MUNI

Annex No. 10 to the MU Directive on Habilitation Procedures and Professor Appointment Procedures

HABILITATION THESIS REVIEWER'S REPORT

Masaryk University

Applicant	Phuoc-Tai Nguyen
Habilitation thesis	Boundary value problems for nonlinear elliptic equations with a Hardy potential
Reviewer	Professor Yehuda Pinchover
Reviewer's home unit, institution	Department of Mathematics, Technion – Israel Institute of Technology, Haifa 3200003, Israel

Review text: Attached in a separate file.

Reviewer's questions for the habilitation thesis defence (number of questions up to the reviewer)

Included in the attached file.

Conclusion

The habilitation thesis entitled "Boundary value problems for nonlinear elliptic equations with a Hardy potential" by Dr. Phuoc-Tai Nguyen **fulfils** requirements expected of a habilitation thesis in the field of Mathematics - Mathematical Analysis.

Date: 20/4/2021 Signature:

Referee report on the habilitation thesis

"Boundary value problems for nonlinear elliptic equations with a Hardy potential"

by Phuoc-Tai Nguyen

The well written habilitation thesis under review presents recent developments on boundary value problems for linear and nonlinear elliptic equations with a Hardy potential and associated measure data.

Let Ω be a C^2-bounded (nonempty) domain in $\mathbb{R}^N$, and consider the Schrödinger operator $L_\mu := \Delta + \mu\delta^{-2}(x)$ defined on Ω, where $\delta = \delta_\Omega$ is the distance function to $\partial\Omega$, i.e. $\delta(x) := \mathrm{dist}(x, \partial\Omega)$. It is well known that there exists $0 < C_H(\Omega) \leq 1/4$, called the Hardy constant, such that the operator $-L_\mu$ is nonnegative (in the sense of the associated quadratic form) iff $\mu \leq C_H(\Omega)$. Moreover, for such μ the operator $-L_\mu$ admits a positive eigenvalue λ_μ for $\mu < C_H(\Omega)$. Throughout the paper it is assumed that $0 \leq \mu \leq C_H(\Omega)$ and $\lambda_\mu > 0$. In other words, it is assumed that the operator $-L_\mu$ is weakly coercive.

The thesis studies the following Dirichlet boundary value problem:

$$\begin{cases} -L_\mu u \pm g(u, \nabla u) = \tau, & \text{where } \tau \in \mathcal{M}(\Omega, \delta^\alpha). \\ \mathrm{tr}_\mu u = \nu, & \text{where } \nu \in \mathcal{M}(\partial\Omega). \end{cases} \tag{1}$$

Here g is a nonnegative function satisfying appropriate regularity and structural conditions, $\alpha := (1 + \sqrt{1 - 4\mu})/2$, and tr_μ is a notion of *μ-boundary trace* introduced by the author and his collaborators. The plus sign in (1) corresponds to the absorption case while the minus sign represents the source case.

Using the celebrated results of Ancona [1] on the Martin boundary of $-L_\mu$ in Ω, and the explicit asymptotic behaviors of the positive minimal Green function and the Martin Kernel of $-L_\mu$ in Ω proved by S. Filippas, L. Moschini and A. Tertikas in [2], the author and his collaborators prove the existence, uniqueness and representations of solutions of the above boundary value problem with measure data. Moreover, for the case $g = \pm|u|^p$ (absorption and source cases) there is a critical exponent p_μ, below it the problem is well-posed and above it nonexistence appears.

The thesis is based on four papers by the authors which appeared in leading journals. One paper is a collaboration with M. Marcus who was Nguyen's mentor during his postdoc at the Technion, two papers are with K. T. Gkikas who is roughly in the same academic group age as Nguyen, and one is a solo paper by Nguyen. The results of the thesis are nice and significant contributions to the theory of positive solutions of important linear, semilinear and quasilinear partial differential equations. In my opinion, Nguyen's habilitation thesis is of high mathematical level and it is my pleasure to give it my warmest recommendation.

Reviewer's questions for the habilitation thesis defense:

1. What about the case $\mu < 0$?

2. Replace the operator L_μ by $L_\mu^V := L_\mu + V$, where $V \in C_c^\infty(\Omega)$, or more generally ,$|V(x)| \leq C\delta^{-2+\varepsilon}$, where $\varepsilon > 0$. Do the results of the thesis change?

3. There is a recent paper [3], where the results of the paper of Marcus-Mizel-Pinchover [4] are extended to $C^{1,\alpha}$ domains with compact boundaries. Could the results of the thesis be extended to this cases (to the bounded domain case and maybe even to the exterior domain case).

4. [4, 3] and other papers consider the p-Laplacian operator with Hardy potentials. What about possible extensions of your results to this family of operators.

Conclusion: The habilitation thesis entitled "Boundary value problems for nonlinear elliptic equations with a Hardy potential" by Dr. Phuoc-Tai Nguyen fulfills requirements expected of a habilitation thesis in the field of Mathematics - Mathematical Analysis.

References

[1] A. Ancona, *Negatively curved manifolds, elliptic operators and the Martin boundary*, Ann. of Math. (2), **125** (1987), 495–536.

[2] S. Filippas, L. Moschini, and A. Tertikas, *Sharp two-sided heat kernel estimates for critical Schrödinger operators on bounded domains*, Comm. Math. Phys. **273** (2007), 237–281.

[3] P. D. Lamberti, and Y. Pinchover, *L^p Hardy inequality on $C^{1,\alpha}$ domains*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **19** (2019), 1135–1159.

[4] M. Marcus, V. J. Mizel and Y. Pinchover, *On the best constant for Hardys inequality in $\mathbb{R}^n$*, Trans. Amer. Math. Soc. **350** (1998), 3237–3255.

19/4/2021