived by using the above procedure. We work out one example for each of the 4-dimensional non-abelian subalgebras in table 4.

In section 5 we have argued that the R-matrices related to the subalgebras $\mathfrak{h}_3 \oplus \mathbb{R}$, $\mathfrak{r}'_{3,0} \oplus \mathbb{R}$, $\mathfrak{r}_{3,-1} \oplus \mathbb{R}$ should produce backgrounds which can be obtained by sequences of TsT-transformations starting from $AdS_5 \times S^5$. We check this explicitly for the backgrounds that we have derived, where we follow the conventions of [32] for the T-duality rules [64–66]. Because the isometries of the first TsT do not commute with those of the second one, we will see that before doing the last step it is necessary to implement a coordinate transformation, which realizes the second pair of isometries as shifts of the corresponding coordinates. Let us mention that since we have chosen to have just one overall deformation parameter η (i.e. we fix some free parameters in the definitions of the possible R-matrices), the shifts of the two TsT-transformations are related to each other. This does not need to be true for generic cases.

6.1 $\mathfrak{h}_3 \oplus \mathbb{R}$

Let us choose the R-matrix (this corresponds to R_1 in table 1 with $x^1 \leftrightarrow x^3$)

$$R = (J_{03} + J_{13}) \wedge (p_0 + p_1) + p_2 \wedge p_3 \,, \tag{6.10}$$

which preserves 4 bosonic isometries

$$p_2 \,, \quad p_3 \,, \quad p_0 + p_1 \,, \quad p_0 - p_1 - 2(J_{02} + J_{12}) \,, \tag{6.11}$$

and 8 supercharges. Clearly, it is convenient to introduce lightcone coordinates $x^{\pm} = x^0 \pm x^1$, since a shift of x^+ will correspond to an isometry. The spacetime metric that we obtain is

$$\begin{aligned} ds^2 &= z^{-2}\left(1 + \frac{4\eta^2}{z^4}\right)^{-1} \left(4\eta^2 z^{-4} x^- dx^- (2dx_2 - x^- dx^-) + dx_2{}^2 + dx_3{}^2\right) \\ &\quad + \frac{-dx^- dx^+ + dz^2}{z^2} + ds_s^2. \end{aligned} \tag{6.12}$$

The dilaton depends only on the z-coordinate, while the B-field also on x^-

$$e^{\phi} = \left(1 + \frac{4\eta^2}{z^4}\right)^{-1/2}, \qquad B = \frac{2\eta(dx_2 - x^- dx^-) \wedge dx_3}{(4\eta^2 + z^4)}. \tag{6.13}$$

The RR-fluxes turn out to be quite simple

$$F^{(5)} = (1+*)\frac{2dx^- \wedge dx^+ \wedge dx_2 \wedge dx_3 \wedge dz}{z(z^4 + 4\eta^2)}, \quad F^{(3)} = \frac{4\eta}{z^5}(2x^- dx_2 - dx^+) \wedge dx^- \wedge dz. \tag{6.14}$$

In order to show that this background can be obtained by a sequence of TsT-transformations, we start from the deformed background and show that we can reach the undeformed $AdS_5 \times S^5$ by TsT-transformations. We will write $T(x_i)$ to indicate that we apply T-duality along the isometric coordinate x_i, and denote by $\tilde{x}_i$ the dual coordinate. In this case we need to do the sequence

$$T(x_2),\ x_3 \to x_3 - 2\eta \tilde{x}_2,\ T(\tilde{x}_2), \qquad T(\psi),\ w^+ \to w^+ - 2\eta\tilde{\psi},\ T(\tilde{\psi}), \tag{6.15}$$

where we need to redefine the coordinates in the 013 space

$$x^{+} = 2(\psi^2 w^{-} + w^{+}), \quad x^{-} = 2w^{-}, \quad x_3 = -2\psi w^{-}, \tag{6.16}$$

before applying the last TsT-transformation. Obviously, starting from $AdS_5 \times S^5$ and applying these TsT-transformations backwards, we find the deformed background presented here.

6.2 $\mathfrak{r}'_{3,0} \oplus \mathbb{R}$

In this case we can choose an R-matrix which involves generators along spacelike directions (R_{11} in table 1)

$$R = J_{12} \wedge p_3 + p_2 \wedge p_1 \,. \tag{6.17}$$

It preserves 3 bosonic isometries

$$J_{12}, \qquad p_0, \qquad p_3, \tag{6.18}$$

and no supercharges. It is more convenient to use the parameterisation

$$g_a = \exp(\xi J_{12}) \cdot \exp(r p_1 + x^0 p_0 + x^3 p_3) \cdot \exp(\log z\, D), \tag{6.19}$$

since ξ will be isometric. In the undeformed case

$$ds^2_{\eta=0} = \frac{-(dx^0)^2 + r^2 d\xi^2 + dr^2 + dx_3{}^2 + dz^2}{z^2} + ds^2_s, \tag{6.20}$$

so that (r, ξ) are a radial and an angular coordinate in the $1,2$ plane. Turning on the deformation parameter we find

$$\begin{aligned} ds^2 = {}& z^{-6}\left(1+\frac{4\eta^2\left(r^2+1\right)}{z^4}\right)^{-1}\left[dr^2\left(4\eta^2 r^2+z^4\right)+r^2 z^4 d\xi^2-8\eta^2 r\, dr dx_3+dx_3{}^2\left(4\eta^2+z^4\right)\right] \\ &+\frac{dz^2-(dx^0)^2}{z^2}+ds^2_s \end{aligned} \tag{6.21}$$

The dilaton and the B-field now depend on r and z

$$e^{\phi} = \left(1+\frac{4\eta^2\left(r^2+1\right)}{z^4}\right)^{-1/2}, \qquad B = \frac{2\eta\, r\, d\xi \wedge (dr + r dx_3)}{z^4 + 4\eta^2\left(r^2+1\right)}. \tag{6.22}$$

For the RR-fluxes we find

$$F^{(5)} = (1+*)\frac{4r\ dx^0 \wedge dr \wedge d\xi \wedge dx_3 \wedge dz}{z\left(z^4 + 4\eta^2\left(r^2+1\right)\right)}, \qquad F^{(3)} = \frac{8\eta}{z^5}(dx_3 - r dr) \wedge dx^0 \wedge dz. \tag{6.23}$$

The sequence of TsT-transformations

$$T(x_3),\ \xi \to \xi + 2\eta\tilde{x}_3,\ T(\tilde{x}_3), \qquad T(x_1),\ x_2 \to x_2 - 2\eta\tilde{x}_1,\ T(\tilde{x}_1), \tag{6.24}$$

(where $r = \sqrt{x_1^2 + x_2^2},\ \xi = \arctan(x_1/x_2)$) yields undeformed $AdS_5 \times S^5$.

6.3 $\mathfrak{r}_{3,-1} \oplus \mathbb{R}$

The R-matrix (R_6 in table 1 with $x^1 \to x^2$, $x^3 \to x^1$)

$$R = J_{01} \wedge p_2 + 2p_0 \wedge p_1 \,, \tag{6.25}$$

preserves 3 bosonic isometries

$$J_{01}, \qquad p_2, \qquad p_3 \,, \tag{6.26}$$

and no supercharges. As before, it is more convenient to parameterise the group element in a different way

$$g_a = \exp(tJ_{01}) \cdot \exp(\rho p_1 + x^2 p_2 + x^3 p_3) \cdot \exp(\log z\, D), \tag{6.27}$$

so that t is an isometry. In the undeformed case we have the spacetime metric

$$ds^2_{\eta=0} = \frac{-\rho^2 dt^2 + d\rho^2 + dx_2{}^2 + dx_3{}^2 + dz^2}{z^2} + ds^2_s, \tag{6.28}$$

while the defomation gives

$$\begin{aligned} ds^2 = z^{-6}\left(1-\frac{4\eta^2\left(\rho^2+4\right)}{z^4}\right)^{-1}&\left(-\rho^2 z^4 dt^2 - 16\eta^2 \rho d\rho dx_2 + dx_2{}^2\left(z^4-16\eta^2\right) + d\rho^2\left(z^4-4\eta^2\rho^2\right)\right) \\ &+ \frac{dx_3{}^2}{z^2} + \frac{dz^2}{z^2} + ds^2_s \,. \end{aligned} \tag{6.29}$$

The dilaton and the B-field depend on ρ and z

$$e^\phi = \left(1 - \frac{4\eta^2(4+\rho^2)}{z^4}\right)^{-1/2}, \qquad B = \frac{2\eta\, \rho\, dt \wedge (2d\rho - \rho dx_2)}{z^4 - 4\eta^2\left(4+\rho^2\right)}\,, \tag{6.30}$$

and the RR-fluxes are

$$F^{(5)} = -(1+*)\frac{4\rho\, dt \wedge d\rho \wedge dx_2 \wedge dx_3 \wedge dz}{z\left(z^4 - 4\eta^2\left(4+\rho^2\right)\right)}\,, \qquad F^{(3)} = \frac{8\eta(2dx_2 + \rho d\rho) \wedge dx_3 \wedge dz}{z^5}\,. \tag{6.31}$$

We can get back the undeformed $AdS_5 \times S^5$ background by applying the sequence of TsT-transformations

$$T(x_2),\ t \to t + 2\eta\tilde{x}_2,\ T(\tilde{x}_2), \qquad T(x_1),\ x_0 \to x_0 - 4\eta\tilde{x}_1,\ T(\tilde{x}_1), \tag{6.32}$$

where $x_1 = \rho\cosh t,\ x_0 = \rho \sinh t$.

6.4 $\mathfrak{n}_4$

Let us consider the R-matrix (R_{15} in table 1 with $x^1 \leftrightarrow x^3$)

$$R = p_1 \wedge p_3 + (p_0 + p_1) \wedge (J_{03} + J_{13}) \tag{6.33}$$

which preserves the 3 bosonic isometries

$$p_0 + p_1\,, \qquad p_2\,, \qquad p_3\,, \tag{6.34}$$

and 8 supercharges. The metric is given by

$$ds^2 = z^{-6}\left(1-\frac{4\eta^2\xi_-}{z^4}\right)^{-1}\left[z^4 dx_3{}^2 - \eta^2(dx^+)^2 - \frac{1}{4}d\xi_-\left(\eta^2\xi_-^2 d\xi_- + 2dx^+\left(z^4 - 2\eta^2\xi_-\right)\right)\right] + \frac{dx_2{}^2 + dz^2}{z^2} + ds_s^2\,, \tag{6.35}$$

where we preferred to redefine $\xi_- = 2x^- - 1$. The dilaton and the B-field depend on ξ_- and z

$$e^\phi = \left(1 - \frac{4\eta^2\xi_-}{z^4}\right)^{-1/2}, \qquad B = \frac{\eta(\xi_- d\xi_- + 2dx^+)\wedge dx_3}{2\left(z^4 - 4\eta^2\xi_-\right)}. \tag{6.36}$$

The RR-fluxes are

$$F^{(5)} = (1+*)\frac{d\xi_- \wedge dx^+ \wedge dx_2 \wedge dx_3 \wedge dz}{z(z^4 - 4\eta^2\xi_-)}, \quad F^{(3)} = \frac{2\eta}{z^5}\left(\xi_- d\xi_- - 2dx^+\right)\wedge dx_2 \wedge dz. \tag{6.37}$$

We have checked that this background is self-dual (after field redefinitions) under a TsT-transformation involving $p_0 + p_1$ and p_3. If we view it as a deformation of AdS_4 there are no other bosonic isometries at our disposal, so it appears that this background cannot be generated by (bosonic) TsT-transformations. As remarked earlier, it would be very interesting to understand if it can be generated by applying non-abelian T-duality.

7 Conclusions

We have derived the target space geometry of the η and λ-deformed type IIB supercoset string sigma models. With this result we have checked that the λ-deformation leads to a (type II*) supergravity background, while in general the η-deformation only to a "generalized" one in the sense of [25, 30]. When this is the case, the sigma model is expected to be scale invariant but not Weyl invariant, and therefore does not seem to define a consistent string theory. We have identified the (necessary and sufficient) condition for the η-model to have a *standard* supergravity background as target space. This is translated into an algebraic condition on the R-matrix, which we refer to as the *unimodularity condition*. It imposes strong restrictions on non-abelian R-matrices, and in fact all non-abelian R-matrices considered in previous works do not lead to supergravity solutions.

We have also analyzed the problem of finding all unimodular R-matrices which solve the CYBE for the bosonic subalgebra $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6) \subset \mathfrak{psu}(2,2|4)$. The complete list of rank four non-abelian R-matrices for $\mathfrak{so}(2,4)$ has been given and we have showed that the only other non-abelian R-matrices in this case have rank six. We have argued that most of these examples should correspond to a sequence of non-commuting TsT-transformations and have verified this explicitly in some cases. It should be possible to understand these deformations in terms of twisted boundary conditions for the string just as in the standard TsT case [44]. There are many similarities between the backgrounds we construct and that of Hashimoto-Itzhaki/Maldacena-Russo [67, 68] and the dual field theories are expected to be certain non-commutative deformations of $\mathcal{N} = 4$ super Yang-Mills, see [69] and in particular [70].

Many interesting open questions remain. It would be important to find all possible unimodular R-matrices of $\mathfrak{psu}(2,2|4)$ to have a complete list of Yang-Baxter deformations of $AdS_5 \times S^5$ with a string theory interpretation. A question is whether any of them are of the Jordanian type. It is particularly interesting to investigate whether it is possible to have unimodular R-matrices that solve the MCYBE rather than the CYBE, to solve one of the puzzles of [22]. One could also try to give an interpretation to backgrounds generated by non-unimodular R-matrices; in many cases one can associate to them a formally T-dual model which does describe a string sigma model, so it is natural to wonder what these backgrounds correspond to. See [41] for some investigations along these lines. It would be also interesting to clarify if these deformed models have a connection to non-abelian T-duality, in view of the similarities between our unimodularity condition and the tracelessness condition of [63].

Our results are also useful to make further progress in the case of the λ-model. In fact, we have written the NSNS and RR background fields in terms of the Lie algebra operators which are used to define the deformation procedure, and after picking a certain parameterisation for the group element this enables to obtain their explicit form. This method is more efficient, albeit equivalent, to the ones used so far e.g. in [22, 26, 41]. One could then check the proposal of [27] for the background of the λ-deformed $AdS_3 \times S^3 \times T^4$ string, and finally derive the one for the $AdS_5 \times S^5$ case. It would be interesting to understand whether there is room to modify the definition of the λ-model, hence realising other possible deformations of the string. In fact, in the current status the λ-model is related through Poisson-Lie T-duality to the η-model based on the MCYBE, but there is no known counterpart for deformations based on the CYBE.

Acknowledgments

We would like to thank Arkady Tseytlin for useful discussions and helpful comments on a first draft of this manuscript. This work was supported by the ERC Advanced grant No. 290456.

A $\mathbb{Z}_4$-graded superisometry algebras

In this appendix we review some facts about the relevant superalgebras and explain our notation and conventions. In [4] it was shown that for all cases of interest here[40] the superisometry algebra — which admits a $\mathbb{Z}_4$-grading that extends the $\mathbb{Z}_2$-grading of the bosonic subalgebra — can be written in the same form. The bosonic subalgebra is of the standard symmetric space form

$$[J_{ab}, P_c] = 2\eta_{c[a}P_{b]}\,, \quad [P_a, P_b] = \frac{1}{2}\widehat{\mathcal{K}}_{ab}{}^{cd}J_{cd}\,,$$
$$[J_{ab}, J_{cd}] = \eta_{ac}J_{bd} - \eta_{bc}J_{ad} - \eta_{ad}J_{bc} + \eta_{bd}J_{ac}\,. \tag{A.1}$$

Here $a, b, c = 0, \ldots, 9$ and J_{ab} generate Lorentz-transformations and rotations while P_a generate translations. Note that since the space is typically a product of factors J_{ab} is

[40] We restrict our attention to models with only RR flux since these have certain simplifying features like $\mathbb{Z}_4$-symmetry.

block-diagonal with components mixing different factors absent and this should be taken into account in interpreting the last commutator above. In the case of RR backgrounds, i.e. no NSNS three-form flux, the commutators involving the supercharges take the form (here and in the rest of the paper we specialize to the type IIB case, but the type IIA case works in the same way)

$$[P_a, Q^I_{\hat\alpha}] = -i(Q^J \widehat{\mathcal{K}}^{JI}\gamma_a)_{\hat\alpha}\,, \qquad [J_{ab}, Q^I_{\hat\alpha}] = -\frac{1}{2}(Q^I\gamma_{ab})_{\hat\alpha}\,, \qquad (I,J=1,2)$$
$$\{Q^1_{\hat\alpha}, Q^1_{\hat\beta}\} = \{Q^2_{\hat\alpha}, Q^2_{\hat\beta}\} = i\gamma^a_{\hat\alpha\hat\beta}\,P_a\,, \qquad \{Q^1_{\hat\alpha}, Q^2_{\hat\beta}\} = (\gamma^a\widehat{\mathcal{K}}^{12}\gamma^b)_{\hat\alpha\hat\beta}\,J_{ab}\,. \tag{A.2}$$

Here $\hat\alpha = 1,\ldots,N$ where $2N$ is the number of supersymmetries preserved by the background. For $AdS_5\times S^5$ ($\mathfrak{psu}(2,2|4)$) $N=16$ and $\gamma^a_{\hat\alpha\hat\beta}$ are the standard 16×16 symmetric Weyl blocks or 'chiral gamma-matrices' (see for example the appendix of [36]). For $AdS_3\times S^3\times T^4$ ($\mathfrak{psu}(1,1|2)^2$) $N=8$ and for $AdS_2\times S^2\times T^6$ ($\mathfrak{psu}(1,1|2)$) $N=4$ and the gamma-matrices $\gamma^a_{\hat\alpha\hat\beta}$ involve an extra projector to make them 8×8 and 4×4 respectively. The $\mathbb{Z}_4$ automorphism acts as

$$J_{ab}\to J_{ab}\,, \qquad P_a\to -P_a\,, \qquad Q^1\to iQ^1\,, \qquad Q^2\to -iQ^2\,. \tag{A.3}$$

We introduce projectors that split the generators $T_A = \{P_a,\, J_{ab},\, Q^I_{\hat\alpha}\}$ according to their $\mathbb{Z}_4$-grading as follows

$$P^{(0)}(T_A) = J_{ab}\,, \qquad P^{(1)}(T_A) = Q^1_{\hat\alpha}\,, \qquad P^{(2)}(T_A) = P_a\,, \qquad P^{(3)}(T_A) = Q^2_{\hat\alpha}\,. \tag{A.4}$$

Finally $\widehat{\mathcal{K}}^{AB}$ appearing on the right-hand-side in (A.1) and (A.2) is the inverse of the Lie algebra metric defined by the supertrace[41]

$$\mathrm{Str}(T_AT_B) = \mathcal{K}_{AB}\,, \qquad T_A = \{P_a,\, J_{ab},\, Q^I_{\hat\alpha}\}\,, \tag{A.5}$$

e.g.

$$\frac{1}{2}\widehat{\mathcal{K}}_{ab}{}^{ef}\mathcal{K}_{ef,cd} = 2\eta_{a[c}\eta_{d]b}\,. \tag{A.6}$$

It can be expressed in terms of the geometry and fluxes of the corresponding symmetric space supergravity background as

$$\widehat{\mathcal{K}}^{ab} = \eta^{ab}\,, \qquad \widehat{\mathcal{K}}_{ab}{}^{cd} = -\underline{R}_{ab}{}^{cd}\,, \qquad \widehat{\mathcal{K}}^{\hat\alpha I\hat\beta J} = \frac{i}{8}\underline{\mathcal{S}}^{\hat\alpha I\hat\beta J}\,, \tag{A.7}$$

where $\underline{R}_{ab}{}^{cd}$ and $\underline{\mathcal{S}}^{IJ}$ are the Riemann curvature and RR field strength bispinor respectively.[42] Let us also note the relation

$$\widehat{\mathcal{K}}_{ab}{}^{cd}(\mathcal{K}^{12}\gamma_{cd})_{\hat\alpha\hat\beta} = 8(\gamma_{[a}\widehat{\mathcal{K}}^{12}\gamma_{b]})_{\hat\alpha\hat\beta}\,. \tag{A.8}$$

[41]Note that our definition of $\mathcal{K}$ differs by a factor of i compared to the definition used in [4].

[42]The curvature of AdS is $R_{ab}{}^{cd} = 2\delta^c_{[a}\delta^d_{b]}$ while that of the sphere is $R_{ab}{}^{cd} = -2\delta^c_{[a}\delta^d_{b]}$ in our conventions. The RR flux takes the form

$$AdS_n\times S^n\times T^{10-2n}: \qquad \underline{\mathcal{S}}^{\hat\alpha I\hat\beta J} = -4i(\sigma^2)^{IJ}(\mathcal{P}\gamma^{01234})^{\hat\alpha\hat\beta}\,,$$

where the projector $\mathcal{P}$, with $Q^I = Q^I\mathcal{P}$, is given by 1 for $n=5$, $\frac{1}{2}(1+\gamma^{6789})$ for $n=3$ and $\frac{1}{2}(1+\gamma^{6789})\frac{1}{2}(1+\gamma^{4568})$ for $n=2$.

Finally for operators acting on the Lie algebra (i.e. endomorphisms) $\mathcal{M} : \mathfrak{g} \to \mathfrak{g}$ we define its components in the following way

$$\mathcal{M}(T_C) = T_D \mathcal{M}^D{}_C \,. \tag{A.9}$$

The transpose operator is defined with respect to the supertrace by

$$\mathrm{Str}(T_A \mathcal{M}(T_B)) = \mathrm{Str}(\mathcal{M}^T(T_A) T_B) \,, \tag{A.10}$$

or

$$\mathcal{M}_{AB} = (-1)^{AB} (\mathcal{M}^T)_{BA} \qquad \mathcal{M}_{AB} = \mathcal{K}_{AC} \mathcal{M}^C{}_B \tag{A.11}$$

e.g.

$$(\mathcal{M}^T)_{a\hat{\beta}1} = \mathcal{K}_{\hat{\beta}1\hat{\gamma}2} M^{\hat{\gamma}2}{}_a \,, \qquad (\mathcal{M}^T)_{a,bc} = \frac{1}{2} \mathcal{K}_{bc,de} \mathcal{M}^{de}{}_a \,. \tag{A.12}$$

The supertrace of the Lie algebra operator $\mathcal{M}$ is given by

$$\mathrm{Str}(\mathcal{M}) = (-1)^A \mathcal{M}^A{}_A = \widehat{\mathcal{K}}^{AB} \mathrm{Str}(T_A \mathcal{M} T_B) \,. \tag{A.13}$$

When we need to raise indices with $\widehat{\mathcal{K}}^{AB}$ we use the convention

$$\mathcal{M}^A = \mathcal{M}_B \widehat{\mathcal{K}}^{BA} \,. \tag{A.14}$$

To conclude, when writing generic commutation relations we write

$$[T_A, T_B] = f^C{}_{AB} T_C \,. \tag{A.15}$$

B Useful results for the deformed models

In this appendix we collect some useful identities and expressions to obtain the results presented in the main text. In the two deformed models, we can relate $\mathcal{O}_\pm^T$ and $\mathcal{O}_\pm$ by

$$\lambda - \textbf{model}: \qquad \mathcal{O}_-^T = \mathrm{Ad}_g^{-1} \mathcal{O}_+, \qquad \eta - \textbf{model}: \qquad \mathcal{O}_-^T \hat{d}^T = \hat{d}^T \mathcal{O}_+ \,. \tag{B.1}$$

Using the definitions of $\mathcal{O}_\pm$, we can express M defined in (2.10) in terms of $\mathcal{O}_\pm$ and projectors only

$$\begin{aligned}
\lambda - \textbf{model}: \qquad & M = -\Omega^T + (\mathcal{O}_+^T)^{-1}(1 - \Omega\Omega^T) = -\Omega^T + (1 - \lambda^{-4})(\mathcal{O}_+^T)^{-1} P^{(2)} \,, \\
\eta - \textbf{model}: \qquad & M = \mathcal{O}_-^{-1}(\mathcal{O}_- + 2\eta R_g \hat{d} P^{(2)}) = 1 - 2P^{(2)} + 2\mathcal{O}_-^{-1} P^{(2)} \,,
\end{aligned} \tag{B.2}$$

which is useful to prove

$$\begin{aligned}
\lambda - \textbf{model}: \qquad & \mathrm{Ad}_h^{-1} P^{(2)} = \mathcal{O}_+ (1 + \Omega(\mathcal{O}_+^T)^{-1}) P^{(2)} = P^{(2)} (1 + (\mathcal{O}_+^T)^{-1}\Omega) \mathcal{O}_+ \,, \\
\eta - \textbf{model}: \qquad & \mathrm{Ad}_h^{-1} P^{(2)} = \mathcal{O}_+ (2P^{(2)} - 1) \mathcal{O}_-^{-1} P^{(2)} \,.
\end{aligned} \tag{B.3}$$

Note that using the expression for M we can express A_- in terms of A_+ as

$$A_- = M A_+ = \begin{cases} A_+ + (M-1) A_+^{(2)} \\ -\Omega^T A_+ + (M + \lambda^{-2}) A_+^{(2)} \end{cases} . \tag{B.4}$$

The rest of this appendix is devoted to the two deformed models separately.

B.1 λ-model

The expressions for $dA_\pm$ in (3.1), (3.2) can be rewritten as

$$dA_+ = -\frac{1}{2}\{A_+, A_+\} - \frac{1}{2}(1-\lambda^{-4})\mathcal{O}_+^{-1}(\{A_+^{(2)}, A_+^{(2)}\} - \lambda^2\{A_+^{(1)}, A_+^{(1)}\} + 2\lambda\{A_+^{(2)}, A_+^{(3)}\}), \tag{B.5}$$

$$dA_- = \frac{1}{2}\{A_-, A_-\} + \frac{1}{2}(1-\lambda^{-4})(\mathcal{O}_+^T)^{-1}(\{A_-^{(2)}, A_-^{(2)}\} - \lambda^2\{A_-^{(3)}, A_-^{(3)}\} + 2\lambda\{A_-^{(2)}, A_-^{(1)}\}), \tag{B.6}$$

if we use

$$\Omega^T\{X, X\} - \{\Omega^T X, \Omega^T X\} = (1-\lambda^{-4})(\{X^{(2)}, X^{(2)}\} - \lambda^2\{X^{(1)}, X^{(1)}\} + 2\lambda\{X^{(2)}, X^{(3)}\}), \tag{B.7}$$

for $X \in \mathfrak{g}$, and the same for Ω but with $X^{(1)}$ and $X^{(3)}$ interchanged.

To calculate the component $T^{\hat{\alpha}1}$ of the torsion, we first need to compute the Lorentz-transformed spin-connection $\mathrm{Ad}_h A_+^{(0)} + dhh^{-1}$. We do this by taking the exterior derivative of both sides of the relation $E^{(2)} = \mathrm{Ad}_h A_-^{(2)}$, from which we find the equation

$$\begin{aligned}
0 &= \{\mathrm{Ad}_h A_+^{(0)} + dhh^{-1} - A_+^{(0)}, E^{(2)}\} + \lambda(1-\lambda^{-4})P^{(2)}\mathrm{Ad}_h(\mathcal{O}_+^T)^{-1}\mathrm{Ad}_h^{-1}\{E^{(2)}, E^{(1)}\} \\
&+ \{E^{(1)}, \mathrm{Ad}_h P^{(1)} M E^{(2)}\} - i\lambda\{E^{(3)}, P^{(3)} M E^{(2)}\} - i\lambda P^{(2)} M^T\{E^{(2)}, E^{(3)}\} \\
&- \{\mathrm{Ad}_h P^{(0)} M E^{(2)}, E^{(2)}\} - \frac{1}{2}\mathrm{Ad}_h\{P^{(1)} M E^{(2)}, P^{(1)} M E^{(2)}\} - \frac{1}{2}\lambda^2\{P^{(3)} M E^{(2)}, P^{(3)} M E^{(2)}\} \\
&- \frac{1}{2}(1-\lambda^{-4})P^{(2)}\mathrm{Ad}_h(\mathcal{O}_+^T)^{-1}\mathrm{Ad}_h^{-1}(\{E^{(2)}, E^{(2)}\} + 2\lambda\{E^{(2)}, \mathrm{Ad}_h P^{(1)} M E^{(2)}\}) \\
&- \frac{1}{2}P^{(2)} M^T\{E^{(2)}, E^{(2)}\} - \lambda^2 P^{(2)} M^T\{E^{(2)}, P^{(3)} M E^{(2)}\},
\end{aligned} \tag{B.8}$$

where we used (3.3) and (B.6). This equation determines $\mathrm{Ad}_h A_+^{(0)} + dhh^{-1}$ completely: this is obvious for the terms involving fermionic vielbeins, while for the terms involving E^a it follows from symmetry in the same way that the condition $T_{ab}{}^c = 0$ determines the spin connection $\Omega_{ab}{}^c$. Using the algebra (A.1), (A.2) as well as (B.3) the result is

$$[dhh^{-1} + \mathrm{Ad}_h A_+]_{ab} = -\Omega_{ab} + \frac{1}{2}E^c H_{abc} + 2i(E^1\gamma_{[a})_{\hat{\alpha}}(\mathrm{Ad}_h M)^{\hat{\alpha}1}{}_{b]}\,. \tag{B.9}$$

Here we have used the fact, which will be proven below, that the expression that we find

$$H_{abc} = 3[\mathrm{Ad}_h M]_{[ab,c]} + 3iM^{\hat{\alpha}1}{}_{[a}[\mathrm{Ad}_h]_{b|d|}\gamma^d_{\hat{\alpha}\hat{\beta}}M^{\hat{\beta}1}{}_{c]}\,, \tag{B.10}$$

is equivalent to the one in (3.8). In fact, if we calculate $H = dB$ using the first definition for B in (3.15) we find

$$
\begin{aligned}
H = dB &= \frac{1}{3}(1-\lambda^{-4})^{-1}\big(\mathrm{Str}(\Omega A_- \wedge \Omega A_- \wedge \Omega A_-) - \mathrm{Str}(A_- \wedge A_- \wedge A_-)\big) \\
&\quad - \frac{1}{2}(1-\lambda^{-4})^{-1}\mathrm{Str}(A_+ \wedge (\Omega\{A_-, A_-\} - \{\Omega A_-, \Omega A_-\})) \\
&= -\mathrm{Str}((A_+^{(0)} + A_-^{(0)}) \wedge A_-^{(2)} \wedge A_-^{(2)}) + \lambda^2 \mathrm{Str}(A_+^{(2)} \wedge A_-^{(3)} \wedge A_-^{(3)}) \\
&\quad - \frac{1}{2}\mathrm{Str}(A_-^{(2)} \wedge \{A_-^{(1)}, A_-^{(1)} + 2\lambda A_+^{(1)}\}) \\
&= \mathrm{Str}(E^{(2)} \wedge E^{(1)} \wedge E^{(1)}) - \mathrm{Str}(E^{(2)} \wedge E^{(3)} \wedge E^{(3)}) - \mathrm{Str}(P^{(0)}\mathrm{Ad}_h M E^{(2)} \wedge E^{(2)} \wedge E^{(2)}) \\
&\quad - \mathrm{Str}(E^{(2)} \wedge P^{(1)}\mathrm{Ad}_h M E^{(2)} \wedge P^{(1)}\mathrm{Ad}_h M E^{(2)}) \\
&= -\frac{i}{2}E^a\, E^1\gamma_a E^1 + \frac{i}{2}E^a\, E^2\gamma_a E^2 + \frac{1}{3!}E^c E^b E^a H_{abc}\,,
\end{aligned} \tag{B.11}
$$

with H_{abc} given by (B.10). On the other hand, if we start from B given in the second line of (3.15), we find a result which is mapped to the previous one by the replacements $A_- \leftrightarrow A_+$, $\Omega \leftrightarrow \Omega^T$ and $A^{(3)} \leftrightarrow A^{(1)}$. This leads to the same form of H except now with H_{abc} given by (3.8), which proves the equivalence of the two expressions. Let us also remark that this computation shows that the NSNS three-form superfield $H = dB$ satisfies the correct superspace constraints.

In order to check that the dilatinos in (3.10), (3.14) are in fact the spinor derivatives of the dilaton ϕ, we start from (3.16) and compute

$$
\begin{aligned}
d\phi &= -\frac{1}{2}\mathrm{STr}(\mathcal{O}_-^{-1}\mathrm{Ad}_g^{-1} d\mathrm{Ad}_g) = -\frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\mathcal{O}_-^{-1}[g^{-1}dg, T_B]) \\
&= -\frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\mathcal{O}_-^{-1}[\mathcal{O}_- A_-, T_B]) \\
&= -\frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\mathcal{O}_-^{-1}[A_-, T_B]) + \frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\mathcal{O}_-^{-1}\mathrm{Ad}_g^{-1}[\Omega A_-, \mathrm{Ad}_g T_B]) \\
&= -\frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\mathcal{O}_-^{-1}[A_-, T_B]) + \frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\mathrm{Ad}_g\mathcal{O}_-^{-1}\mathrm{Ad}_g^{-1}[\Omega A_-, T_B]) \\
&= -\frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\mathcal{O}_-^{-1}[A_-, T_B]) + \frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\mathrm{Ad}_g(\mathcal{O}_+^T)^{-1}[\Omega A_-, T_B]) \\
&= -\frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\mathcal{O}_-^{-1}[A_-, T_B]) + \frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\Omega\mathcal{O}_-^{-1} Ad_g^{-1}[\Omega A_-, T_B]) \\
&= -\frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\mathcal{O}_-^{-1}[A_-, T_B]) + \frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\Omega\mathcal{O}_-^{-1}\Omega^T[\Omega A_-, T_B]) \\
&\quad + \frac{1}{2}(1-\lambda^{-4})\lambda^2\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\Omega\mathcal{O}_-^{-1}P^{(2)}[\Omega A_-, T_B])\,,
\end{aligned} \tag{B.12}
$$

where we used (A.13) and in the last step we inserted $1 = 1 - \Omega\Omega^T + \Omega\Omega^T = (1-\lambda^{-4})P^{(2)} + \Omega\Omega^T$. It is easy to see that the $A_-^{(0)}$-terms cancel, as they must since they transform as a connection.

JHEP10(2016)045

B.2 η-model

The expressions for $dA_\pm$ in (4.1), (4.3) can be rewritten as

$$\begin{aligned} dA_+ =& \frac{1}{2}\{A_+, A_+\} - \frac{1}{2}c\eta^2\{\hat{d}^T A_+, \hat{d}^T A_+\} + (\mathcal{O}_+^{-1} - 1)\Big(4\{A_+^{(2)}, A_+^{(3)}\} + \hat{\eta}^2\{A_+^{(1)}, A_+^{(1)}\}\Big) \\ &+ \eta\mathcal{O}_+^{-1}R_g\{A_+^{(2)}, \hat{d}^T A_+^{(2)}\}\,, \end{aligned} \tag{B.13}$$

$$\begin{aligned} dA_- =& \frac{1}{2}\{A_-, A_-\} - \frac{1}{2}c\eta^2\{\hat{d}A_-, \hat{d}A_-\} + (\mathcal{O}_-^{-1} - 1)\Big(4\{A_-^{(2)}, A_-^{(1)}\} + \hat{\eta}^2\{A_-^{(3)}, A_-^{(3)}\}\Big) \\ &- \eta\mathcal{O}_-^{-1}R_g\{A_-^{(2)}, \hat{d}A_-^{(2)}\}\,, \end{aligned} \tag{B.14}$$

where we have rewritten e.g. the last term in the expression for dA_+ as

$$\begin{aligned} &\eta\mathcal{O}_+^{-1}R_g\{A_+^{(2)}, \hat{d}^T A_+^{(2)}\} \\ &+ (1 - \mathcal{O}_+^{-1})\Big(\frac{1}{2}\{A_+, A_+\} - \frac{1}{2}c\eta^2\{\hat{d}^T A_+, \hat{d}^T A_+\} - 4\{A_+^{(2)}, A_+^{(3)}\} - \hat{\eta}^2\{A_+^{(1)}, A_+^{(1)}\}\Big)\,. \end{aligned} \tag{B.15}$$

As in the case of the λ-model, to calculate the component $T^{\hat{\alpha}1}$ of the torsion we must first find the Lorentz-transformed spin connection $\mathrm{Ad}_h A_+^{(0)} - dhh^{-1}$ (note the difference in sign between the two models). We use the same method explained in the previous subsection and we find

$$[\mathrm{Ad}_h A_+^{(0)} - dhh^{-1}]_{ab} = \Omega_{ab} - \frac{1}{2}E^c H_{abc} + 2i\hat{\eta}(\gamma_{[a}E^1)_{\hat{\alpha}}[\mathrm{Ad}_h M]^{\hat{\alpha}1}{}_{b]}\,, \tag{B.16}$$

where we write the components of H_{abc} as

$$H_{abc} = 3[\mathrm{Ad}_h M]_{[ab,c]} - 3i\hat{\eta}^2[\mathrm{Ad}_h]_{[a|d|}M^{\hat{\alpha}1}{}_b\gamma^d_{\hat{\alpha}\hat{\beta}}M^{\hat{\beta}1}{}_{c]}\,. \tag{B.17}$$

This expression is equivalent to the one in (4.7), which is easy to verify by a calculation similar to the one performed for the λ-model: the B-field written as in the first way of (4.13) leads to H_{abc} of the form (4.7), while the second way leads to the form in (B.17). The same calculation also shows that the remaining components of the superform H satisfy the standard supergravity constraints.

If we take (4.14) as the definition of the dilaton in the case of the η-model we find

$$\begin{aligned} d\phi =& -\frac{1}{2}\eta\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\hat{d}^T\mathcal{O}_+^{-1}R_g[g^{-1}dg, T_B]) + \frac{1}{2}\eta\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A R_g\hat{d}^T\mathcal{O}_+^{-1}[g^{-1}dg, T_B]) \\ =& -\frac{1}{2}\eta\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\hat{d}^T\mathcal{O}_+^{-1}R_g[A_+, T_B]) - \frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\mathcal{O}_+^{-1}[A_+, T_B]) \\ &- \frac{1}{2}\eta\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\mathcal{O}_+^{-1}[R_g\hat{d}^T A_+, T_B]) - \frac{1}{2}\eta^2\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\hat{d}^T\mathcal{O}_+^{-1}R_g[R_g\hat{d}^T A_+, T_B]) \\ =& -\frac{1}{2}\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\mathcal{O}_+^{-1}[A_+, T_B]) + \frac{1}{2}c\eta^2\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\hat{d}^T\mathcal{O}_+^{-1}[\hat{d}^T A_+, T_B]) \\ &- \frac{1}{2}\eta\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\hat{d}^T\mathcal{O}_+^{-1}R_g[A_+, T_B]) - \frac{1}{2}\eta\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A\mathcal{O}_+^{-1}R_g[\hat{d}^T A_+, T_B]) \\ &+ \frac{1}{2}\eta\widehat{\mathcal{K}}^{AB}\mathrm{STr}(T_A R_g[\hat{d}^T A_+, T_B])\,, \end{aligned} \tag{B.18}$$

where we used the (M)CYBE (1.7) in the last step. It is again easy to verify that the $A^{(0)}$-terms cancel, as they must.

Using (4.16) and (4.15) and (B.16), the explicit result for the vector K^a in (4.19) is

$$
\begin{aligned}
K^a &= \frac{i}{32}\eta(\gamma^a)^{\hat\alpha\hat\beta}\widehat{\mathcal{K}}^{AB}\mathrm{STr}\\
&\quad\times\Big([T_A,RT_B]\mathrm{Ad}_g\big([(1-\eta R_g)\mathrm{Ad}_h^{-1}Q^1_{\hat\alpha},\mathrm{Ad}_h^{-1}Q^1_{\hat\beta}]+\hat\eta^{-2}[(1+\eta R_g)Q^2_{\hat\alpha},Q^2_{\hat\beta}]\big)\Big)\\
&\quad+\text{fermions}\\
&=-\frac{\eta}{2}[\hat\eta^{-2}+\mathrm{Ad}_h]^a{}_b\widehat{\mathcal{K}}^{AB}\mathrm{STr}([T_A,RT_B]gP_bg^{-1})\\
&\quad-\frac{\eta^2}{32}\big(\hat\eta^{-2}(\gamma^a\gamma^c)^{\hat\alpha}{}_{\hat\beta}[R_g]^{\hat\beta2}{}_{\hat\alpha2}-[\mathrm{Ad}_h]^a{}_b(\gamma^b\gamma^c)^{\hat\alpha}{}_{\hat\beta}[R_g]^{\hat\beta1}{}_{\hat\alpha1}\big)\widehat{\mathcal{K}}^{AB}\mathrm{STr}([T_A,RT_B]gP_cg^{-1})\\
&\quad-\frac{i\eta^2}{32}\big([\mathrm{Ad}_h]^a{}_b(\gamma^b\gamma^cK^{12}\gamma^d)^{\hat\alpha}{}_{\hat\beta}[R_g]^{\hat\beta2}{}_{\hat\alpha1}\\
&\qquad\qquad-\hat\eta^{-2}(\gamma^a\gamma^cK^{21}\gamma^d)^{\hat\alpha}{}_{\hat\beta}[R_g]^{\hat\beta1}{}_{\hat\alpha2}\big)\widehat{\mathcal{K}}^{AB}\mathrm{STr}([T_A,RT_B]gJ_{cd}g^{-1})\\
&\quad+\text{fermions}\,.
\end{aligned}
\tag{B.19}
$$

References

[1] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev.* **D 69** (2004) 046002 [hep-th/0305116] [INSPIRE].

[2] G. Arutyunov and S. Frolov, *Foundations of the $AdS_5 \times S^5$ superstring. Part I*, *J. Phys.* **A 42** (2009) 254003 [arXiv:0901.4937] [INSPIRE].

[3] L. Wulff, *Superisometries and integrability of superstrings*, *JHEP* **05** (2014) 115 [arXiv:1402.3122] [INSPIRE].

[4] L. Wulff, *On integrability of strings on symmetric spaces*, *JHEP* **09** (2015) 115 [arXiv:1505.03525] [INSPIRE].

[5] F. Delduc, M. Magro and B. Vicedo, *An integrable deformation of the $AdS_5 \times S^5$ superstring action*, *Phys. Rev. Lett.* **112** (2014) 051601 [arXiv:1309.5850] [INSPIRE].

[6] T.J. Hollowood, J.L. Miramontes and D.M. Schmidtt, *An integrable deformation of the $AdS_5 \times S^5$ superstring*, *J. Phys.* **A 47** (2014) 495402 [arXiv:1409.1538] [INSPIRE].

[7] I.V. Cherednik, *Relativistically invariant quasiclassical limits of integrable two-dimensional quantum models*, *Theor. Math. Phys.* **47** (1981) 422 [*Teor. Mat. Fiz.* **47** (1981) 225] [INSPIRE].

[8] C. Klimčík, *Yang-Baxter σ-models and dS/AdS T duality*, *JHEP* **12** (2002) 051 [hep-th/0210095] [INSPIRE].

[9] C. Klimčík, *On integrability of the Yang-Baxter σ-model*, *J. Math. Phys.* **50** (2009) 043508 [arXiv:0802.3518] [INSPIRE].

[10] F. Delduc, M. Magro and B. Vicedo, *On classical q-deformations of integrable σ-models*, *JHEP* **11** (2013) 192 [arXiv:1308.3581] [INSPIRE].

[11] K. Sfetsos, *Integrable interpolations: from exact CFTs to non-Abelian T-duals*, *Nucl. Phys.* **B 880** (2014) 225 [arXiv:1312.4560] [INSPIRE].

[12] T.J. Hollowood, J.L. Miramontes and D.M. Schmidtt, *Integrable deformations of strings on symmetric spaces*, *JHEP* **11** (2014) 009 [arXiv:1407.2840] [INSPIRE].

[13] A.A. Tseytlin, *On a 'universal' class of WZW type conformal models*, *Nucl. Phys.* **B 418** (1994) 173 [hep-th/9311062] [INSPIRE].

[14] F. Delduc, M. Magro and B. Vicedo, *Derivation of the action and symmetries of the q-deformed $AdS_5 \times S^5$ superstring*, *JHEP* **10** (2014) 132 [arXiv:1406.6286] [INSPIRE].

[15] T.J. Hollowood, J.L. Miramontes and D.M. Schmidtt, *S-matrices and quantum group symmetry of k-deformed σ-models*, arXiv:1506.06601 [INSPIRE].

[16] C. Klimčík and P. Ševera, *Dual non-Abelian duality and the Drinfel'd double*, *Phys. Lett.* **B 351** (1995) 455 [hep-th/9502122] [INSPIRE].

[17] C. Klimčík and P. Ševera, *Poisson-Lie T duality and loop groups of Drinfel'd doubles*, *Phys. Lett.* **B 372** (1996) 65 [hep-th/9512040] [INSPIRE].

[18] B. Vicedo, *Deformed integrable σ-models, classical R-matrices and classical exchange algebra on Drinfel'd doubles*, *J. Phys.* **A 48** (2015) 355203 [arXiv:1504.06303] [INSPIRE].

[19] B. Hoare and A.A. Tseytlin, *On integrable deformations of superstring σ-models related to $AdS_n \times S^n$ supercosets*, *Nucl. Phys.* **B 897** (2015) 448 [arXiv:1504.07213] [INSPIRE].

[20] C. Klimčík, *η and λ deformations as $\mathcal{E}$-models*, *Nucl. Phys.* **B 900** (2015) 259 [arXiv:1508.05832] [INSPIRE].

[21] G. Arutyunov, R. Borsato and S. Frolov, *S-matrix for strings on η-deformed $AdS_5 \times S^5$*, *JHEP* **04** (2014) 002 [arXiv:1312.3542] [INSPIRE].

[22] G. Arutyunov, R. Borsato and S. Frolov, *Puzzles of η-deformed $AdS_5 \times S^5$*, *JHEP* **12** (2015) 049 [arXiv:1507.04239] [INSPIRE].

[23] B. Hoare, R. Roiban and A.A. Tseytlin, *On deformations of $AdS_n \times S^n$ supercosets*, *JHEP* **06** (2014) 002 [arXiv:1403.5517] [INSPIRE].

[24] R. Borsato, *Integrable strings for AdS/CFT*, arXiv:1605.03173 [INSPIRE].

[25] G. Arutyunov, S. Frolov, B. Hoare, R. Roiban and A.A. Tseytlin, *Scale invariance of the η-deformed $AdS_5 \times S^5$ superstring, T-duality and modified type-II equations*, *Nucl. Phys.* **B 903** (2016) 262 [arXiv:1511.05795] [INSPIRE].

[26] R. Borsato, A.A. Tseytlin and L. Wulff, *Supergravity background of λ-deformed model for $AdS_2 \times S^2$ supercoset*, *Nucl. Phys.* **B 905** (2016) 264 [arXiv:1601.08192] [INSPIRE].

[27] Y. Chervonyi and O. Lunin, *Supergravity background of the λ-deformed $AdS_3 \times S^3$ supercoset*, *Nucl. Phys.* **B 910** (2016) 685 [arXiv:1606.00394] [INSPIRE].

[28] K. Sfetsos and D.C. Thompson, *Spacetimes for λ-deformations*, *JHEP* **12** (2014) 164 [arXiv:1410.1886] [INSPIRE].

[29] S. Demulder, K. Sfetsos and D.C. Thompson, *Integrable λ-deformations: squashing coset CFTs and $AdS_5 \times S^5$*, *JHEP* **07** (2015) 019 [arXiv:1504.02781] [INSPIRE].

[30] L. Wulff and A.A. Tseytlin, *κ-symmetry of superstring σ-model and generalized* 10d *supergravity equations*, *JHEP* **06** (2016) 174 [arXiv:1605.04884] [INSPIRE].

[31] A. Mikhailov, *Cornering the unphysical vertex*, JHEP **11** (2012) 082 [arXiv:1203.0677] [INSPIRE].

[32] B. Hoare and A.A. Tseytlin, *Type IIB supergravity solution for the T-dual of the η-deformed $AdS_5 \times S^5$ superstring*, JHEP **10** (2015) 060 [arXiv:1508.01150] [INSPIRE].

[33] I. Kawaguchi, T. Matsumoto and K. Yoshida, *Jordanian deformations of the $AdS_5 \times S^5$ superstring*, JHEP **04** (2014) 153 [arXiv:1401.4855] [INSPIRE].

[34] T. Matsumoto and K. Yoshida, *Integrable deformations of the $AdS_5 \times S^5$ superstring and the classical Yang-Baxter equation — towards the gravity/CYBE correspondence*, J. Phys. Conf. Ser. **563** (2014) 012020 [arXiv:1410.0575] [INSPIRE].

[35] S.J. van Tongeren, *On classical Yang-Baxter based deformations of the $AdS_5 \times S^5$ superstring*, JHEP **06** (2015) 048 [arXiv:1504.05516] [INSPIRE].

[36] L. Wulff, *The type-II superstring to order θ^4*, JHEP **07** (2013) 123 [arXiv:1304.6422] [INSPIRE].

[37] H. Kyono and K. Yoshida, *Supercoset construction of Yang-Baxter deformed $AdS_5 \times S^5$ backgrounds*, arXiv:1605.02519 [INSPIRE].

[38] O.A. Bedoya, L.I. Bevilaqua, A. Mikhailov and V.O. Rivelles, *Notes on β-deformations of the pure spinor superstring in $AdS_5 \times S^5$*, Nucl. Phys. **B 848** (2011) 155 [arXiv:1005.0049] [INSPIRE].

[39] B. Hoare and S.J. van Tongeren, *Non-split and split deformations of AdS_5*, arXiv:1605.03552 [INSPIRE].

[40] B. Hoare and S.J. van Tongeren, *On Jordanian deformations of AdS_5 and supergravity*, arXiv:1605.03554 [INSPIRE].

[41] D. Orlando, S. Reffert, J.-I. Sakamoto and K. Yoshida, *Generalized type IIB supergravity equations and non-Abelian classical r-matrices*, arXiv:1607.00795 [INSPIRE].

[42] O. Lunin and J.M. Maldacena, *Deforming field theories with $\mathrm{U}(1) \times \mathrm{U}(1)$ global symmetry and their gravity duals*, JHEP **05** (2005) 033 [hep-th/0502086] [INSPIRE].

[43] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for superconformal deformations of $N = 4$ super Yang-Mills theory*, JHEP **07** (2005) 045 [hep-th/0503192] [INSPIRE].

[44] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, JHEP **05** (2005) 069 [hep-th/0503201] [INSPIRE].

[45] U. Gürsoy and C. Núñez, *Dipole deformations of $N = 1$ SYM and supergravity backgrounds with $\mathrm{U}(1) \times \mathrm{U}(1)$ global symmetry*, Nucl. Phys. **B 725** (2005) 45 [hep-th/0505100] [INSPIRE].

[46] R.R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, Nucl. Phys. **B 625** (2002) 70 [hep-th/0112044] [INSPIRE].

[47] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, Nucl. Phys. **B 533** (1998) 109 [hep-th/9805028] [INSPIRE].

[48] G. Arutyunov and S. Frolov, *Superstrings on $AdS_4 \times CP^3$ as a coset σ-model*, JHEP **09** (2008) 129 [arXiv:0806.4940] [INSPIRE].

[49] B. Stefanski, jr, *Green-Schwarz action for type IIA strings on $AdS_4 \times CP^3$*, Nucl. Phys. **B 808** (2009) 80 [arXiv:0806.4948] [INSPIRE].

JHEP10(2016)045

[50] J. Gomis, D. Sorokin and L. Wulff, *The complete $AdS_4 \times CP^3$ superspace for the type IIA superstring and D-branes*, JHEP **03** (2009) 015 [arXiv:0811.1566] [INSPIRE].

[51] A. Babichenko, B. Stefanski, Jr. and K. Zarembo, *Integrability and the AdS_3/CFT_2 correspondence*, JHEP **03** (2010) 058 [arXiv:0912.1723] [INSPIRE].

[52] D. Sorokin, A. Tseytlin, L. Wulff and K. Zarembo, *Superstrings in $AdS_2 \times S^2 \times T^6$*, J. Phys. **A 44** (2011) 275401 [arXiv:1104.1793] [INSPIRE].

[53] V.N. Tolstoy, *Chains of extended Jordanian twists for Lie superalgebras*, math/0402433.

[54] A. Stolin, *On rational solutions of Yang-Baxter equation for $\mathfrak{sl}(n)$*, Math. Scand. **69** (1991) 57.

[55] A. Stolin, *Rational solutions of the classical Yang-Baxter equation and quasi Frobenius Lie algebras*, J. Pure Appl. Alg. **137** (1999) 285.

[56] M. Gerstenhaber and A. Giaquinto, *Boundary solutions of the classical Yang-Baxter equation*, Lett. Math. Phys. **40** (1997) 337.

[57] A. Stolin, *Constant solutions of Yang-Baxter equation for $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$*, Math. Scand. **69** (1991) 81.

[58] A. Lichnerowicz and A. Medina, *On Lie groups with left-invariant symplectic or Kählerian structures*, Lett. Math. Phys. **16** (1988) 225.

[59] G. Ovando, *Four dimensional symplectic Lie algebras*, Beiträge Alg. Geom. **47** (2006) 419.

[60] J. Patera, P. Winternitz and H. Zassenhaus, *The maximal solvable subgroups of the $\mathrm{SU}(p,q)$ groups and all subgroups of $\mathrm{SU}(2,1)$*, J. Math. Phys. **15** (1974) 1378 [INSPIRE].

[61] J. Patera, P. Winternitz and H. Zassenhaus, *The maximal solvable subgroups of $\mathrm{SO}(p,q)$ groups*, J. Math. Phys. **15** (1974) 1932.

[62] L.F. Alday, G. Arutyunov and S. Frolov, *Green-Schwarz strings in TsT-transformed backgrounds*, JHEP **06** (2006) 018 [hep-th/0512253] [INSPIRE].

[63] S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer and G. Veneziano, *Remarks on non-Abelian duality*, Nucl. Phys. **B 435** (1995) 147 [hep-th/9409011] [INSPIRE].

[64] E. Bergshoeff, C.M. Hull and T. Ortín, *Duality in the type-II superstring effective action*, Nucl. Phys. **B 451** (1995) 547 [hep-th/9504081] [INSPIRE].

[65] M.B. Green, C.M. Hull and P.K. Townsend, *D-brane Wess-Zumino actions, t duality and the cosmological constant*, Phys. Lett. **B 382** (1996) 65 [hep-th/9604119] [INSPIRE].

[66] S.F. Hassan, *T duality, space-time spinors and RR fields in curved backgrounds*, Nucl. Phys. **B 568** (2000) 145 [hep-th/9907152] [INSPIRE].

[67] A. Hashimoto and N. Itzhaki, *Noncommutative Yang-Mills and the AdS/CFT correspondence*, Phys. Lett. **B 465** (1999) 142 [hep-th/9907166] [INSPIRE].

[68] J.M. Maldacena and J.G. Russo, *Large-N limit of noncommutative gauge theories*, JHEP **09** (1999) 025 [hep-th/9908134] [INSPIRE].

[69] T. Matsumoto and K. Yoshida, *Integrability of classical strings dual for noncommutative gauge theories*, JHEP **06** (2014) 163 [arXiv:1404.3657] [INSPIRE].

[70] S.J. van Tongeren, *Yang-Baxter deformations, AdS/CFT and twist-noncommutative gauge theory*, Nucl. Phys. **B 904** (2016) 148 [arXiv:1506.01023] [INSPIRE].

JHEP10(2016)045