MASARYK UNIVERSITY · FACULTY OF SCIENCE
HABILITATION THESIS

Odd Scalar Curvature in Batalin-Vilkovisky Geometry

BY

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Abstract

After a brief introduction to Batalin-Vilkovisky (BV) formalism, we treat aspects of supermathematics in algebra and differential geometry, such as, stratification theorems, Frobenius theorem and Darboux theorem on supermanifolds. We use Weinstein's splitting principle to prove Darboux theorem for regular, possible degenerate, even and odd Poisson supermanifolds. Khudaverdian's nilpotent Δ_E operator (which takes semidensities into semidensities of opposite Grassmann-parity) is introduced on both (i) an atlas of Darboux coordinates and (ii) in arbitrary coordinates. An odd scalar function ν_ρ is defined and it is shown that it has a geometric interpretation as an odd scalar curvature.

Keywords: Supermathematics; Supermanifolds; Odd Poisson Geometry; Darboux Theorem; Antibracket; Batalin-Vilkovisky Geometry; Curvature;

Contents

1 Introduction

Batalin-Vilkovisky (BV) formalism [1, 2, 3] was originally proposed as a recipe to provide a Bechi-Rouet-Stora-Tyutin (BRST) formulation [4, 5] of an arbitrary local Lagrangian gauge field theory [6].

In physics, a field $\varphi^a(x)$ is a map φ from an n-dimensional spacetime/world-volume with local coordinates x^μ to a target space/field configuration space. The index a labels the coordinates of the fields in the target space.

BV formalism often uses deWitt's condensed notation $\varphi^i \equiv \varphi^a(x)$, where all discrete and continuous indices are collected together into a single index $i = (a, x)$.

In physics the field φ^i therefore contains infinite degrees of freedom. We may therefore formally view φ^i as coordinates on an infinite-dimensional manifold.

A more rigorous field-theoretic approach (which we shall not pursue here) properly tracks the local spacetime structure of the field theory jet bundle construction, see *e.g.*, Refs. [7, 8]. The fact that renormalization naturally fits into the BV formalism contributes to its versatility and state-of-the-art in gauge field theory quantization, see *e.g.*, Refs. [9, 10, 11].

A gauge theory is a theory with a groupoid of gauge transformations that typically leave the action invariant up to boundary terms. Gauge symmetry signifies a redundant formulation, where field configurations on the same gauge orbit are physically equivalent. The redundant formulation can typically not be removed without rendering the theory non-local or destroy manifest Lorentz symmetry.

A BRST transformation is a nilpotent, Grassmann-odd transformation that encodes the gauge-symmetry. The corresponding BRST-cohomology is needed in order to consistently define the spectrum of physical states of the theory. (By the way, BRST symmetry should not be confused with Poincare supersymmetry, which may or may not be present.)

Besides the original fields φ^i, the BV recipe introduces new fields, such as, *e.g.*, Faddeev-Popov ghosts c^a, and Lagrange multipliers. For each field $\phi^\alpha = \{\varphi^i, c^a, \ldots\}$ of Grassmann-parity ε_α, there is introduced an antifield ϕ^*_α of opposite Grassmann-parity $\varepsilon_\alpha + 1$. Supernumbers and Grassmann-variables will be discussed in section 2.

The original BV formulation [1, 2, 3] introduced two new interesting mathematical structures.

1. The Δ operator
$$\Delta \;=\; \frac{(-1)^{\varepsilon_\alpha}}{\sigma} \frac{\overrightarrow{\delta^\ell}}{\delta\phi^\alpha} \frac{\overrightarrow{\delta^\ell}}{\delta\phi^*_\alpha} \sigma \;, \tag{1.1}$$
which is nilpotent and Grassmann-odd
$$\Delta^2 \;=\; 0 \;, \qquad \varepsilon(\Delta) \;=\; 1 \;. \tag{1.2}$$

2. The antibracket
$$(\phi^\alpha, \phi^*_\beta) \;=\; \delta^\alpha_\beta \;. \tag{1.3}$$

Here $\sigma = \sigma(\phi)$ is a density in field configuration space. It is needed in order to ensure that the Δ-operator (1.1) takes scalars in scalars. The antibracket is an antisymplectic structure on the Grassmann-parity-inverted cotangent bundle.

In the BV recipe the quantum action

$$W \;=\; S+\sum_{n=1}^{\infty}\hbar^{n}M_{n} \tag{1.4}$$

satisfies the quantum master equation (QME)

$$\Delta e^{\frac{i}{\hbar}W} \;=\; 0\ . \tag{1.5}$$

To be concrete, the partition function/functional integral/Feynman path integral of the theory is formally given as

$$Z_{\psi} \;=\; \int\sigma[d\phi]\ e^{\frac{i}{\hbar}W}\Big|_{\phi^{*}=\frac{\delta\psi}{\delta\phi}}\ , \tag{1.6}$$

where ψ is a gauge-fixing Fermion. It is one of the powerful virtues of the BV formalism, that one may formally demonstrate that Z_{ψ} does not depend on ψ.

The BV formalism leads to two new mathematical disciplines:

- (i) BV algebra, and its generalization, such as, BV homotopy algebra [12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23], which we shall not discuss further here, and
- (ii) BV geometry [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49], which is the main topic of this thesis, cf. section 4.

At this point, it is natural to generalize the Grassmann-parity-inverted cotangent bundle with Darboux coordinates $\{\phi^{\alpha};\phi^{*}_{\beta}\}$ and density σ^{2} into a general anti-Poisson supermanifold $(M;E)$ with local coordinates z^{A} and equipped with a density $\rho=\rho(z)$.

Supermanifolds will be discussed in subsection 2.3. We shall also for simplicity from now on assume that the manifolds are finite-dimensional, despite that the original motivation in field theory deals with infinite-dimensional configuration spaces.

Anti-Poisson geometry shares many properties with Grassmann-even Poisson geometry. E.g. they both sport versions of the Jacobi identity and Darboux' theorem, cf. section 3.

Somewhat surprisingly, anti-Poisson geometry also has parallels to (pseudo)Riemannian geometry [46] (paper V). The analogue of the Laplace-Beltrami operator in (pseudo)Riemannian geometry is an odd Laplacian [29]

$$\Delta_{\rho} \;:=\; \frac{(-1)^{\varepsilon_{A}}}{2\rho}\overrightarrow{\partial^{\ell}_{A}}\rho E^{AB}\overrightarrow{\partial^{\ell}_{B}}\ , \qquad \overrightarrow{\partial^{\ell}_{A}} \;\equiv\; \frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}\ . \tag{1.7}$$

It is Grassmann-odd but *not* necessarily nilpotent.

One of our main contributions to the topic was to realize [42, 43] (paper I & II) that for regular BV geometries (M,E,ρ) there exists a canonical odd scalar ν_{ρ} (which depends only on the geometric data E and ρ) such that the Δ operator

$$\Delta \;=\; \Delta_{\rho}+\nu_{\rho} \tag{1.8}$$

is nilpotent,

$$\Delta^{2} \;=\; 0\ , \tag{1.9}$$

cf. section 4.7. The nilpotency is important in order to define a chain complex.

Furthermore, one of our main conclusions is that the odd scalar

$$\nu_{\rho} \;=\; -\frac{R}{8} \tag{1.10}$$

has a geometric interpretation as odd scalar curvature for an arbitrary connection ∇ that is

1. anti-Poisson,
2. torsionfree, and
3. ρ-compatible,

cf. section 5.

This is a quite remarkable result because there are infinitely many connections that satisfies these 3 conditions.

The QME (1.5) is modified with a ν_{ρ}-term at the two-loop order $\mathcal{O}(\hbar^2)$:

$$\frac{1}{2}(W,W) \;=\; i\hbar\Delta_{\rho}W + \hbar^2\nu_{\rho} \;. \tag{1.11}$$

Expanding the quantum action (1.4) in powers of Planck's constant $\hbar$ leads to an infinite tower of master equations

$$(S,S) \;=\; 0 \;, \tag{1.12}$$

$$(M_1,S) \;=\; i(\Delta_{\rho}S) \;, \tag{1.13}$$

$$(M_2,S) \;=\; i(\Delta_{\rho}M_1) + \nu_{\rho} - \frac{1}{2}(M_1,M_1) \;, \tag{1.14}$$

$$(M_n,S) \;=\; i(\Delta_{\rho}M_{n-1}) - \frac{1}{2}\sum_{r=1}^{n-1}(M_r,M_{n-r}) \;, \qquad n \geq 3 \;. \tag{1.15}$$

Eq. (1.12) is known as the classical master equation (CME). Eq. (1.13) determines the one-loop contribution M_1. This is where possible quantum anomalies may appear in quantum field theory. Interestingly, the odd scalar ν_{ρ} appears in the eq. (1.14) for the two-loop contribution M_2, see *e.g.*, Ref. [45].

General remark about notation. We have two types of grading: A Grassmann grading ε and an exterior form degree p. The sign conventions are such that two exterior forms ξ and η, of Grassmann parity ε_{ξ}, ε_{η} and exterior form degree p_{ξ}, p_{η}, respectively, commute in the following graded sense:

$$\eta \wedge \xi \;=\; (-1)^{\varepsilon_{\xi}\varepsilon_{\eta}+p_{\xi}p_{\eta}}\xi \wedge \eta \tag{1.16}$$

inside the exterior algebra. We will often not write the exterior wedges "$\wedge$" explicitly.

$[A,B]$ and $\{A,B\}$ denote the graded commutator $[A,B] \equiv AB - (-1)^{\varepsilon_A\varepsilon_B}BA$ and the graded anticommutator $\{A,B\} \equiv AB + (-1)^{\varepsilon_A\varepsilon_B}BA$, respectively.

We often omit the prefix "super" from various words in supermathematics, such as in, *e.g.*, supermanifold, supercommutator, etc.

2 Aspects of Supermathematics

Before we can begin, we need supermathematics [50, 51, 52, 53, 54, 55, 56, 57], *i.e.*, Grassmann-odd variables.

From the physics side, this is mainly because matter particles/fields in Nature, (such as, *e.g.*, electrons) are Fermions, which obey Fermi-Dirac statistics and Pauli's exclusion principle. In particular, their multi-particle wave-function needs to be antisymmetric under particle exchange. To properly describe interactions of Fermionic fields at the classical level, we therefore need to use anticommuting variables. In other words, we need to use supernumbers.

A second optional source is Poincare supersymmetric field theories.

A third typical (this time mathematical) source of Grassmann-odd variables is exponentiations of determinants and Pfaffians, *e.g.*,

$$\mathrm{Pf}(A) \;=\; \int d\theta^n \ldots d\theta^1 \;\exp\left(\frac{1}{2}\theta^i A_{ij}\theta^j\right) \;, \qquad \varepsilon(A_{ij}) \;=\; 0, \qquad A_{ij} \;=\; -A_{ji}\;. \tag{2.1}$$

E.g. Faddeev-Popov ghosts are often introduced in this manner: The argument of the exponentials, such as, *e.g.*, eq. (2.1), can be interpreted as new action terms in the path integral Z.

The topic of supermathermatics contains several non-intuitive surprises, which we will try to expose in this chapter.

2.1 Grassmann-numbers and Supernumbers

Here is one approach to Grassmann-numbers and supernumbers [53].

If V_1 is an infinite-dimensional* $\mathbb{C}$-vector space, then the exterior algebra $\Lambda_\infty := \bigwedge^\bullet V_1$ is (a copy of) the algebra of **supernumbers**.

Moreover,

- $\bigwedge^0 V_1 \cong \mathbb{C}$ is called the **body**,
- $\bigwedge^{>0} V_1$ is the **soul**,
- $\mathbb{C}^{1|0} \cong \bigwedge^{\mathrm{even}} V_1$ is the Grassmann-even/Bosonic part, and
- $\mathbb{C}^{0|1} \cong \bigwedge^{\mathrm{odd}} V_1$ is the Grassmann-odd/Fermionic part.

Let $\varepsilon_z \equiv \varepsilon(z) \in \{0,1\}$ mod 2 denote the Grassmann-parity of a supernumber z (with definite Grassmann grading). And let

$$m(z) \equiv z_B \tag{2.2}$$

denote the body of a supernumber z.

*Here we are caught between a rock and a hard place: If V_1 is finite-dimensional, there appear unwanted truncation phenomena. If V_1 is infinite-dimensional, delicate problems of analysis arise. (For definiteness, assume that the body algebra $\bigwedge^0 V_1 \cong \mathbb{C}$ is equipped with the standard norm, while infinite sums in the soul vector space $\bigwedge^{>0} V_1$ are formal.)

Complex conjugation $z \mapsto \bar{z}$ is an anti-involution $\overline{zw} = \bar{w}\bar{z}$ on the algebra of supernumbers. We define real and imaginary part

$$\mathrm{Re} z := \frac{z+\bar{z}}{2} , \qquad \mathrm{Im} z := \frac{z-\bar{z}}{2i} , \tag{2.3}$$

of a supernumber z in the standard way. Let from now on $\mathbb{F}$ denote the field of either the real or complex numbers, $\mathbb{R}$ or $\mathbb{C}$.

Left (right) differentiation satisfies a left (right) graded Leibniz rule. Left and right differentiation of a function $z \mapsto f(z)$ are connected via the rule

$$(\frac{\overrightarrow{\partial^{\ell}}}{\partial z} f(z)) = (-1)^{\varepsilon_z(\varepsilon_f+1)} (f(z) \frac{\overleftarrow{\partial^{r}}}{\partial z}) . \tag{2.4}$$

This is to ensure compatibility

$$(\frac{\overrightarrow{\partial^{\ell}}}{\partial z} z) = 1 = (z \frac{\overleftarrow{\partial^{r}}}{\partial z}) . \tag{2.5}$$

(The outer parentheses in eqs. (2.4) and (2.5) are supposed to notationally indicate that the differential operators don't act beyond the parentheses.)

2.2 Definite Contour Integral for Supernumbers?

It is well-known that if a function $z \mapsto f(z)$ has an antiderivative $z \mapsto F(z)$, then a contour integral

$$\int_{\gamma} dz \ f(z) := \int_a^b dt \ \gamma'(t) \ F'(\gamma(t)) = \int_a^b dt \ (F \circ \gamma)'(t) = F(\gamma(b)) - F(\gamma(a)) , \qquad t \in [a,b] \subset \mathbb{R} , \tag{2.6}$$

can only depend on the endpoints of the curve. All analytic functions $f(x) = \sum_{n=0}^{\infty} a_n x^n$ of a Grassmann-even supernumber x has an antiderivative $F(x) = \sum_{n=0}^{\infty} a_n x^{n+1}/(n+1)$. However the Grassmann-odd function $f : \theta \mapsto \theta$ has no antiderivative, and it is easy to see that the contour integral $\int_{\gamma} d\theta \ \theta$ depends on the contour beyond the endpoints.

Example 2.1

$$\gamma(t) = t\theta_1 + t(1-t)\theta_2 , \qquad t \in [0,1] , \qquad \gamma(t=0) = 0, \qquad \gamma(t=1) = \theta_1 , \tag{2.7}$$

$$\int_{\gamma} d\theta \ \theta = \int_0^1 dt \ \gamma'(t) \ \gamma(t) = \int_0^1 dt \ (\theta_1 + (1-2t)\theta_2)(t\theta_1 + t(1-t)\theta_2) = \frac{1}{3}\theta_1\theta_2 , \tag{2.8}$$

which does not just depend on the endpoints.

For this and similar reasons, we give up to try to assign a value to a Grassmann-odd number, whatever that is supposed to mean. In terms of topology, DeWitt [53] assigned to Grassmann-odd directions the coarsest topology, *i.e.*, the trivial topology.

The paradigm is that the specific choice of the vector space V_1 is meant to be forgotten/is not important. The soul part is not quantifiable/does not have a value. An element $\xi \in V_1$ should be thought of as an *indeterminate.*

Normally the word 'indeterminate' is just another word for a 'variable', but here we use the word in a stronger sense closer to the etymology of the word: Unlike a variable, which can be determined/evaluated/given a value later-on by replacing the variable with a number, this is not possible for the indeterminate $\xi \in V_1$.

Put succinctly: There is no physical quantity or physical measurement device in a physical system that measures a soul-valued output.

But that begs the question: *How does the soul then affect a physical system at all?* The answer is: Via Berezin integration:

$$\begin{aligned} \int_{\mathbb{F}^{0|1}} d\theta \ f(\theta) \ &:= \ \frac{\overrightarrow{\partial^\ell}}{\partial\theta} f(\theta) \ , \qquad && \varepsilon(\theta) \ = \ 1 \ , \\ \int_{\mathbb{R}^{1|0}} dx \ f(x) \ &:= \ \int_{\mathbb{R}} dx_B \ f(x_B) \qquad && \varepsilon(x) \ = \ 0 \ . \end{aligned} \tag{2.9}$$

In other words, all souls are ultimately integrated out.

This leads to another question: *How do we then understand 'external Grassmann-odd constants' in a physical or mathematical model?* Answer: The model should be understood as a subsector of a bigger structure with only intrinsic souls.

2.3 Supermanifolds

There are roughly speaking two equivalent† concepts of supermanifolds M of finite dimensions $(n|m)$, at least within the applications to physics. Here we will briefly review their main features, cf. the next two subsections 2.4 and 2.5. Let $\mathbb{F}^{n|m} := (\mathbb{F}^{1|0})^{\times m} \times (\mathbb{F}^{0|1})^{\times n}$.

2.4 Differential Geometric Supermanifolds

On one hand, there is a differential geometric definition, pioneered by deWitt [53], which considers atlases of local $\mathbb{F}^{n|m}$ coordinate neighborhoods, which treats Grassmann-even and Grassmann-odd variables on the same footing.

A map between supermanifolds becomes within local coordinates a map

$$f: \quad V \subseteq \mathbb{F}^{n|m} \to V' \subseteq \mathbb{F}^{n'|m'} \tag{2.10}$$

of the form

$$z'^A \ = \ \text{function of the } z\text{-coordinates.} \tag{2.11}$$

2.5 Algebro-Geometric/Sheaf-Theoretic Supermanifolds

On the other hand, there is a sheaf-theoretic definition (M_B, R) of a ringed space over an underlying ordinary n-dimensional manifold M_B called the body (hence the subscript B), pioneered by Kostant and Leites [50, 51, 52, 55]. If $U \subseteq \mathbb{R}^n$ is an open coordinate neighborhood for the body M_B, then $R(U) = R_B(U)[\theta^1, \ldots, \theta^m]$ is the polynomial ring, where $\theta^1, \ldots, \theta^m$ are anti-commuting

†Note however that the second approach has a vast generalization to category theory and Grothendieck's schemes.

indeterminates. Furthermore, $R_B(U)$ is the ring of, *e.g.*, smooth functions $C^\infty(U)$, real analytic functions $\mathbb{R}_{\rm an}(U)$, or holomorphic functions $H(U)$. (In this work, we shall assume the category of real analytic functions, to keep proofs as simple as possible.) This construction treats Grassmann-even and Grassmann-odd variables very differently.

A morphism between supermanifolds consists within local coordinate neighborhoods an underlying body map $f: \quad U \subseteq \mathbb{F}^n \to U' \subseteq \mathbb{F}^{n'}$ together with a sheaf map, *i.e.*, an algebra homomorphism $f^*: \quad R(U') \to R(U)$, uniquely specified by it action on a local basis

$$\begin{aligned} f^*(x_B^{i\prime}) &= \text{Grassmann-even function of } (x_B, \theta), \\ f^*(\theta^{a\prime}) &= \text{Grassmann-odd function of } (x_B, \theta). \end{aligned} \tag{2.12}$$

(The sheaf map should be compatible with the underlying body map.)

Given a fixed supermanifold M we can consider the so-called functor of points

$$\mathrm{Hom}(\cdot, M): \quad \mathbf{SMan}^{\rm op} \ni S \mapsto \underbrace{\mathrm{Hom}(S, M)}_{S\text{-point of } M} \in \mathbf{Set} . \tag{2.13}$$

In particular, the indeterminates $\theta^1, \ldots, \theta^m$ are viewed as morphisms from S. (Incidentally, that is almost the perfect metaphor for passing the buck to S if asked *What is* θ^i*?*) Morphisms $M \to N$ between supermanifolds M and N are in bijective correspondence with natural transformations $\mathrm{Hom}(\cdot, M) \to \mathrm{Hom}(\cdot, N)$, cf. Yoneda's Lemma.

3 Differential Geometry for Supermanifolds

In this section, we would like to mention the generalization to supermanifolds of a couple of well-known local theorems from the differential geometry of ordinary manifolds. We need in particular a superversion of the Darboux theorem to define Khudaverdian's Δ_E operator in the next section 4. In turn, the Darboux theorem relies on the Frobenius theorem.

Another purpose of this section is to provide proofs that would be accessible for a physics student.

Students of BV formalism will have noticed that Bosonic and Fermionic variables surprisingly often can be treated in a collective and uniform manner in calculations, merely by keeping track of Grassmann-parity sign factors. This is one of key paradigms that underlies most of the presentation in this thesis.

One difference between Bosonic and Fermionic directions is that Bosonic directions may feature non-trivial topology, while Fermionic directions cannot, as we saw in the last section 2. Since the results will be local, this difference is irrelevant, and we can conveniently to a large extent treat Bosonic and Fermionic variables on equal footing.

The biggest difference between Bosonic and Fermionic operators is that a Bosonic operator trivially (super)commutes with itself, while this is not necessarily so for a Fermionic operator.

In the following we need that the **rank** of a (possible rectangular) super matrix is the dimension $(n|m)$ of its image. In particular, $n+m$ is the rank of the body of the super matrix.

3.1 Stratification Theorem and Frobenius Theorems

Theorem 3.1 (Stratification theorem for vector field) *Given an $(n|m)$-dimensional supermanifold M. Let $N := n+m$. Given a self-(super)commuting vector field X of definite parity with non-zero rank in a point $p \in M$. Then there exists a local coordinate system $(z^1, z^2, \ldots, z^N)$ such that $X = \frac{\overrightarrow{\partial^\ell}}{\partial z^1}$.*

Induction Proof: Given a local coordinate system $(z^1, z^2, \ldots, z^N)$. After a possible translation, we may assume that the point p is at the origin. There exists a coordinate (which we relabel as) z^1 such that $m(X_p[z^1]) \neq 0$. By a rigid linear transformation, we may assume that $X_p[z^1] = 1$.

Induction assumption: The components X^A of the vector field $X = X^A \overrightarrow{\partial^\ell_A}$ are of the form

$$X^A \;=\; \delta^A_1 + X^A_{[k]} \;, \qquad X^A_{[k]} \;=\; \mathcal{O}((z)^k) \;, \tag{3.1}$$

for some integer $k \in \mathbb{N}$.

In the odd case $\varepsilon(z^1) = 1$, nilpotency additionally yields

$$0 \;=\; X[X^A] \;=\; (\overrightarrow{\partial^\ell_1} X^A_{[k]}) + \mathcal{O}((z)^{2k-1}) \;. \tag{3.2}$$

Choose

$$f^B_{[k+1]}(z^1, z^2, \ldots, z^N) \;:=\; \left\{ \begin{array}{ll} -\int_0^{z^1} d\tilde{z}^1 \; X^B_{[k]}(\tilde{z}^1, z^2, \ldots, z^N) & \text{if } \varepsilon(z^1) \;=\; 0 \text{ even} \;, \\ -z^1 X^B_{[k]}(z^1, z^2, \ldots, z^N) & \text{if } \varepsilon(z^1) \;=\; 1 \text{ odd} \;. \end{array} \right. \tag{3.3}$$

Then

$$f^B_{[k+1]} \;=\; \mathcal{O}((z)^{k+1}) \;, \qquad (\overrightarrow{\partial^{\ell}_1} f^B_{[k+1]}) \;=\; -X^B_{[k]} + \mathcal{O}((z)^{2k}) \;. \tag{3.4}$$

We now change coordinates

$$z'^B \;=\; z^B + f^B_{[k+1]} \;. \tag{3.5}$$

Then the new components satisfies an even higher induction assumption (3.1):

$$X'^B \;=\; X[z'^B] \;=\; (\delta^A_1 + X^A_{[k]})\overrightarrow{\partial^{\ell}_A}(z^B + f^B_{[k+1]}) \;=\; \delta^B_1 + X^B_{[k]} + (\overrightarrow{\partial^{\ell}_1} f^B_{[k+1]}) + \mathcal{O}((z)^{2k}) \;=\; \delta^B_1 + \mathcal{O}((z)^{2k}) \;. \tag{3.6}$$

□

Theorem 3.2 (Abelian Frobenius theorem) *If $X_{(1)}, \ldots, X_{(r)}$ are strongly commuting vector fields,*

$$[X_{(a)}, X_{(b)}] \;=\; 0 \;, \qquad a, b \;\in\; \{1, \ldots, r\} \;, \tag{3.7}$$

pointwise linearly independent, of definite Grassmann parity, then there exists a local coordinate neighborhood $(z^1, z^2, \ldots, z^N)$ such that

$$X_{(a)} \;=\; \overrightarrow{\partial^{\ell}_a} \;, \qquad a \;\in\; \{1, \ldots, r\} \;. \tag{3.8}$$

Splitting proof: From the stratification theorem 3.1 we may assume that $X_{(1)} = \overrightarrow{\partial^{\ell}_1}$. Then the components $X^A_{(a)}$ cannot depend on the coordinate z^1. Define vector fields

$$Y_{(a>1)} \;:=\; X_{(a)} - X_{(a)}[z^1]X_{(1)} \;=\; X_{(a)} - X^1_{(a)}\overrightarrow{\partial^{\ell}_{(1)}} \;\in\; \mathrm{span}(\overrightarrow{\partial^{\ell}_{A>1}}). \tag{3.9}$$

Straightforward calculations then shows that

$$[X_{(1)}, Y_{(a\neq 1)}] \;=\; 0 \;, \tag{3.10}$$

$$[Y_{(a\neq 1)}, Y_{(b\neq 1)}] \;=\; -X_{(a)}[X^1_{(b)}]\overrightarrow{\partial^{\ell}_{(1)}} - (-1)^{\varepsilon_a \varepsilon_b}(a \leftrightarrow b) \;=\; -[X_{(a)}, X_{(b)}][z^1] \;=\; 0 \;. \tag{3.11}$$

So $Y_{(2)}, \ldots, Y_{(r)}$ is the same type of problem with rank $r-1$ and dimension $N-1$ one less each.

□

Theorem 3.3 (Non-Abelian Frobenius theorem) *Given vector fields $X_{(1)}, \ldots, X_{(r)}$, pointwise linearly independent, of definite Grassmann parity, such that*

$$\exists f_{(ab)}{}^{(c)} : \quad [X_{(a)}, X_{(b)}] \;=\; \sum_{c=1}^{r} f_{(ab)}{}^{(c)} \; X_{(c)} \;, \qquad a, b \;\in\; \{1, \ldots, r\} \;. \tag{3.12}$$

Then it is locally integrable, i.e., locally there exists a coordinate system $(z^1, z^2, \ldots, z^N)$ and a matrix function

$$S_{(a)}{}^{(b)} \;, \qquad a, b \;\in\; \{1, \ldots, r\} \;, \tag{3.13}$$

such that

$$X'_{(a)} \;=\; \sum_{b=1}^{r} S_{(a)}{}^{(b)} \; \frac{\overrightarrow{\partial^{\ell}}}{\partial z^a} \;. \tag{3.14}$$

PROOF: Note that the non-Abelian involution property (3.12) is preserved by taking linear combination of vector fields

$$X'_{(a)} \;=\; \sum_{b=1}^{r} S_{(a)}{}^{(b)}\, X_{(b)}\;, \qquad a \;\in\; \{1,\ldots,r\}\;. \tag{3.15}$$

After such multiplication (3.15) of the vector fields with an invertible matrix $S_{(a)}{}^{(b)}$, we may assume that they are of the form

$$X'_{(a)} \;=\; \overrightarrow{\partial^{\ell}_{a}} + \sum_{A>r} X'^{A}_{(a)}\, \overrightarrow{\partial^{\ell}_{A}}\;, \qquad a \;\in\; \{1,\ldots,r\}\;. \tag{3.16}$$

Straightforward calculations then shows that

$$\sum_{c=1}^{r} f'_{(ab)}{}^{(c)}\, X'_{(c)} \;=\; [X'_{(a)}, X'_{(b)}] \;\in\; \mathrm{span}(\overrightarrow{\partial^{\ell}_{A>r}}) \qquad \Rightarrow \qquad f'_{(ab)}{}^{(c)} \;=\; 0\;. \tag{3.17}$$

Now use the Abelian Frobenius theorem 3.2.

□

3.2 Poisson Structure

We next consider a possibly degenerate Poisson manifold (M,E), with Poisson bracket

$$(f,g) \;=\; (f\frac{\overleftarrow{\partial^{r}}}{\partial z^{A}})E^{AB}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{B}}g)\;. \tag{3.18}$$

Here the bi-vector

$$E^{AB} \;=\; (z^{A}, z^{B}) \tag{3.19}$$

is graded skewsymmetric

$$E^{AB} \;=\; -(-1)^{(\varepsilon_{A}+\varepsilon_{E})(\varepsilon_{B}+\varepsilon_{E})}E^{BA}\;, \tag{3.20}$$

and has Grassmann parity

$$\varepsilon(E^{AB}) \;=\; \varepsilon_{A}+\varepsilon_{B}+\varepsilon_{E}\;. \tag{3.21}$$

It satisfies the Jacobi identity

$$0 \;=\; \sum_{f,g,h\ \mathrm{cycl.}} (-1)^{(\varepsilon_{f}+\varepsilon_{E})(\varepsilon_{h}+\varepsilon_{E})}((f,g),h)\;. \tag{3.22}$$

Here ε_{E} is the (internal) Grassmann-parity of the Poisson bracket. The case $\varepsilon_{E}=1$ is often called an antibracket, anti-Poisson bracket, odd Poisson bracket, or a Gerstenhaber bracket.

In general, a Poisson manifold can have singular points where the rank of E^{AB} jumps.

3.3 2-form

Consider a 2-form

$$E \;=\; \frac{1}{2}dz^{A}\; E_{AB}\wedge dz^{B} \;=\; -\frac{1}{2}(-1)^{\varepsilon_{A}(1-\varepsilon_{E})}E_{AB}\; dz^{B}\wedge dz^{A} \tag{3.23}$$

of (internal) Grassmann-parity ε_E. By definition the tensor E_{AB} has Grassmann parity

$$\varepsilon(E_{AB}) \;=\; \varepsilon_A{+}\varepsilon_B{+}\varepsilon_E \;, \tag{3.24}$$

and is graded skewsymmetric,

$$E_{AB} \;=\; -(-1)^{\varepsilon_A\varepsilon_B+(1-\varepsilon_E)(\varepsilon_A+\varepsilon_B)}E_{BA} \;. \tag{3.25}$$

(Here the exterior form-degree and Grassmann-parity are considered to be non-correlated, independent gradings. Hopefully, it will not cause confusion that we use the same letter E for both a Poisson bi-vector E^{AB} with upper indices and a 2-form E_{AB} with lower indices.)

3.4 Pre-Symplectic Structure

Definition 3.4 *A* **pre-symplectic** 2**-form** *is a (not necessarily non-degenerate) closed 2-form.*

The closedness relation

$$dE \;=\; 0 \tag{3.26}$$

reads in components

$$0 \;=\; \sum_{A,B,C \text{ cycl.}} (-1)^{(\varepsilon_A+1-\varepsilon_E)\varepsilon_C}(\overrightarrow{\partial^{\ell}_A}E_{BC}) \;. \tag{3.27}$$

3.5 Symplectic Structure

A non-degenerate Poisson manifold (M,E) is called a **symplectic** manifold. By the non-degeneracy assumption, there exists an inverse tensor E_{AB} such that

$$E^{AB}E_{BC} \;=\; \delta^A_C \;=\; E_{CB}E^{BA} \;. \tag{3.28}$$

In other words, E_{AB} is a two-form (3.23). The Jacobi identity (3.22) is equivalent to the closedness relations (3.26) and (3.27).

3.6 Stratification Lemmas

Consider a possibly degenerate Poisson manifold (M,E) with a point $p\in M$. Let there be given a local coordinate system $(z^1,z^2,\ldots,z^N)$ of definite Grassmann degree in a neighborhood of $p\in M$.

Assumption 3.5 (Seed 1) *There exists one local coordinate (which we w.l.o.g. can rename z^1), such that the body*

$$m((z^1,z^1)_p)) \;\neq\; 0 \tag{3.29}$$

does not vanish.

Assumption 3.6 (Seed 2) *There exist two different local coordinates (which we can w.l.o.g. can rename z^1 and z^2), such that the body*

$$m((z^1,z^2)_p) \;\neq\; 0 \tag{3.30}$$

does not vanish.

Remark 3.7 *Seed 1 is only possible for an even Poisson bracket with a Grassmann-odd* z^1*; For an odd Poisson bracket* $m((z^1, z^1)_p) = 0$ *automatically.*

Remark 3.8 *Seed 2 with an even Poisson bracket with two Grassmann-odd variables* z^1 *and* z^2 *can be traded for a seed 1 by possibly taking a constant linear combination* $z'^1 = az^1 + bz^2$*, so that* $m((z'^1, z'^1)_p) \neq 0$*.*

Remark 3.9 *For the remaining cases of seed 2 we may assume (possibly after relabeling* $z^1 \leftrightarrow z^2$*) that* z^1 *is a Boson.*

Lemma 3.10 (Stratification lemma for seed 1) *Given seed 1 then there locally exists a Grassmann-odd variable* Q *such that*

$$(Q, Q) \;=\; \pm 1 \tag{3.31}$$

in a local neighborhood, and such that

$$m\left(\left.\frac{\overrightarrow{\partial^{\ell}}}{\partial z^1} Q\right|_p\right) \;\neq\; 0 \;, \tag{3.32}$$

i.e., $(Q, z^2, \ldots, z^N)$ *is a local coordinate system.*

Lemma 3.11 (Stratification lemma for seed 2) *Given seed 2 (excluding the case in remark 3.8, and possibly after relabeling* $z^1 \leftrightarrow z^2$*), then* z^1 *is a Boson and there locally exists a Bosonic variable* S *such that*

$$(S, S) \;=\; 0 \tag{3.33}$$

in a local neighborhood, and such that

$$m\left(\left.\frac{\overrightarrow{\partial^{\ell}}}{\partial z^1} S\right|_p\right) \;\neq\; 0 \;, \qquad m((S, z^2)_p) \;\neq\; 0 \;, \tag{3.34}$$

i.e., $(S, z^2, \ldots, z^N)$ *is a local coordinate system.*

Induction proof for stratification lemma for seed 1: By a rigid affine transformation $v^1 := az^1 + b$, we may assume that $(v^1, v^1)_p = \pm 1$. By shifting the other coordinates

$$v^A \;=\; z^A - c^A v^1 \;, \qquad A \;\in\; \{2, \ldots, N\} \;, \tag{3.35}$$

by an appropriate constant multipla of v^1, we may assume that

$$(v^A, v^1)_p \;=\; 0, \qquad A \;\in\; \{2, \ldots, N\} \;. \tag{3.36}$$

After a possible translation, we may assume that the point p is at the origin.

Induction assumption: There is a Grassmann-odd variable

$$Q \;=\; v^1 + Q_{[2]} \;, \qquad Q_{[2]} \;=\; \mathcal{O}((v)^2) \;, \tag{3.37}$$

such that

$$(Q, Q) \;=\; \pm 1 + B_{[k]} \;, \qquad B_{[k]} \;=\; \mathcal{O}((v)^k) \;, \tag{3.38}$$

for some integer $k \in \mathbb{N}$.

We now change the variable

$$Q' \;=\; Q + f_{[k+1]}\;, \qquad f_{[k+1]} \;:=\; \mp\frac{1}{2}v^1 B_{[k]} \;=\; \mathcal{O}((v)^{k+1})\;. \tag{3.39}$$

Then

$$(Q',\cdot) \;=\; (v^1 + Q_{[2]} + f_{[k+1]}, v^A)\frac{\overrightarrow{\partial^\ell}}{\partial v^A} \;=\; \pm\frac{\overrightarrow{\partial^\ell}}{\partial v^1} + \mathcal{O}((v)^0)\;, \tag{3.40}$$

and

$$\begin{aligned}(Q',Q') \;&=\; (Q,Q) + 2(v^1 + Q_{[2]}, v^A)(\frac{\overrightarrow{\partial^\ell}}{\partial v^A} f_{[k+1]}) + (f_{[k+1]}, f_{[k+1]}) \\ &=\; \pm 1 + B_{[k]} \pm 2(\frac{\overrightarrow{\partial^\ell}}{\partial v^1} f_{[k+1]}) + \mathcal{O}((v)^{k+1}) && (3.41)\\ &\overset{(3.39)+(3.43)}{=}\; \pm 1 + \mathcal{O}((v)^{k+1})\;. && (3.42)\end{aligned}$$

The last equality completes the induction step. It follows with the help of the Jacobi identity

$$0 \overset{(3.22)}{=} (Q',(Q',Q')) \overset{(3.40)+(3.41)}{=} \pm(\frac{\overrightarrow{\partial^\ell}}{\partial v^1} B_{[k]}) + \mathcal{O}((v)^k)\;. \tag{3.43}$$

□

Induction proof for stratification lemma for seed 2: There are two remaining possibilities, cf. eq. (3.30). In the first case, the Poisson bracket, z^1 and z^2 are all Grassmann-even. Then the lemma is trivial. Assume from now on the second case: The Poisson bracket and z^2 are both Grassmann-odd.

By shifting the coordinates

$$v^A \;=\; z^A - c^A z^2, \qquad A \;\in\; \{1,\ldots,N\}\backslash\{2\}\;, \tag{3.44}$$

by an appropriate constant multipla of z^2, we may assume that

$$(v^A, v^1)_p \;=\; 0, \qquad A \;\in\; \{1,\ldots,N\}\backslash\{2\}\;. \tag{3.45}$$

(Note that eq. (3.45) with index $A=1$ remains linear in $c^{A=1}$ because we excluded the case of remark 3.8.) By a rigid affine transformation $v^2 := az^2 + b$, we may assume that $(v^1,v^2)_p = 1$. After possible translation, we may assume that the point p is at the origin.

Induction assumption: There is a variable

$$S \;=\; v^1 + S_{[2]}\;, \qquad S_{[2]} \;=\; \mathcal{O}((v)^2)\;, \tag{3.46}$$

such that

$$(S,S) \;=\; \mathcal{O}((v)^k)\;, \tag{3.47}$$

for some integer $k \in \mathbb{N}$.

We now change the variable

$$S' \;=\; S + f_{[k+1]}\;, \qquad f_{[k+1]} \;:=\; -\frac{1}{2}v^2(S,S) \;=\; \mathcal{O}((v)^{k+1})\;. \tag{3.48}$$

Then

$$(S',\cdot) \;=\; (v^1+S_{[2]}+f_{[k+1]},v^A)\frac{\overrightarrow{\partial^\ell}}{\partial v^A} \;=\; \frac{\overrightarrow{\partial^\ell}}{\partial v^2}+\mathcal{O}((v)^0)\;, \tag{3.49}$$

and

$$\begin{aligned}(S',S') \quad &= \quad (S,S)+2(v^1+S_{[2]},v^A)(\frac{\overrightarrow{\partial^\ell}}{\partial v^A}f_{[k+1]})+(f_{[k+1]},f_{[k+1]}) \\ &= \quad (S,S)+2(\frac{\overrightarrow{\partial^\ell}}{\partial v^2}f_{[k+1]})+\mathcal{O}((v)^{k+1}) &(3.50)\\ &\overset{(3.48)+(3.52)}{=} \quad \mathcal{O}((v)^{k+1})\;. &(3.51)\end{aligned}$$

The last equality completes the induction step. It follows with the help of the Jacobi identity

$$0 \overset{(3.22)}{=} (S',(S',S')) \overset{(3.49)+(3.50)}{=} (\frac{\overrightarrow{\partial^\ell}}{\partial v^2}(S,S))+\mathcal{O}((v)^k)\;. \tag{3.52}$$

□

3.7 Darboux Theorem via Weinstein Splitting Method

In this subsection we prove a superversion of Weinstein's splitting proof [58, 59] for Poisson manifolds, namely theorem 3.12. We believe this is a novel result. Most super Darboux theorems in the literature only discuss the non-generate symplectic case [39, 60]. In particular, we allow that the rank of the Poisson structure can jump, *i.e.*, the Poisson structure need not be regular.

Theorem 3.12 (Super Darboux theorem) *Given a $(n|m)$-dimensional supermanifold (M,E) with a Poisson structure E of (internal) Grassmann-parity ε_E. Let $N:=n+m$. If the rank of the Poisson tensor E_p at a point $p\in M$ is r, then there exists a local coordinate system $(z^1,z^2,\ldots,z^N)$ around the point p, such that the first r coordinates $(z^1,z^2,\ldots,z^r)$ are Darboux coordinates, i.e., the sub-matrix $(E^{AB})_{1\leq A,B\leq r}$ is a constant invertible matrix.*

Remark 3.13 *To give a full proof of Darboux theorem 3.12, it is enough to consider seed 1 and 2.*

Splitting proof of Darboux theorem 3.12 with seed 1: There exists a Grassmann-odd coordinate v^1 such that

$$(v^1,v^1) \;=\; \pm 1\;, \tag{3.53}$$

and such that

$$m\left(\left.\frac{\overrightarrow{\partial^\ell}}{\partial z^1}v^1\right|_p\right) \;\neq\; 0\;, \tag{3.54}$$

and so $(v^1,z^2,\ldots,z^N)$ is a coordinate system, cf. the stratification lemma 3.10.

Next $X:=(v^1,\cdot)$ is a self-(super)commuting left Hamiltonian vector field. It has non-zero rank in the point p because of eq. (3.29). Then there exists a new coordinate system $(w^1,w^2,\ldots,w^N)$, such that

$$\frac{\overrightarrow{\partial^\ell}}{\partial w^1} \;=\; X \;=\; (v^1,\cdot) \tag{3.55}$$

because of the stratification theorem 3.1. Then

$$(v^1, w^{A\geq 2}) \ = \ 0 \ . \tag{3.56}$$

Then

$$(\frac{\overrightarrow{\partial^\ell}}{\partial w^1}v^1) \ = \ X[v^1] \ = \ (v^1, v^1) \ = \ \pm 1 \ . \tag{3.57}$$

So $(v^1, w^2, \ldots, w^N)$ is also a coordinate system. From the Jacobi identity

$$\begin{aligned} 0 \stackrel{(3.22)+(3.56)}{=} (v^1, (w^A, w^B)) \ &= \ \underbrace{(v^1, v^1)}_{=\pm 1} \frac{\overrightarrow{\partial^\ell}}{\partial v^1}(w^A, w^B) + \sum_{C\geq 2} \underbrace{(v^1, w^C)}_{=0} \frac{\overrightarrow{\partial^\ell}}{\partial w^C}(w^A, w^B) \\ &\stackrel{(3.53)+(3.56)}{=} \ \pm \frac{\overrightarrow{\partial^\ell}}{\partial v^1}(w^A, w^B) \ , \qquad A, B \ \in \ \{2, \ldots, N\} \ , \end{aligned} \tag{3.58}$$

we see that (w^A, w^B), $A, B \in \{2, \ldots, N\}$, cannot depend on v^1. In conclusion, the coordinate v^1 is a decoupled Darboux coordinate. So we are left with the same problem in one less dimension.

□

Splitting proof of Darboux theorem 3.12 with seed 2 (excluding the case in remark 3.8): According to the stratification lemma 3.11, there exists a new coordinate system $(u^1, u^2, \ldots, u^N)$, where u^1 is a Boson, such that

$$(u^1, u^1) \ = \ 0 \ , \qquad m\left((u^1, u^2)_p\right) \ \neq \ 0 \ . \tag{3.59}$$

The left Hamiltonian vector field $X := (u^1, \cdot)$ is self-(super)commuting because of the stratification lemma 3.11. It has non-zero rank in the point p because of eq. (3.30). Then there exists a new coordinate system $(v^1, v^2, \ldots, v^N)$, such that

$$\frac{\overrightarrow{\partial^\ell}}{\partial v^2} \ = \ X \ = \ (u^1, \cdot) \tag{3.60}$$

because of the stratification theorem 3.1. Since $\varepsilon(v^2) = \varepsilon_E$, then

$$(v^2, v^2) \ = \ 0 \tag{3.61}$$

follows from symmetry (3.20). Then

$$(u^1, v^2) \ = \ X[v^2] \ = \ \frac{\overrightarrow{\partial^\ell}}{\partial v^2}v^2 \ = \ 1 \ . \tag{3.62}$$

There should exists a coordinate v^A, $A \in \{1, \ldots, N\}$, such that

$$m\left(\left. \frac{\overrightarrow{\partial^\ell}}{\partial v^A}u^1 \right|_p \right) \ \neq \ 0 \ . \tag{3.63}$$

It cannot be v^2 because

$$\frac{\overrightarrow{\partial^\ell}}{\partial v^2}u^1 \ = \ X[u^1] \ = \ (u^1, u^1) \ = \ 0 \ . \tag{3.64}$$

We can w.l.o.g. rename v^A as v^1. So $(u^1, v^2, \ldots, v^N)$ is a coordinate system.

Consider next the left Hamiltonian vector field $Y := (v^2, \cdot)$. It super-commutes with Y and X because of eqs. (3.61) and (3.62). So by the Abelian Frobenius theorem 3.2 there exist coordinates $(w^1, \ldots, w^N)$, such that

$$\frac{\overrightarrow{\partial^\ell}}{\partial w^2} \;=\; X \;:=\; (u^1, \cdot) \;, \qquad \frac{\overrightarrow{\partial^\ell}}{\partial w^1} \;=\; Y \;:=\; (v^2, \cdot) \;. \tag{3.65}$$

Then

$$(u^1, w^{A\geq 3}) \;=\; 0 \;, \qquad (v^2, w^{A\geq 3}) \;=\; 0 \;. \tag{3.66}$$

One may show that for fixed $(w^3, \ldots, w^N)$, the map $(w^1, w^2) \mapsto (u^1, v^2)$ is locally bijective (because the Jacobian matrix is bijective). Hence $(u^1, v^2, w^3, \ldots, w^N)$ is a coordinate system.

From the Jacobi identity

$$\begin{aligned} 0 \;&\stackrel{(3.22)+(3.66)}{=}\; (u^1, (w^A, w^B)) \\ &=\; \underbrace{(u^1, u^1)}_{=0} \frac{\overrightarrow{\partial^\ell}}{\partial u^1}(w^A, w^B) + \underbrace{(u^1, v^2)}_{=1} \frac{\overrightarrow{\partial^\ell}}{\partial v^2}(w^A, w^B) + \sum_{C\geq 3} \underbrace{(u^1, w^C)}_{=0} \frac{\overrightarrow{\partial^\ell}}{\partial w^C}(w^A, w^B) \\ &\stackrel{(3.62)+(3.66)}{=}\; \frac{\overrightarrow{\partial^\ell}}{\partial v^2}(w^A, w^B), \qquad A, B \;\in\; \{3, \ldots, N\} \;, \end{aligned} \tag{3.67}$$

$$\begin{aligned} 0 \;&\stackrel{(3.22)+(3.66)}{=}\; (v^2, (w^A, w^B)) \\ &=\; \underbrace{(v^2, u^1)}_{=\pm 1} \frac{\overrightarrow{\partial^\ell}}{\partial u^1}(w^A, w^B) + (v^2, v^2) \underbrace{\frac{\overrightarrow{\partial^\ell}}{\partial v^2}(w^A, w^B)}_{=0} + \sum_{C\geq 3} \underbrace{(v^2, w^C)}_{=0} \frac{\overrightarrow{\partial^\ell}}{\partial w^C}(w^A, w^B) \\ &\stackrel{(3.62)+(3.66)+(3.67)}{=}\; \pm\frac{\overrightarrow{\partial^\ell}}{\partial u^1}(w^A, w^B), \qquad A, B \;\in\; \{3, \ldots, N\} \;, \end{aligned} \tag{3.68}$$

we see that (w^A, w^B), $A, B \in \{3, \ldots, N\}$, cannot depend on u^1 and v^2. In conclusion, the coordinates u^1 and v^2 are decoupled Darboux coordinates. So we are left with the same problem in two less dimensions.

□

3.8 Regular Poisson Structure

Definition 3.14 *A Poisson manifold (M, E) is called* **regular** *if the Poisson structure E has constant rank, i.e., the rank r does not jump.*

Theorem 3.15 (Darboux theorem for regular Poisson manifolds) *A regular N-dimensional Poisson manifold (M, E) has an atlas of Darboux coordinates*

$$\{z^1, \ldots, z^N\} \;=\; \{x^1, \ldots, x^r, c^1, \ldots, c^{N-r}\} \tag{3.69}$$

such that the sub-block

$$(x^A, x^B)_{1\leq A,B\leq r} \tag{3.70}$$

is constant and invertible, while c^A are Casimir coordinates

$$(c^A, \cdot) \;=\; 0 \;, \qquad A \;\in\; \{1, \ldots, N-r\} \;. \tag{3.71}$$

Remark 3.16 *One may show that under a coordinate transformation $z^A \to z'^B$ between two Darboux charts,*

$$\frac{\overrightarrow{\partial^{\ell}}}{\partial x^A} c'^B \;=\; 0 \; . \tag{3.72}$$

This shows that the new Casimir coordinates c'^B are a function of only the old Casimir coordinates c'^A, i.e., we have an r-dimensional foliation with r-dimensional symplectic leaves. In other words, the Hamiltonian vector fields form an r-dimensional integrable distribution, cf. the Frobenius theorem 3.3.

3.9 Poisson Structure with Compatible 2-form

Definition 3.17 *A globally defined 2-form E_{AB} of a Poisson manifold (M, E) with same internal Grassmann parity ε_E is called* **compatible** *if*

$$E^{AB} E_{BC} E^{CD} \;=\; E^{AD} \; , \tag{3.73}$$

$$E_{AB} E^{BC} E_{CD} \;=\; E_{AD} \; . \tag{3.74}$$

The existence of a compatible 2-form is a relatively mild requirement. However, there could be global obstructions. The existence of a compatible 2-form is always automatically satisfied for a Dirac bracket on symplectic manifolds with globally defined second-class constraints [29, 34, 41, 43] (paper II). Note that the 2-form E_{AB} is neither unique nor necessarily closed.

Obviously, an antisymplectic structure E^{AB} has always a unique compatible 2-form, namely its inverse structure E_{AB}.

One can define a $(1,1)$ tensor field as

$$P^A{}_C \;\equiv\; E^{AB} E_{BC} \; , \tag{3.75}$$

or equivalently,

$$P_A{}^C \;\equiv\; E_{AB} E^{BC} \;=\; (-1)^{\varepsilon_A(\varepsilon_C+1)} P^C{}_A \; . \tag{3.76}$$

It then follows from either of the compatibility relations (3.73) and (3.74) that $P^A{}_B$ is an idempotent

$$P^A{}_B P^B{}_C \;=\; P^A{}_C \; . \tag{3.77}$$

4 Batalin-Vilkovisky Geometry

Definition 4.1 *A* **BV manifold** (M,E,ρ) *is an anti-Poisson manifold* (M,E) *with odd internal Grassmann parity* $\varepsilon_E = 1$ *equipped with a density* ρ.

4.1 Scalars, Densities and Semidensities

Recall that a scalar function $f=f(z)$, a density $\rho=\rho(z)$ and a semidensity $\sigma=\sigma(z)$ are by definition quantities that transform as

$$f \longrightarrow f' = f , \qquad \rho \longrightarrow \rho' = \frac{\rho}{J} , \qquad \sigma \longrightarrow \sigma' = \frac{\sigma}{\sqrt{J}} , \tag{4.1}$$

respectively, under general coordinate transformations $z^A \to z'^A$, where $J \equiv \mathrm{sdet}\frac{\partial z'^A}{\partial z^B}$ denotes the Jacobian.

We shall ignore the global issues of orientability of M and the choice of square root for semidensities.

In principle the above f, ρ and σ could either be Bosons or Fermions, however normally we shall require the densities ρ to be invertible, and therefore Bosons.

The fundamental object in BV geometry is the nilpotent Δ operator, but first we should introduce the odd Laplacian Δ_F.

4.2 Is there a Canonical Density on an Antisymplectic Manifold?

Recall that for an even symplectic manifold with an even symplectic two-form $E = \frac{1}{2}dz^A E_{AB} dz^B$, there exists a canonical density given by the Pfaffian $\rho = \mathrm{Pf}(E_{AB})$, *i.e.*, there is a natural notion of volume on an even symplectic manifold. A related fact is the Liouville Theorem for even symplectic manifolds, which states that Hamiltonian vector fields are divergenceless.

On the other hand, the situation is completely different for an odd symplectic manifold endowed with an odd antisymplectic two-form $E = \frac{1}{2}dz^A E_{AB} dz^B$. It turns out that there is *no* canonical choice of density ρ in this case, as, for instance, the above Pfaffian. This is tied to the fact that there is no meaningful notion of a superdeterminant/Berezinian for a matrix that is intrinsically Grassmann-odd. However, the upset runs deeper. In fact, a density ρ can never be a function of the antisymplectic matrix E_{AB}. Phrased differently, a density ρ always carries information that cannot be deduced from the antisymplectic structure E alone [35].

Example 4.2 *Consider* $\mathbb{R}^{1|1}$ *endowed with the antisymplectic* 2*-form* $E = \mathrm{d}x \wedge \mathrm{d}\theta$ *and* $\rho = 1$. *The antisymplectic structure is invariant under an anti-canonical transformation*

$$x' = \frac{1}{2}x, \qquad \theta' = 2\theta , \tag{4.2}$$

but the density $\rho' = 4$ *in the new coordinates is now 4 times bigger!*

4.3 The odd Laplacians Δ_ρ and Δ_F on Scalars

Given a choice of density ρ, one may define the odd Laplacian [29]

$$\Delta_\rho := \frac{(-1)^{\varepsilon_A}}{2\rho} \overrightarrow{\partial}^{\ell}_A \rho E^{AB} \overrightarrow{\partial}^{\ell}_B , \qquad \overrightarrow{\partial}^{\ell}_A \equiv \frac{\overrightarrow{\partial}^{\ell}}{\partial z^A} , \tag{4.3}$$

that takes scalars to scalars of opposite Grassmann parity.

A natural generalization of the odd Laplacian (4.3) is [44] (paper III)

$$\Delta_F \;\equiv\; \frac{(-1)^{\varepsilon_A}}{2}(\overrightarrow{\partial^{\ell}_A}+F_A)E^{AB}\overrightarrow{\partial^{\ell}_B} \; . \tag{4.4}$$

where $F_A = F_A(z)$ is a line bundle connection with Grassmann parity $\varepsilon(F_A) \;=\; \varepsilon_A$. A line bundle connection F_A transforms under general coordinate transformations $z^A \to z'^B$ as

$$F_A \;=\; (\overrightarrow{\partial^{\ell}_A}z'^B)F'_B + (\overrightarrow{\partial^{\ell}_A}\ln J) \; , \qquad J \;\equiv\; \mathrm{sdet}\frac{\partial z'^B}{\partial z^A} \; . \tag{4.5}$$

This transformation property (4.5) guarantees that the expression (4.4) remains invariant under general coordinate transformations.

The curvature tensor for the line bundle connection F_A is

$$\mathcal{R}_{AB} \;\equiv\; [\overrightarrow{\partial^{\ell}_A}+F_A, \overrightarrow{\partial^{\ell}_B}+F_B] \;=\; (\overrightarrow{\partial^{\ell}_A}F_B) - (-1)^{\varepsilon_A\varepsilon_B}(A \leftrightarrow B) \; . \tag{4.6}$$

The line bundle is called flat if the curvature vanishes

$$\mathcal{R}_{AB} \;=\; 0 \; . \tag{4.7}$$

The flatness condition (4.7) is an integrability conditions for the local existence of ρ. The odd Laplacian Δ_F reduces to Δ_ρ for a flat line bundle of the form

$$F_A \;=\; (\overrightarrow{\partial^{\ell}_A}\ln\rho) \; . \tag{4.8}$$

Recall that the super-commutator of an n-order differential operator and an m-order differential operator is at most an $(n+m-1)$-order differential operator. Because the odd Laplacian Δ_F is a second-order differential operator, it follows that the square operator

$$\Delta_F^2 \;=\; \frac{1}{2}[\Delta_F, \Delta_F] \tag{4.9}$$

(which happens to be half the supercommutator) is at most a third-order operator. The vanishing of the third-order terms is equivalent to the Jacobi identity (3.22), so Δ_F^2 is actually at most a second-order operator. It can also not have any zero-order terms, since its expression (4.4) has a derivative to the right.

The vanishing of the second-order terms in Δ_F^2 means that that the curvature tensor (4.6) (projected into the range of the anti-Poisson structure) vanish [44] (paper III)

$$\mathcal{R}^{AD} \;\equiv\; E^{AB}\mathcal{R}_{BC}E^{CD}(-1)^{\varepsilon_C} \;=\; 0 \; . \tag{4.10}$$

There exists another descriptive characterization: The second-order terms of Δ_F^2 vanish if and only if there is a Leibniz rule for the interplay of the so-called “one-bracket” Δ_F and the “two-bracket” $(\cdot,\cdot)$

$$\Delta_F(f,g) \;=\; (\Delta_F f, g) - (-1)^{\varepsilon_f}(f, \Delta_F g) \; . \tag{4.11}$$

See Refs. [16, 20] for more details.

For the above reasons, we will from now on make the following equivalent assumption 4.3.

Assumption 4.3 (Modular vector field) *The square (4.9) of the odd Laplacian Δ_F is a linear derivation, i.e., a first-order differential operator, or equivalently, a vector field [37]*

$$\Delta_F^2(fg) \; = \; \Delta_F^2(f)\; g + f\; \Delta_F^2(g)\; . \qquad (4.12)$$

This assumption 4.3 is automatically satisfied for a flat line bundle.

Conventionally, one imposes additionally that the Δ_F operator should be nilpotent $\Delta_F^2 = 0$. However, it is one of the main points of this thesis that this is not necessary, cf. eq. (4.58) below.

The odd Laplacian (4.4) has a geometric interpretation as a divergence of a Hamiltonian vector field [25, 28]

$$\Delta_F \Psi \; = \; -\frac{1}{2}\mathrm{div}_F(X_\Psi)\; , \qquad \varepsilon(\Psi) \; = \; 1\; . \qquad (4.13)$$

Here $X_\Psi := (\Psi, \cdot)$ denotes a Hamiltonian vector field with a Grassmann-odd Hamiltonian Ψ, and the divergence $\mathrm{div}_F X$ of a vector field X, with respect to the density ρ, is

$$\mathrm{div}_F X \; := \; (-1)^{\varepsilon_A}((\overrightarrow{\partial^\ell_A} + F_A)X^A)\; , \qquad \varepsilon(X) \; = \; 0\; . \qquad (4.14)$$

The fact that the odd Laplacian (4.13) is non-zero, shows that antisymplectic manifolds do not have an analogue of the Liouville Theorem, cf. subsection 4.2.

4.4 Khudaverdian's Δ_E Operator on Semidensities

Khudaverdian [39] only considered the antisymplectic case, but we show here that his construction actually works for any regular anti-Poisson manifold.

Khudaverdian showed that one may define a Grassmann-odd, nilpotent, second-order operator Δ_E *without* a choice of density ρ. This Δ_E operator does not take scalars to scalars like the odd Laplacian (4.3), but instead takes semidensities to semidensities of opposite Grassmann parity. Equivalently, the Δ_E operator transforms as

$$\Delta_E \; \longrightarrow \; \Delta'_E \; = \; \frac{1}{\sqrt{J}}\Delta_E\sqrt{J} \qquad (4.15)$$

under general coordinate transformations $z^A \to z'^B$, cf. eq. (4.1). Khudaverdian's construction relies first of all on an atlas of Darboux charts, which is granted by the Darboux Theorem 3.15, and secondly, on a Lemma by Batalin and Vilkovisky about the possible form of the Jacobians for anticanonical transformations, also known as antisymplectomorphisms.

Lemma 4.4 (The Batalin-Vilkovisky Lemma) *[3, 6, 37, 38, 39, 41]. Consider a finite anti-canonical transformation between initial Darboux coordinates $z^A_{(i)}$ and final Darboux coordinates $z^A_{(f)}$. Then the Jacobian $J \equiv \mathrm{sdet}(\partial z^A_{(f)}/\partial z^B_{(i)})$ satisfies*

$$\Delta_1^{(i)}\sqrt{J} \; = \; 0\; . \qquad (4.16)$$

Here $\Delta_1^{(i)}$ refers to the odd Laplacian (4.3) with $\rho{=}1$ in the initial Darboux coordinates $z^A_{(i)}$.

A simple proof of the Batalin-Vilkovisky Lemma for finite anticanonical transformations can be found in Ref. [41].

Definition 4.5 (The Δ_E operator in Darboux coordinates) *Given Darboux coordinates z^A, the Δ_E operator is defined on a semidensity σ as [36, 37, 38, 39, 41]*

$$(\Delta_E \sigma) \ := \ (\Delta_1 \sigma) \ , \tag{4.17}$$

where Δ_1 is the Δ_ρ operator (4.3) with $\rho = 1$.

Remark 4.6 *It is important in eq. (4.17) that the formula for the Δ_1 operator (4.3) and the semidensity σ both refer to the same Darboux coordinates z^A. The parentheses in eq. (4.17) indicate that the equation should be understood as an equality among semidensities (in the sense of zeroth-order differential operators) rather than an identity among differential operators.*

Theorem 4.7 *The Δ_E operator (4.17) does not depend on choice of Darboux coordinates z^A, and it takes semidensities to semidensities. Concretely, this means in terms of formulas, that the right–hand side of the definition (4.17) transforms as a semidensity*

$$(\Delta_1^{(f)} \sigma_{(f)}) \ = \ \frac{1}{\sqrt{J}} (\Delta_1^{(i)} \sigma_{(i)}) \tag{4.18}$$

under an anticanonical transformation between any two Darboux coordinates $z^A_{(i)}$ and $z^A_{(f)}$.

Finite transformation proof of theorem 4.7: One uses the Batalin-Vilkovisky Lemma to argue that the definition (4.17) does not depend on the choices of Darboux coordinates z^A. One calculates:

$$\sqrt{J}(\Delta_1^{(f)} \sigma_{(f)}) \ = \ \sqrt{J}(\Delta_J^{(i)} \sigma_{(f)}) \ = \ \sqrt{J}(\Delta_J^{(i)} \frac{\sigma_{(i)}}{\sqrt{J}}) \ = \ (\Delta_1^{(i)} \sigma_{(i)}) - \frac{1}{\sqrt{J}}(\Delta_1^{(i)} \sqrt{J}) \sigma_{(i)} \ = \ (\Delta_1^{(i)} \sigma_{(i)}) \ . \tag{4.19}$$

The third equality is a non-trivial property of the odd Laplacian (4.3). The Batalin-Vilkovisky Lemma is used in the fourth equality.

□

Infinitesimal transformation proof of theorem 4.7: Strictly speaking, it is enough to consider infinitesimal anticanonical transformations to justify the definition (4.17). The proof of the infinitesimal version of the Batalin-Vilkovisky Lemma goes like this: An infinitesimal anticanonical coordinate transformation $\delta z^A = X^A$ is generated by a Grassmann-even vector field $X = X^A \overrightarrow{\partial_A^\ell}$ that preserves the antibracket

$$X[(f,g)] \ = \ (X[f],g) + (f, X[g]) \ , \tag{4.20}$$

which implies that

$$E^{AC}(\overrightarrow{\partial_C^\ell} X^B) \ \equiv \ (z^A, X^B) \ = \ -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}(A \leftrightarrow B) \ . \tag{4.21}$$

So the Jacobian becomes

$$\ln J \ \approx \ (-1)^{\varepsilon_A}(\overrightarrow{\partial_A^\ell} X^A) \ = \ \mathrm{div}_1(X) \ , \tag{4.22}$$

and hence

$$\Delta_1 \sqrt{J} \ \approx \ \frac{1}{2}\Delta_1 \mathrm{div}_1(X) \ = \ \frac{(-1)^{\varepsilon_A+\varepsilon_B}}{4}(\overrightarrow{\partial_A^\ell} E^{AC} \overrightarrow{\partial_C^\ell} \overrightarrow{\partial_B^\ell} X^B) \ \overset{(4.21)}{=} \ 0 \ . \tag{4.23}$$

Here we used repeatedly that E^{AB} is constant in Darboux coordinates. The "$\approx$" sign is used to indicate that equality only holds at the infinitesimal level. (Here we are guilty of mixing active and passive pictures; the active vector field is properly speaking *minus* X.)

□

Theorem 4.8 (Nilpotency) *Given an atlas of Darboux coordinates, the Δ_E operator (4.17) is nilpotent,*

$$\Delta_E^2 \;=\; \frac{1}{2}[\Delta_E, \Delta_E] \;=\; 0 \;, \tag{4.24}$$

i.e., it squares to zero, or equivalently, it super-commutes with itself.

PROOF OF THEOREM 4.8: The Δ_E operator super-commutes with itself, because the z^A-derivatives have no z^A's to act on in Darboux coordinates.

□

4.5 The Odd Scalar

The plan is now to define the Δ_E operator in arbitrary coordinates, but first we need to define an odd scalar.

Theorem 4.9 (Odd Scalar) *[43] (paper II). Given a (not necessarily flat) line bundle connection F_A on an anti-Poisson manifold (M, E) with a compatible 2-form, then the following Grassmann-odd quantity is a scalar:*

$$\nu_F \;:=\; \nu_F^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{8} - \frac{\nu^{(3)}}{24} + \frac{\nu^{(4)}}{24} + \frac{\nu^{(5)}}{12} \;, \tag{4.25}$$

where

$$\nu_F^{(0)} \;:=\; \frac{(-1)^{\varepsilon_A}}{4}(\overrightarrow{\partial^{\ell}_A} + \frac{F_A}{2})(E^{AB}F_B) \;, \tag{4.26}$$

$$\nu^{(1)} \;:=\; (-1)^{\varepsilon_A}(\overrightarrow{\partial^{\ell}_B}\overrightarrow{\partial^{\ell}_A}E^{AB}) \;, \tag{4.27}$$

$$\nu^{(2)} \;:=\; (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^{\ell}_D}E^{AB})E_{BC}(\overrightarrow{\partial^{\ell}_A}E^{CD}) \;, \tag{4.28}$$

$$\nu^{(3)} \;:=\; (-1)^{\varepsilon_B}(\overrightarrow{\partial^{\ell}_A}E_{BC})E^{CD}(\overrightarrow{\partial^{\ell}_D}E^{BA}) \;, \tag{4.29}$$

$$\nu^{(4)} \;:=\; (-1)^{\varepsilon_B}(\overrightarrow{\partial^{\ell}_A}E_{BC})E^{CD}(\overrightarrow{\partial^{\ell}_D}E^{BF})P_F{}^A \;, \tag{4.30}$$

$$\begin{aligned}\nu^{(5)} \;&:=\; (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^{\ell}_D}E^{AB})E_{BC}(\overrightarrow{\partial^{\ell}_A}E^{CF})P_F{}^D \\ &=\; (-1)^{(\varepsilon_A+1)\varepsilon_B}E^{AD}(\overrightarrow{\partial^{\ell}_D}E^{BC})(\overrightarrow{\partial^{\ell}_C}E_{AF})P^F{}_B \;.\end{aligned} \tag{4.31}$$

Remark 4.10 *Despite this definition (4.25) seems to depend on the choice of compatible 2-form, we shall later see that it has a geometric meaning as odd scalar curvature.*

SKETCHED PROOF OF THEOREM 4.9: Under an arbitrary infinitesimal coordinate transformation $\delta z^A = X^A$, one calculates

$$\delta\nu_F^{(0)} \;=\; -\frac{1}{2}\Delta_1 \mathrm{div}_1 X \;, \tag{4.32}$$

$$\delta\nu^{(1)} \;=\; 4\Delta_1 \mathrm{div}_1 X + (-1)^{\varepsilon_A}(\overrightarrow{\partial^{\ell}_C}E^{AB})(\overrightarrow{\partial^{\ell}_B}\overrightarrow{\partial^{\ell}_A}X^C) \;, \tag{4.33}$$

$$\delta\nu^{(2)} \;=\; (-1)^{\varepsilon_A}(\overrightarrow{\partial^{\ell}_D}E^{AB})\left(2P_B{}^C(\overrightarrow{\partial^{\ell}_C}\overrightarrow{\partial^{\ell}_A}X^D) + (\overrightarrow{\partial^{\ell}_B}\overrightarrow{\partial^{\ell}_A}X^C)P_C{}^D\right) \;, \tag{4.34}$$

$$\delta\nu^{(3)} \;=\; (-1)^{\varepsilon_B}(\overrightarrow{\partial^{\ell}_A}E_{BC})E^{CD}\left((\overrightarrow{\partial^{\ell}_D}X^B\overleftarrow{\partial^{r}_F})E^{FA} - (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}(\overrightarrow{\partial^{\ell}_D}X^A\overleftarrow{\partial^{r}_F})E^{FB}\right)$$

$$-\frac{3}{2}(-1)^{\varepsilon_A} P_C{}^D (\overrightarrow{\partial^{\ell}_D} E^{AB})(\overrightarrow{\partial^{\ell}_B}\overrightarrow{\partial^{\ell}_A} X^C) \ , \tag{4.35}$$

$$\begin{aligned} \delta\nu^{(4)} \ = \ & -2(-1)^{\varepsilon_B}(\overrightarrow{\partial^{\ell}_A}\overrightarrow{\partial^{\ell}_B} X^C) P_C{}^D (\overrightarrow{\partial^{\ell}_D} E^{BF}) P_F{}^A \\ & +(-1)^{\varepsilon_B}(\overrightarrow{\partial^{\ell}_A} E_{BC}) E^{CD} (\overrightarrow{\partial^{\ell}_D} X^B \overleftarrow{\partial^r_F}) E^{FA} \\ & +(-1)^{(\varepsilon_B+1)\varepsilon_F} P_F{}^A (\overrightarrow{\partial^{\ell}_A} E_{BC}) E^{CD} (\overrightarrow{\partial^{\ell}_D} X^F \overleftarrow{\partial^r_G}) E^{GB} \\ & +\frac{1}{2}(-1)^{\varepsilon_A} P_C{}^D (\overrightarrow{\partial^{\ell}_D} E^{AB})(\overrightarrow{\partial^{\ell}_B}\overrightarrow{\partial^{\ell}_A} X^C) \ , \end{aligned} \tag{4.36}$$

$$\begin{aligned} \delta\nu^{(5)} \ = \ & -(-1)^{\varepsilon_A(\varepsilon_B+1)}(\overrightarrow{\partial^{\ell}_A} E_{BC}) E^{CD} (\overrightarrow{\partial^{\ell}_D} X^A \overleftarrow{\partial^r_F}) E^{FB} \\ & +2(-1)^{\varepsilon_B}(\overrightarrow{\partial^{\ell}_A}\overrightarrow{\partial^{\ell}_B} X^C) P_C{}^D (\overrightarrow{\partial^{\ell}_D} E^{BF}) P_F{}^A \ . \end{aligned} \tag{4.37}$$

A proof of eqs. (4.32) and (4.33) can be found in Ref. [42], and eqs. (4.34)–(4.37) are proven in Ref. [43]. One may verify that while the six constituents $\nu^{(0)}_F$, $\nu^{(1)}$, $\nu^{(2)}$, $\nu^{(3)}$, $\nu^{(4)}$ and $\nu^{(5)}$ separately have non-trivial transformation properties, the linear combination ν_F in eq. (4.25) is indeed a scalar.

□

Lemma 4.11 *On an antisymplectic manifold (M,E), we have*

$$\nu^{(5)} \ = \ \nu^{(2)} \ = \ -\nu^{(3)} \ = \ -\nu^{(4)} \ . \tag{4.38}$$

Proof of lemma 4.11: Straightforward calculation.

□

Because of lemma 4.11, the theorem 4.9 simplifies in the antisymplectic case.

Theorem 4.12 (Odd Scalar) *[42] (paper I). Given a (not necessarily flat) line bundle connection F_A on an antisymplectic manifold (M,E), then the following Grassmann-odd quantity is a scalar:*

$$\nu_F \ := \ \nu^{(0)}_F + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24} \ , \tag{4.39}$$

where

$$\nu^{(0)}_F \ := \ \frac{(-1)^{\varepsilon_A}}{4}(\overrightarrow{\partial^{\ell}_A} + \frac{F_A}{2})(E^{AB} F_B) \ , \tag{4.40}$$

$$\nu^{(1)} \ := \ (-1)^{\varepsilon_A}(\overrightarrow{\partial^{\ell}_B}\overrightarrow{\partial^{\ell}_A} E^{AB}) \ , \tag{4.41}$$

$$\nu^{(2)} \ := \ -(-1)^{\varepsilon_B}(z^C,(z^B,z^A))(\overrightarrow{\partial^{\ell}_A} E_{BC}) \ = \ (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^{\ell}_A} E^{CD})(\overrightarrow{\partial^{\ell}_D} E^{AB}) E_{BC} \ . \tag{4.42}$$

Sketched proof of theorem 4.12: One should check that ν_F is a scalar under general infinitesimal coordinate transformations. Under an arbitrary infinitesimal coordinate transformation $\delta z^A = X^A$, one calculates [42] (paper I)

$$\delta\nu^{(0)}_F \ = \ -\frac{1}{2}\Delta_1 \mathrm{div}_1 X \ , \tag{4.43}$$

$$\delta\nu^{(1)} \ = \ 4\Delta_1 \mathrm{div}_1 X + (-1)^{\epsilon_A}(\overrightarrow{\partial^{\ell}_C} E^{AB})(\overrightarrow{\partial^{\ell}_B}\overrightarrow{\partial^{\ell}_A} X^C) \ , \tag{4.44}$$

$$\delta\nu^{(2)} \ = \ 3(-1)^{\epsilon_A}(\overrightarrow{\partial^{\ell}_C} E^{AB})(\overrightarrow{\partial^{\ell}_B}\overrightarrow{\partial^{\ell}_A} X^C) \ . \tag{4.45}$$

One easily sees that while the three constituents $\nu_F^{(0)}$, $\nu^{(1)}$ and $\nu^{(2)}$ separately have non-trivial transformation properties, the linear combination ν_F in eq. (4.25) is indeed a scalar.

$\square$

Corollary 4.13 (Odd Scalar) *[42] (paper I). Given a density ρ on an anti-Poisson manifold (M,E) with a compatible 2-form, then the following Grassmann-odd quantity is a scalar:*

$$\nu_\rho \;:=\; \nu_\rho^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{8} - \frac{\nu^{(3)}}{24} + \frac{\nu^{(4)}}{24} + \frac{\nu^{(5)}}{12} \;, \tag{4.46}$$

where

$$\nu_\rho^{(0)} \;:=\; \frac{1}{\sqrt{\rho}}(\Delta_1\sqrt{\rho}) \;. \tag{4.47}$$

Corollary 4.14 (Odd Scalar) *[42] (paper I). Given a density ρ on an antisymplectic manifold (M,E), then the following Grassmann-odd quantity is a scalar:*

$$\nu_\rho \;:=\; \nu_\rho^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24} \;. \tag{4.48}$$

4.6 The Δ_E Operator in General Coordinates

We now give a definition of the Δ_E operator whose definition does not rely on Darboux coordinates. (However, we should emphasize that we have only been able to prove nilpotency by assuming that the Poisson manifold is regular.)

Definition 4.15 (The Δ_E operator in arbitrary coordinates) *[43] (paper II). Given an anti-Poisson manifold (M,E) with a compatible 2-form, then the Δ_E operator is defined on a semidensity σ in an arbitrary coordinate system as*

$$(\Delta_E\sigma) \;:=\; (\Delta_1\sigma) + \left(\frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{8} - \frac{\nu^{(3)}}{24} + \frac{\nu^{(4)}}{24} + \frac{\nu^{(5)}}{12}\right)\sigma \;, \tag{4.49}$$

where Δ_1 is the Δ_ρ operator (4.3) with $\rho{=}1$.

Recalling lemma 4.11, we have the following simplification in the antisymplectic case.

Definition 4.16 (The Δ_E operator in arbitrary coordinates) *[42] (paper I). Given an antisymplectic manifold (M,E), then the Δ_E operator is defined on a semidensity σ in an arbitrary coordinate system as*

$$(\Delta_E\sigma) \;:=\; (\Delta_1\sigma) + \left(\frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24}\right)\sigma \;. \tag{4.50}$$

Remark 4.17 *Notice that in Darboux coordinates, where E^{AB} is constant, i.e., independent of the coordinates z^A, then $\nu^{(1)}$, $\nu^{(2)}$, $\nu^{(3)}$, $\nu^{(4)}$ and $\nu^{(5)}$ vanish. Hence the definitions 4.15 and 4.16 of the Δ_E operator agree with Khudaverdian's definition 4.5.*

Theorem 4.18 *The Δ_E operator (4.49) and (4.50) does not depend on choice of coordinates z^A, and it takes semidensities to semidensities, i.e., the right–hand side of eqs. (4.49) and (4.50) behaves as a semidensity under general coordinate transformations.*

PROOF OF THEOREM 4.18: Here we will only explicitly consider the case where the semidensity σ is invertible to simplify the presentation. (The non-invertible case is fundamentally no different.) In the invertible case, we customarily write the semidensity $\sigma = \sqrt{\rho}$ as a square root of a density ρ. If we divide the definition (4.49) and (4.50) with the square root of a density ρ, we obtain the odd scalar

$$\frac{1}{\sqrt{\rho}}(\Delta_E\sqrt{\rho}) \;=\; \nu_\rho \;, \tag{4.51}$$

which is independent of coordinate system, cf. corollaries 4.13 and 4.14. This proves the theorem.

□

Corollary 4.19 *The odd scalar is connected to the Δ_E operator and the density ρ via*

$$\nu_\rho \;=\; \frac{1}{\sqrt{\rho}}(\Delta_E\sqrt{\rho}) \;. \tag{4.52}$$

Now what about nilpotency of the definitions 4.15 and 4.16? Nilpotency is clearly a local statement. In Darboux coordinates, the nilpotency is obvious. But what about in general coordinates?

More specifically, given an arbitrary coordinate system, could it be that an ingenious repeated use of the Jacobi identity (3.22) and properties (3.73) and (3.74) of the compatible 2-form, could yield a proof of the nilpotency of the Δ_E operator *without* resorting to Darboux coordinates? In the antisymplectic case, this was successfully done in our paper Ref. [44] (paper III). However, a preliminary investigation strongly suggests that this approach does not generalize to the degenerated case. In the end, we have not been able to generalize the nilpotency theorem 4.8 to situations where Darboux coordinates do not exist.

In quantum field theory applications in physics the manifolds typically are infinite dimensional, and going to Darboux coordinates typically violates locality. Therefore it is of interests of considering general coordinates, even if Darboux coordinates are formally available.

4.7 The Nilpotent Δ Operator

Definition 4.20 *Given a nilpotent Δ_E operator and a density ρ, then a nilpotent Δ operator, that takes scalars in scalars, can be defined as*

$$\Delta \;:=\; \frac{1}{\sqrt{\rho}}\Delta_E\sqrt{\rho} \;=\; \Delta_\rho + \nu_\rho \tag{4.53}$$

via conjugation of Khudaverdian's Δ_E operator with the square root of the density.

The second equality in (4.53) is a non-trivial property of the odd Laplacian (4.3).

Nilpotent operators play a prominent role in (co)homology theory. For this reason, it is of interest to more generally add an odd vector field $V = V^A \overrightarrow{\partial^\ell_A}$ and an odd scalar function ν to the odd Laplacian Δ_F,

$$\Delta \;=\; \Delta_F + V + \nu \tag{4.54}$$

such that the total Δ operator is nilpotent

$$\Delta^2 \;=\; \frac{1}{2}[\Delta,\Delta] \;=\; 0 \;, \tag{4.55}$$

i.e., supercommutes with itself. In physics, the nilpotency (4.55) encodes Becchi-Rouet-Stora-Tyutin (BRST) symmetry.

The antibracket (f,g) of two functions $f=f(z)$ and $g=g(z)$ can be defined as a double commutator [16] with the Δ-operator, acting on the constant unit function 1,

$$\begin{aligned}(f,g) \;&=\; (f\overleftarrow{\partial^r_A})E^{AB}(\overrightarrow{\partial^\ell_B}g) \;=\; (-1)^{\varepsilon_f}[[\overrightarrow{\Delta},f],g]1 \\ &=\; (-1)^{\varepsilon_f}\Delta(fg)-(-1)^{\varepsilon_f}(\Delta f)g-f(\Delta g)+(-1)^{\varepsilon_g}fg(\Delta 1) \;.\end{aligned} \tag{4.56}$$

Let us assume from now on that the line bundle connection F is flat. Recall that then the square Δ_F^2 of the odd Laplacian is a first-order operator, cf. assumption 4.3. It follows that the square Δ^2 is at most a second-order operator.

The vanishing of the second-order terms in Δ^2 is equivalent to that the odd vector field V preserves the antibracket

$$V(f,g) \;=\; (V[f],g)-(-1)^{\varepsilon_f}(f,V[g]) \;, \tag{4.57}$$

i.e., it belongs to the first Poisson cohomology group. The vector field V has seen applications in physics in the context of Sp(2)-symmetric BRST/anti-BRST quantization [61]. In the non-degenerate case, it is locally a Hamiltonian vector field $v=(H,\cdot)$, and it can be viewed as part of the line bundle connection $F_A+V^BE_{BA}$. For this reason, we shall put $V=0$ in what follows.

With $V=0$ in eq. (4.54), the nilpotency condition (4.55) becomes equivalent to two conditions (4.58) and (4.59) as follows.

- At first order,

$$\Delta_F^2 \;=\; (\nu,\cdot) \;, \tag{4.58}$$

  i.e., the modular vector field Δ_F^2 is a Hamiltonian vector field with the ν as odd Hamiltonian.

- At zeroth order

$$(\Delta_F\nu) \;=\; 0 \;. \tag{4.59}$$

Equation (4.59) is not an independent condition but it follows instead automatically from the previous requirements. PROOF:

$$\begin{aligned}-(\Delta_F\nu) \;&=\; \frac{(-1)^{\varepsilon_A}}{2}(\overrightarrow{\partial^\ell_A}+F_A)(\nu,z^A) \;=\; \frac{(-1)^{\varepsilon_A}}{2}(\overrightarrow{\partial^\ell_A}+F_A)\Delta_F^2z^A \\ &=\; \frac{(-1)^{\varepsilon_A+\varepsilon_B}}{4}(\overrightarrow{\partial^\ell_A}+F_A)(\overrightarrow{\partial^\ell_B}+F_B)(z^B,\Delta_Fz^A) \\ &=\; -\frac{(-1)^{\varepsilon_A}}{8}(\overrightarrow{\partial^\ell_A}+F_A)(\overrightarrow{\partial^\ell_B}+F_B)\Delta_F(z^B,z^A) \\ &=\; \frac{(-1)^{\varepsilon_A\varepsilon_C}}{16}(\overrightarrow{\partial^\ell_A}+F_A)(\overrightarrow{\partial^\ell_B}+F_B)(\overrightarrow{\partial^\ell_C}+F_C)(z^C,(z^B,z^A))(-1)^{(\varepsilon_A+1)(\varepsilon_C+1)} \;=\; 0 \;.\end{aligned} \tag{4.60}$$

Here, the ν eq. (4.58) is used in the second equality, the Leibniz rule (4.11) in the fourth equality, the Jacobi identity (3.22) in the sixth (=last) equality, and the zero curvature condition (4.7) in the second, fourth and sixth equality.

□

Proposition 4.21 *[44] (paper III). For an antisymplectic manifold with a flat line bundle connection F, the odd scalar ν_F from eq. (4.39) is a solution to ν to the differential eq. (4.58).*

The proof is given in Ref. [44] (paper III). It is equivalent to the proof for the Δ_E operator in arbitrary coordinates.

From proposition 4.21, it follows that the difference $\nu - \nu_F$, must satisfy $(\nu - \nu_F, \cdot) = 0$, *i.e.*, the difference $\nu - \nu_F$ is a Grassmann-odd constant.

Altogether, it follows for a non-degenerate BV geometry (M, E, ρ), that the Δ operator (4.54) must be equal to $\Delta_\rho + \nu_\rho$ up to an odd constant. (The undetermined odd constant comes from the fact that the square $\Delta^2 = \frac{1}{2}[\Delta, \Delta]$ does not change if Δ is shifted by an odd constant.)

5 Odd Scalar Curvature

5.1 Connection

We now introduce a connection $\nabla : TM \times TM \to TM$ on the tangent bundle. See Ref. [35, 62] for related discussions. The left covariant derivative $(\nabla_A X)^B$ of a left vector field X^A is defined as [35]

$$(\nabla_A X)^B \;\equiv\; (\overrightarrow{\partial^{\ell}_A} X^B) + (-1)^{\varepsilon_X(\varepsilon_B+\varepsilon_C)}\Gamma_A{}^B{}_C\, X^C \;, \qquad \varepsilon(X^A) \;=\; \varepsilon_X + \varepsilon_A \;, \tag{5.1}$$

The word "left" implies that X^A and $(\nabla_A X)^B$ transform with left derivatives

$$X'^B \;=\; X^A(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^A} z'^B) \;, \qquad (\frac{\overrightarrow{\partial^{\ell}}}{\partial z^A} z'^B)(\nabla_{'B} X)'^C \;=\; (\nabla_A X)^B(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^B} z'^C) \;, \tag{5.2}$$

under general coordinate transformations $z^A \to z'^B$. It is convenient to introduce a reordered Christoffel symbol

$$\Gamma^A{}_{BC} \;\equiv\; (-1)^{\varepsilon_A\varepsilon_B}\Gamma_B{}^A{}_C \tag{5.3}$$

to minimize the appearances of sign factors.

Definition 5.1 *A connection $\Gamma_A{}^B{}_C$ is called* **anti-Poisson** *if it preserves the anti-Poisson structure E^{AB}, i.e., definition [35]*

$$0 \;=\; (\nabla_A E)^{BC} \;\equiv\; (\overrightarrow{\partial^{\ell}_A} E^{BC}) + \Big(\Gamma_A{}^B{}_D\, E^{DC} - (-1)^{(\varepsilon_B+1)(\varepsilon_C+1)}(B \leftrightarrow C)\Big) \;. \tag{5.4}$$

The torsion tensor $T : TM \times TM \to TM$ is defined as

$$T(X,Y) \;=\; \nabla_X Y - (-1)^{\varepsilon_X\varepsilon_Y}\nabla_Y X \;=\; -(-1)^{\varepsilon_X\varepsilon_Y} T(Y,X) \;. \tag{5.5}$$

A *torsion-free* connection with $T = 0$ has the following symmetry in the lower indices:

$$\Gamma^A{}_{BC} \;=\; -(-1)^{(\varepsilon_B+1)(\varepsilon_C+1)}\Gamma^A{}_{CB} \;. \tag{5.6}$$

On one hand, a connection ∇ can be used to define a divergence of a Bosonic vector field X^A as

$$\mathrm{str}(\nabla X) \;\equiv\; (-1)^{\varepsilon_A}(\nabla_A X)^A \;=\; ((-1)^{\varepsilon_A}\overrightarrow{\partial^{\ell}_A} + \Gamma^B{}_{BA})X^A \;, \qquad \varepsilon_X \;=\; 0 \;. \tag{5.7}$$

On the other hand, the divergence is defined in terms of the line bundle connection F as

$$\mathrm{div}_F X \;\equiv\; (-1)^{\varepsilon_A}(\overrightarrow{\partial^{\ell}_A} + F_A)X^A \;. \tag{5.8}$$

See Ref. [63] for a mathematical exposition of divergence operators on supermanifolds.

Definition 5.2 *The tangent bundle connection ∇ is called* **compatible** *with the line bundle connection F if their divergences (5.7) and (5.8) are the same, i.e., if*

$$\Gamma^B{}_{BA} \;=\; (-1)^{\varepsilon_A} F_A \;. \tag{5.9}$$

In the affirmative case, we have a unique divergence operator (and hence a unique notion of volume).

We shall only consider anti-Poisson, torsion-free, and F-compatible connections ∇, *i.e.*, connections that satisfy the three conditions (5.4), (5.6) and (5.9).

For connections satisfying the three conditions, the odd Laplacian Δ_F operator can be written on a manifestly covariant form

$$\Delta_F \;=\; \frac{(-1)^{\varepsilon_A}}{2}\nabla_A E^{AB}\nabla_B \;=\; \frac{(-1)^{\varepsilon_B}}{2}E^{BA}\nabla_A\nabla_B \; . \tag{5.10}$$

5.2 Curvature

The Riemann curvature tensor $R_{AB}{}^{C}{}_{D}$ is defined as the commutator of the ∇ connection

$$([\nabla_A,\nabla_B]X)^C \;=\; R_{AB}{}^{C}{}_{D}X^D(-1)^{\varepsilon_X(\varepsilon_C+\varepsilon_D)} \; , \tag{5.11}$$

so that

$$R_{AB}{}^{C}{}_{D} \;=\; (\overrightarrow{\partial^{\ell}_A}\Gamma_B{}^{C}{}_{D}) + (-1)^{\varepsilon_B\varepsilon_C}\Gamma_A{}^{C}{}_{E}\Gamma^{E}{}_{BD} - (-1)^{\varepsilon_A\varepsilon_B}(A\leftrightarrow B) \; . \tag{5.12}$$

It is useful to define a reordered Riemann curvature tensor $R^{A}{}_{BCD}$ as

$$R^{A}{}_{BCD} \;\equiv\; (-1)^{\varepsilon_A(\varepsilon_B+\varepsilon_C)}R_{BC}{}^{A}{}_{D} \;=\; (-1)^{\varepsilon_A\varepsilon_B}(\overrightarrow{\partial^{\ell}_B}\Gamma^{A}{}_{CD}) + \Gamma^{A}{}_{BE}\Gamma^{E}{}_{CD} - (-1)^{\varepsilon_B\varepsilon_C}(B\leftrightarrow C) \; . \tag{5.13}$$

It is interesting to consider the various contractions of the Riemann curvature tensor. There are two possibilities. Firstly, there is the Ricci two-form

$$\mathcal{R}_{AB} \;\equiv\; R_{AB}{}^{C}{}_{C}(-1)^{\varepsilon_C} \;=\; (\overrightarrow{\partial^{\ell}_A}F_B) - (-1)^{\varepsilon_A\varepsilon_B}(A\leftrightarrow B) \; . \tag{5.14}$$

However, the Ricci two-form $\mathcal{R}_{AB}$ typically vanishes, cf. eq. (4.7), and even if it does not vanish, its antisymmetry means that $\mathcal{R}_{AB}$ cannot successfully be contracted with the anti-Poisson tensor E^{AB} to yield a non-zero scalar curvature, cf. eq. (3.20). Secondly, there is the Ricci tensor

$$R_{AB} \;\equiv\; R^{C}{}_{CAB} \;=\; (-1)^{\varepsilon_C}(\overrightarrow{\partial^{\ell}_C} + F_C)\Gamma^{C}{}_{AB} - (\overrightarrow{\partial^{\ell}_A}F_B)(-1)^{\varepsilon_B} - \Gamma_A{}^{C}{}_{D}\Gamma^{D}{}_{CB} \; . \tag{5.15}$$

Note that when the torsion tensor and Ricci two-form vanish, the Ricci tensor R_{AB} possesses exactly the same $A\leftrightarrow B$ symmetry (3.20) as the anti-Poisson tensor E^{AB}

$$R_{AB} \;=\; -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}R_{BA} \; . \tag{5.16}$$

The *odd scalar curvature* R is therefore defined in anti-Poisson geometry as the contraction of the Ricci tensor R_{AB} and the antisymplectic metric E^{BA},

$$R \;\equiv\; R_{AB}E^{BA} \;=\; E^{AB}R_{BA} \; . \tag{5.17}$$

Theorem 5.3 *[45] (paper IV). Given an anti-Poisson manifold (M,E) with a compatible 2-form, and a line bundle connection F, then for an arbitrary, anti-Poisson, torsion-free, and F-compatible connection ∇, the scalar curvature R does only depend on E and F through the odd scalar ν_F,*

$$R \;=\; -8\nu_F \; , \tag{5.18}$$

even if the line bundle connection F is not flat.

Theorem 5.3 is proven in Ref. [45] (paper IV). In particular, one concludes that the scalar curvature R does not depend on the connection $\Gamma^{A}{}_{BC}$ used.

One can perform various consistency checks on the formalism. Here, let us just mention one. For an antisymplectic connection ∇, one has

$$0 \;=\; [\nabla_A, \nabla_B] E^{CD} \;=\; R_{AB}{}^{C}{}_{F} E^{FD} - (-1)^{(\varepsilon_C+1)(\varepsilon_D+1)} (C \leftrightarrow D) \;, \tag{5.19}$$

or, equivalently,

$$R^{C}{}_{ABF} E^{FD} \;=\; -(-1)^{\varepsilon_A \varepsilon_B + (\varepsilon_C+1)(\varepsilon_D+1) + (\varepsilon_A+\varepsilon_B)(\varepsilon_C+\varepsilon_D)} R^{D}{}_{BAF} E^{FC} \;. \tag{5.20}$$

Contracting the $A \leftrightarrow C$ and $B \leftrightarrow D$ indices in eq. (5.20) indeed produces the identity $R = R$. Had the signs turn out differently, the odd scalar curvature (5.17) would have been stillborn, *i.e.*, always zero.

6 Discussions

One important check of the formulas for the odd scalar in the degenerate and non-degenerate case comes from conversion [64, 65, 66, 67, 68] of antisymplectic second-class constraints with a Dirac antibracket [34] into first-class constraints into an extended antisymplectic phase space. Moreover, it is interesting to check how the construction reacts to reparametrization of the second class constraints. These investigations were successfully undertaken in our paper [43] (paper II).

7 Conclusions

In this thesis, we have briefly reviewed the Batalin-Vilkovisky (BV) formalism, and treated aspects of supermathematics in algebra and differential geometry, such as, integration theory, stratification theorems, Frobenius theorem and Darboux theorem on supermanifolds.

We used Weinstein's splitting principle to prove Darboux theorem 3.15 for regular, possible degenerate, even and odd Poisson manifolds.

Khudaverdian's nilpotent Δ_E operator was introduced on both

- (i) an atlas of Darboux coordinates, cf. definition 4.5; and
- (ii) in arbitrary coordinates, cf. definition 4.49.

To express Δ_E in arbitrary coordinates (ii) in the degenerate case, we relied on the existence of a non-unique choice of compatible 2-form E_{AB}. This comes back to haunt us, since we are unable to prove nilpotency of Δ_E without appealing to Darboux coordinates (i). Hence the case (ii) is de facto not more general than the case (i).

Nevertheless, even in the second scenario (ii) with a compatible 2-form E_{AB}, we were able to define an odd scalar function ν_F, cf. theorem 4.9; and show that it has a geometric interpretation as an odd scalar curvature, cf. theorem 5.3.

Acknowledgement: K.B. would like to thank Igor Batalin, Poul Henrik Damgaard and Hovhannes Khudaverdian for fruitful discussions. The work of K.B. is supported by the Grant Agency of the Czech Republic (GACR) under the grant P201/12/G028. K.B. also received financial support from the faculty of science at Masaryk University under the grant 1102/033.

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Paper I

A Note on Semidensities in Antisymplectic Geometry

BY

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J. Math. Phys. **47** (2006) 123513

arXiv:hep-th/0604117

A Note on Semidensities in Antisymplectic Geometry

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July 24, 2013

Abstract

We revisit Khudaverdian's geometric construction of an odd nilpotent operator Δ_E that sends semidensities to semidensities on an antisymplectic manifold. We find a local formula for the Δ_E operator in arbitrary coordinates and we discuss its connection to Batalin-Vilkovisky quantization.

MCS number(s): 53A55; 58A50; 58C50; 81T70.
Keywords: Batalin-Vilkovisky Field-Antifield Formalism; Odd Laplacian; Antisymplectic Geometry; Semidensity; Halfdensity.

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1 Introduction

Recall that for a symplectic manifold with an even symplectic two-form $\omega = \frac{1}{2}dz^A\omega_{AB}dz^B$, there exists a canonical measure density given by the Pfaffian $\rho = \mathrm{Pf}(\omega_{AB})$, *i.e.* there is a natural notion of volume in a symplectic manifold. A related fact is the Liouville Theorem, which states that Hamiltonian vector fields are divergenceless. On the other hand, the situation is completely different for an odd symplectic manifold, also known as an antisymplectic manifold and endowed with an odd antisymplectic two-form $E = \frac{1}{2}d\Gamma^A E_{AB} d\Gamma^B$. These geometries for instance show up in the Lagrangian quantization method of Batalin and Vilkovisky [1]. It turns out that there is *no* canonical choice of measure density ρ in this case, as, for instance, the above Pfaffian. This is tied to the fact that there is no meaningful notion of a superdeterminant/Berezinian for a matrix that is intrinsically Grassmann-odd. However, the upset runs deeper. In fact, a density ρ can never be a function of the antisymplectic matrix E_{AB}. Phrased differently, a density ρ always carries information that cannot be deduced from the antisymplectic structure E alone [2]. Within the standard Batalin-Vilkovisky framework, the possible choices of a density ρ is only partially determined by a requirement of gauge symmetry.

Around 1992 Batalin-Vilkovisky quantization took a more geometric form, in particular with the work of Schwarz [3]. The concensus was that the geometric setting requires two independent structures: an odd symplectic, non-degenerate two-form E and a measure density ρ. From these two structures, one may build a Grassmann-odd, second-order operator Δ_ρ, known as the odd Laplacian. Alternatively, one can view the odd Laplacian Δ_ρ itself as *the* fundamental structure of Batalin-Vilkovisky geometry [4, 5], which is conventionally required to be nilpotent.

Khudaverdian has constructed [6, 7, 8, 9] a Grassmann-odd, nilpotent, second-order operator Δ_E that does not rely on a choice of density ρ. The caveat is that the Δ_E operator is defined on semidensities rather than on scalars. (The notion of a semidensity is explained in eq. (3.1) below.) In retrospect, many pieces of Khudaverdian's construction were known to physicists, see for instance Ref. [10], p.440. In this short note we reconsider Khudaverdian's construction and find a local formula for the Δ_E operator that applies to arbitrary coordinate systems. The ability to work in any coordinates, not just Darboux coordinates, is important, since if one first has to search for a set of Darboux coordinates to the system that one is studying, symmetries (such as, *e.g.* , Lorentz covariance) or locality that one would like to preserve during the quantization process, are often lost.

The paper is organized as follows: We consider the antisymplectic structure in Section 2; the odd Laplacian Δ_ρ in Section 3; and in Sections 4 and 5, the Δ_E operator using Darboux coordinates and general coordinates, respectively. Finally, in Section 6 we analyze a modified Batalin-Vilkovisky scheme based on the Δ_E operator.

General remark about notation. We have two types of grading: A Grassmann grading ϵ and an exterior form degree p. The sign conventions are such that two exterior forms ξ and η, of Grassmann parity ϵ_ξ, ϵ_η and exterior form degree p_ξ, p_η, respectively, commute in the following graded sense

$$\eta \wedge \xi \ = \ (-1)^{\epsilon_\xi\epsilon_\eta + p_\xi p_\eta}\xi \wedge \eta \tag{1.1}$$

inside the exterior algebra. We will often not write the exterior wedges "$\wedge$" explicitly.

2 Antisymplectic Geometry

Consider an antisymplectic manifold (M, E) and let Γ^A denote local coordinates of Grassmann parity $\epsilon_A \equiv \epsilon(\Gamma^A)$ (and exterior form degree $p(\Gamma^A) = 0$). The antisymplectic two-form can locally be written as

$$E \;=\; \frac{1}{2}d\Gamma^{A}\;E_{AB}\;d\Gamma^{B} \;=\; -\frac{1}{2}E_{AB}\;d\Gamma^{B}\;d\Gamma^{A}\;, \tag{2.1}$$

where $E_{AB}=E_{AB}(\Gamma)$ is the corresponding matrix representation. Besides carrying gradings $\epsilon(E)=1$ and $p(E)=2$, the antisymplectic two-form E has two defining properties. First, E is closed,

$$dE \;=\; 0\;, \tag{2.2}$$

where the grading conventions for the exterior derivative

$$d \;=\; d\Gamma^{A}\frac{\overrightarrow{\partial^{l}}}{\partial\Gamma^{A}} \tag{2.3}$$

are $\epsilon(d)=0$ and $p(d)=1$. Secondly, E is non-degenerate, *i.e.* the antisymplectic matrix E_{AB} has an inverse matrix E^{AB},

$$E^{AB}E_{BC} \;=\; \delta^{A}_{C} \;=\; E_{CB}E^{BA}\;. \tag{2.4}$$

Instead of the compact exterior form notation E, one may equivalently formulate the above conditions with all the indices written out explicitly in terms of the matrices E_{AB} or E^{AB}. In detail, the gradings are

$$\begin{array}{rcccl} \epsilon(E_{AB}) &=& \epsilon_{A}+\epsilon_{B}+1 &=& \epsilon(E^{AB})\;,\\ p(E_{AB}) &=& 0 &=& p(E^{AB})\;, \end{array} \tag{2.5}$$

the skew-symmetries are

$$\begin{array}{rcl} E_{BA} &=& -(-1)^{\epsilon_{A}\epsilon_{B}}E_{AB}\;,\\ E^{BA} &=& -(-1)^{(\epsilon_{A}+1)(\epsilon_{B}+1)}E^{AB}\;, \end{array} \tag{2.6}$$

while the closeness condition and the equivalent Jacobi identity read

$$\sum_{\text{cycl. }A,B,C}(-1)^{\epsilon_{A}\epsilon_{C}}(\frac{\overrightarrow{\partial^{l}}}{\partial\Gamma^{A}}E_{BC}) \;=\; 0\;, \tag{2.7}$$

$$\sum_{\text{cycl. }A,B,C}(-1)^{(\epsilon_{A}+1)(\epsilon_{C}+1)}E^{AD}(\frac{\overrightarrow{\partial^{l}}}{\partial\Gamma^{D}}E^{BC}) \;=\; 0\;, \tag{2.8}$$

respectively. The inverse matrix E^{AB} with upper indices gives rise to the antibracket [1]

$$(F,G) \;=\; (F\frac{\overleftarrow{\partial^{r}}}{\partial\Gamma^{A}})E^{AB}(\frac{\overrightarrow{\partial^{l}}}{\partial\Gamma^{B}}G)\;, \tag{2.9}$$

which satisfies a graded skew-symmetry and a graded Jacobi identity as a consequence of eqs. (2.6) and (2.8). There is an antisymplectic analogue of Darboux's Theorem that states that locally there exist Darboux coordinates $\Gamma^{A}=\{\phi^{\alpha};\phi^{*}_{\alpha}\}$, such that the only non-vanishing antibrackets between the coordinates are $(\phi^{\alpha},\phi^{*}_{\beta})=\delta^{\alpha}_{\beta}=-(\phi^{*}_{\beta},\phi^{\alpha})$. In Darboux coordinates the antisymplectic two-form is simply $E=d\phi^{*}_{\alpha}\wedge d\phi^{\alpha}$.

3 Odd Laplacian Δ_{ρ} on Scalars

A scalar function $F=F(\Gamma)$, a density $\rho=\rho(\Gamma)$ and a semidensity $\sigma=\sigma(\Gamma)$ are by definition quantities that transform as

$$F \;\longrightarrow\; F' \;=\; F\;, \qquad \rho \;\longrightarrow\; \rho' \;=\; \frac{\rho}{J}\;, \qquad \sigma \;\longrightarrow\; \sigma' \;=\; \frac{\sigma}{\sqrt{J}}\;, \tag{3.1}$$

respectively, under general coordinate transformations $\Gamma^A \to \Gamma'^A$, where $J \equiv \mathrm{sdet}\frac{\partial \Gamma'^A}{\partial \Gamma^B}$ denotes the Jacobian. We shall ignore the global issues of orientation and choice of square root. In principle the above F, ρ and σ could either be bosons or fermions, however normally we shall require the densities ρ to be invertible, and therefore bosons.

Given a choice of density ρ one may define the odd Laplacian [4]

$$\Delta_\rho \;:=\; \frac{(-1)^{\epsilon_A}}{2\rho} \frac{\overrightarrow{\partial^l}}{\partial \Gamma^A} \rho E^{AB} \frac{\overrightarrow{\partial^l}}{\partial \Gamma^B} \;, \tag{3.2}$$

that takes scalars to scalars of opposite Grassmann parity. The odd Laplacian (3.2) has a geometric interpretation as a divergence of a Hamiltonian vector field [3, 11]

$$\Delta_\rho \Psi \;=\; -\frac{1}{2}\mathrm{div}_\rho(X_\Psi) \;, \qquad\qquad \epsilon(\Psi) \;=\; 1 \;. \tag{3.3}$$

Here $X_\Psi := (\Psi, \cdot)$ denotes a Hamiltonian vector field with a Grassmann-odd Hamiltonian Ψ, and the divergence $\mathrm{div}_\rho X$ of a vector field X, with respect to the measure density ρ, is

$$\mathrm{div}_\rho X \;:=\; \frac{(-1)^{\epsilon_A}}{\rho} \frac{\overrightarrow{\partial^l}}{\partial \Gamma^A}(\rho X^A) \;, \qquad\qquad \epsilon(X) \;=\; 0 \;. \tag{3.4}$$

The fact that the odd Laplacian (3.3) is non-zero, shows that antisymplectic manifolds do not have an analogue of the Liouville Theorem mentioned in the Introduction. As a consequence of the Jacobi identity eq. (2.8), the square operator $\Delta_\rho^2 = \frac{1}{2}[\Delta_\rho, \Delta_\rho]$ becomes a linear derivation, *i.e.* a first-order differential operator,

$$\Delta_\rho^2(FG) \;=\; \Delta_\rho^2(F)\, G + F\, \Delta_\rho^2(G) \;. \tag{3.5}$$

Conventionally, one imposes additionally that the Δ_ρ operator is nilpotent $\Delta_\rho^2 = 0$, but this is not necessary for our purposes.

4 Khudaverdian's Δ_E Operator on Semidensities

Khudaverdian showed that one may define a Grassmann-odd, nilpotent, second-order operator Δ_E *without* a choice of density ρ. This Δ_E operator does not take scalars to scalars like the odd Laplacian (3.2), but instead takes semidensities to semidensities of opposite Grassmann parity. Equivalently, the Δ_E operator transforms as

$$\Delta_E \;\longrightarrow\; \Delta'_E \;=\; \frac{1}{\sqrt{J}}\Delta_E\sqrt{J} \tag{4.1}$$

under general coordinate transformations $\Gamma^A \to \Gamma'^A$, cf. eq. (3.1). Khudaverdian's construction relies first of all on an atlas of Darboux charts, which is granted by an antisymplectic analogue of Darboux's Theorem, and secondly, on a Lemma by Batalin and Vilkovisky about the possible form of the Jacobians for anticanonical transformations, also known as antisymplectomorphisms.

Lemma 4.1 *"The Batalin-Vilkovisky Lemma" [12, 10, 7, 8, 9, 13]. Consider a finite anticanonical transformation between initial Darboux coordinates $\Gamma^A_{(i)}$ and final Darboux coordinates $\Gamma^A_{(f)}$. Then the Jacobian $J \equiv \mathrm{sdet}(\partial\Gamma^A_{(f)}/\partial\Gamma^B_{(i)})$ satisfies*

$$\Delta_1^{(i)}\sqrt{J} \;=\; 0 \;. \tag{4.2}$$

Here $\Delta_1^{(i)}$ refers to the odd Laplacian (3.2) with $\rho{=}1$ in the initial Darboux coordinates $\Gamma^A_{(i)}$.

Given Darboux coordinates Γ^A the Δ_E operator is defined on a semidensity σ as [6, 7, 8, 9, 13]

$$(\Delta_E\sigma) \;:=\; (\Delta_1\sigma) \;, \tag{4.3}$$

where Δ_1 is the Δ_ρ operator (3.2) with $\rho{=}1$. It is important in eq. (4.3) that the formula for the Δ_1 operator (3.2) and the semidensity σ both refer to the same Darboux coordinates Γ^A. The parentheses in eq. (4.3) indicate that the equation should be understood as an equality among semidensities (in the sense of zeroth-order differential operators) rather than an identity among differential operators. One next uses the Batalin-Vilkovisky Lemma to argue that the definition (4.3) does not depend on the choices of Darboux coordinates Γ^A. What this means is, that the right-hand side of the definition (4.3) transforms as a semidensity

$$(\Delta_1^{(f)}\sigma_{(f)}) \;=\; \frac{1}{\sqrt{J}}(\Delta_1^{(i)}\sigma_{(i)}) \tag{4.4}$$

under an anticanonical transformation between any two Darboux coordinates $\Gamma^A_{(i)}$ and $\Gamma^A_{(f)}$. Proof:

$$\sqrt{J}(\Delta_1^{(f)}\sigma_{(f)}) \;=\; \sqrt{J}(\Delta_J^{(i)}\sigma_{(f)}) \;=\; \sqrt{J}(\Delta_J^{(i)}\frac{\sigma_{(i)}}{\sqrt{J}}) \;=\; (\Delta_1^{(i)}\sigma_{(i)}) - \frac{1}{\sqrt{J}}(\Delta_1^{(i)}\sqrt{J})\sigma_{(i)} \;=\; (\Delta_1^{(i)}\sigma_{(i)}) \;. \tag{4.5}$$

The third equality is a non-trivial property of the odd Laplacian (3.2). The Batalin-Vilkovisky Lemma is used in the fourth equality. Strictly speaking, it is enough to consider infinitesimal anticanonical transformations to justify the definition (4.3). The proof of the infinitesimal version of the Batalin-Vilkovisky Lemma goes like this: An infinitesimal anticanonical coordinate transformation $\delta\Gamma^A = X^A$ is necessarily a Hamiltonian vector field $X^A = (\Psi, \Gamma^A) \equiv X^A_\Psi$ with an infinitesimal, Grassmann-odd Hamiltonian Ψ, where $\epsilon(\Psi) = 1$. So

$$\ln J \;\approx\; (-1)^{\epsilon_A}(\frac{\stackrel{\rightarrow}{\partial^l}}{\partial\Gamma^A}X^A) \;=\; \mathrm{div}_1(X_\Psi) \;=\; -\,2\Delta_1\Psi \;, \tag{4.6}$$

and hence

$$\Delta_1\sqrt{J} \;\approx\; -\,\Delta_1^2\Psi \;=\; 0 \;, \tag{4.7}$$

due to the nilpotency of the Δ_1 operator in Darboux coordinates. The "$\approx$" sign is used to indicate that equality only holds at the infinitesimal level. (Here we are guilty of mixing active and passive pictures; the active vector field is properly speaking *minus* X.) A simple proof of the Batalin-Vilkovisky Lemma for finite anticanonical transformations can be found in Ref. [13].

On the other hand, once the definition (4.3) is justified, it is obvious that the Δ_E operator supercommutes with itself, because the Γ^A-derivatives have no Γ^A's to act on in Darboux coordinates. Therefore Δ_E is nilpotent,

$$\Delta_E^2 \;=\; \frac{1}{2}[\Delta_E, \Delta_E] \;=\; 0 \;. \tag{4.8}$$

Same sort of reasoning shows that $\Delta_E = \Delta_E^T$ is symmetric.

5 The Δ_E Operator in General Coordinates

We now give a definition of the Δ_E operator that does not rely on Darboux coordinates. We claim that in arbitrary coordinates the Δ_E operator is given as

$$(\Delta_E\sigma) \;:=\; (\Delta_1\sigma) + \left(\frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24}\right)\sigma \;, \tag{5.1}$$

where

$$\nu^{(1)} := (-1)^{\epsilon_A}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}E^{AB}) , \tag{5.2}$$

$$\nu^{(2)} := -(-1)^{\epsilon_B}(\Gamma^C,(\Gamma^B,\Gamma^A))(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}E_{BC}) = (-1)^{\epsilon_A\epsilon_C}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}E^{CD})(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^D}E^{AB})E_{BC} . \tag{5.3}$$

Eq. (5.1) is the main result of this paper. Notice that in Darboux coordinates, where E^{AB} is constant, *i.e.* independent of the coordinates Γ^A, the last two terms $\nu^{(1)}$ and $\nu^{(2)}$ vanish. Hence the definition (5.1) agrees in this case with Khudaverdian's Δ_E operator (4.3).

It remains to be shown that the right-hand side of eq. (5.1) behaves as a semidensity under general coordinate transforms. Here we will only explicitly consider the case where σ is invertible to simplify the presentation. (The non-invertible case is fundamentally no different.) In the invertible case, we customarily write the semidensity $\sigma = \sqrt{\rho}$ as a square root of a density ρ, and define a Grassmann-odd quantity

$$\nu_\rho := \frac{1}{\sqrt{\rho}}(\Delta_E\sqrt{\rho}) = \nu_\rho^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24} , \tag{5.4}$$

by dividing both sides of the definition (5.1) with the semidensity σ. Here we have defined

$$\nu_\rho^{(0)} := \frac{1}{\sqrt{\rho}}(\Delta_1\sqrt{\rho}) . \tag{5.5}$$

Hence, to justify the definition (5.1), one should check that ν_ρ is a scalar under general infinitesimal coordinate transformations. Under an arbitrary infinitesimal coordinate transformation $\delta\Gamma^A = X^A$, one calculates

$$\delta\nu_\rho^{(0)} = -\frac{1}{2}\Delta_1\mathrm{div}_1 X , \tag{5.6}$$

$$\delta\nu^{(1)} = 4\Delta_1\mathrm{div}_1 X + (-1)^{\epsilon_A}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^C}E^{AB})(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}X^C) , \tag{5.7}$$

$$\delta\nu^{(2)} = 3(-1)^{\epsilon_A}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^C}E^{AB})(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}X^C) , \tag{5.8}$$

cf. Appendices A–C. One easily sees that while the three constituents $\nu_\rho^{(0)}$, $\nu^{(1)}$ and $\nu^{(2)}$ separately have non-trivial transformation properties, the linear combination ν_ρ in eq. (5.4) is indeed a scalar.

The new definition (5.1) is clearly symmetric $\Delta_E = \Delta_E^T$ and one may check that the nilpotency (4.8) of the Δ_E operator (5.1) precisely encodes the Jacobi identity (2.8). The odd Laplacian Δ_ρ can be expressed entirely by the Δ_E operator and a choice of density ρ,

$$(\Delta_\rho F) = (\Delta_1 F) + \frac{1}{\sqrt{\rho}}(\sqrt{\rho}, F) = \frac{1}{\sqrt{\rho}}[\overrightarrow{\Delta_1}, F]\sqrt{\rho} = \frac{1}{\sqrt{\rho}}[\overrightarrow{\Delta_E}, F]\sqrt{\rho} . \tag{5.9}$$

Since $\nu^{(2)}$ depends on the antisymplectic matrix E_{AB} with lower indices, it is not clear how the formula (5.1) extends to the degenerate anti-Poisson case.

6 Application to Batalin-Vilkovisky Quantization

It is interesting to transcribe the Batalin-Vilkovisky quantization, based on the odd Laplacian Δ_ρ, into a quantization scheme that is based on the Δ_E operator, with the added benefit that no choice of measure density ρ is needed. Since the Δ_E operator takes semidensities to semidensities, this suggests that the Boltzmann factor $\exp[\frac{i}{\hbar}W_E]$ that appears in the Quantum Master Equation

$$\Delta_E \exp\left[\frac{i}{\hbar}W_E\right] \;=\; 0 \tag{6.1}$$

should now be a semidensity, where

$$W_E \;=\; S + \sum_{n=1}^{\infty}(i\hbar)^n W_n \tag{6.2}$$

denotes the quantum action. In fact, this was a common interpretation (when restricting to Darboux coordinates) prior to the introduction of a density ρ around 1992, see for instance Ref. [10], p.440-441. If one only considers $\hbar$-independent coordinate transformations $\Gamma^A \to \Gamma'^A$ for simplicity, this implies that the one-loop factor e^{-W_1} is a semidensity, while the rest of the quantum action, *i.e.* the classical action S and the higher loop corrections W_n, $n \geq 2$, are scalars as usual. For instance, the nilpotent operator $F \mapsto e^{W_1}\Delta_E(e^{-W_1}F)$ takes scalars F to scalars.

At this stage it might be helpful to compare the above Δ_E approach to the Δ_ρ formalism. To this end, fix a density ρ. Then one can define a bona fide scalar quantum action W_ρ as

$$W_\rho \;:=\; W_E + (i\hbar)\ln\sqrt{\rho}\;, \tag{6.3}$$

or equivalently,

$$e^{\frac{i}{\hbar}W_E} \;=\; \sqrt{\rho}\; e^{\frac{i}{\hbar}W_\rho}\;. \tag{6.4}$$

This scalar action W_ρ satisfies the Modified Quantum Master Equation

$$(\Delta_\rho + \nu_\rho)\exp\left[\frac{i}{\hbar}W_\rho\right] \;=\; 0\;, \tag{6.5}$$

cf. eq. (5.4), (5.9), (6.1) and (6.4). One may obtain the Quantum Master Equation $\Delta_\rho \exp[\frac{i}{\hbar}W_\rho] = 0$ by additionally imposing the covariant condition $\nu_\rho = 0$, or equivalently $\Delta_E\sqrt{\rho} = 0$. However this step is not necessary.

Returning now to the pure Δ_E approach with no ρ, the finite Δ_E-exact transformations of the form

$$e^{\frac{i}{\hbar}W'_E} \;=\; e^{-[\overrightarrow{\Delta}_E,\Psi]}e^{\frac{i}{\hbar}W_E}\;, \tag{6.6}$$

play an important rôle in taking solutions W_E to the Quantum Master Equation (6.1) into new solutions W'_E. It is implicitly understood that all objects in eq. (6.6) refer to the same (but arbitrary) coordinate frame. In general, Ψ is a Grasmann-odd operator that takes semidensities to semidensities. If Ψ is a scalar function (=zeroth-order operator), one derives

$$W'_E \;=\; e^{X_\Psi}W_E + (i\hbar)\frac{e^{X_\Psi}-1}{X_\Psi}\Delta_E\Psi\;. \tag{6.7}$$

The formula (6.7) is similar to the usual formula in the Δ_ρ formalism [13]. One may check that eq. (6.7) is covariant with respect to general coordinate transformations.

The W-X formulation discussed in Ref. [5] and Ref. [13] carries over with only minor modifications, since the Δ_E operator is symmetric $\Delta_E^T = \Delta_E$. In short, the W-X formulation is a very general field-antifield formulation, based on two Master actions, W_E and X_E, each satisfying a Quantum Master Equation. At the operational level, *symmetric* means that the Δ_E operator, sandwiched between two semidensities under a (path) integral sign, may be moved from one semidensity to the other, using integration by part. This is completely analogous to the symmetry of the odd Laplacian $\Delta_\rho = \Delta_\rho^T$ itself. The X_E quantum action is a gauge-fixing part,

$$X_E \;=\; G_\alpha \lambda^\alpha + (i\hbar) H_E + \mathcal{O}(\lambda^*) \;, \tag{6.8}$$

which contains the gauge-fixing constraints G_α in involution,

$$(G_\alpha, G_\beta) \;=\; G_\gamma U^\gamma_{\alpha\beta} \;. \tag{6.9}$$

The gauge-fixing functions G_α implement a generalization of the standard Batalin-Vilkovisky gauge-fixing procedure $\phi^*_\alpha = \partial\Psi/\partial\phi^\alpha$. In the simplest cases, the gauge-fixing conditions $G_\alpha = 0$ are enforced by integration over the Lagrange multipliers λ^α. See Ref. [13] for further details on the W-X formulation. The pertinent measure density in the partition function

$$\mathcal{Z} \;=\; \int [d\Gamma][d\lambda] \; e^{\frac{i}{\hbar}(W_E + X_E)} \tag{6.10}$$

is now located inside the one-loop parts of the W_E and the X_E actions. For instance, an on-shell expression for the one-loop factor e^{-H_E} is

$$e^{-H_E} \;=\; \sqrt{J \;\mathrm{sdet}(F^\alpha, G_\beta)} \;, \tag{6.11}$$

where $J = \mathrm{sdet}(\partial\bar{\Gamma}^A/\partial\Gamma^B)$ denotes the Jacobian of the transformation $\Gamma^A \to \bar{\Gamma}^A$ and $\bar{\Gamma}^A \equiv \{F^\alpha; G_\alpha\}$. The formula (6.11) differs from the original square root formula [14, 15, 13] by not depending on a ρ density, consistent with the fact that e^{-H_E} is no longer a scalar but a semidensity. We recall here the main point that the one-loop factor e^{-H_E} is independent of the F^α's and the partition function $\mathcal{Z}$ is independent of the G_α's in involution, cf. eq. (6.9).

To summarize, the density ρ can altogether be avoided in the field-antifield formalism, at the cost of more complicated transformation rules. We stress that the above transcription has no consequences for the physics involved. For instance, the ambiguity that existed in the density ρ is still present in the choice of W_E and X_E.

Acknowledgement: The Author thanks I.A. Batalin and P.H. Damgaard for discussions. The Author also thanks the organizers of the workshop "Gerbes, Groupoids, and Quantum Field Theory, May–July 2006" at the Erwin Schrödinger Institute for warm hospitality. This work is supported by the Ministry of Education of the Czech Republic under the project MSM 0021622409.

A Proof of eq. (5.6)

Consider a general (not necessarily infinitesimal) coordinate transformation $\Gamma^A \to \Gamma'^A$ between an "unprimed" and an "primed" coordinate systems Γ^A and Γ'^A, respectively, cf. eq. (3.1). The primed $\nu^{(0)}_\rho$ quantity (5.5) can be re-expressed with the help of the unprimed coordinates as

$$\nu'^{(0)}_{\rho'} \;:=\; \frac{1}{\sqrt{\rho'}}(\Delta'_1\sqrt{\rho'}) \;=\; \frac{1}{\sqrt{\rho'}}(\Delta_J\sqrt{\rho'}) \;=\; \frac{1}{\sqrt{\rho}}(\Delta_1\sqrt{\rho}) - \frac{1}{\sqrt{J}}(\Delta_1\sqrt{J}) \;=\; \nu^{(0)}_\rho - \nu^{(0)}_J \;, \tag{A.1}$$

where it is convenient (and natural) to introduce the quantity

$$\nu^{(0)}_J \;:=\; \frac{1}{\sqrt{J}}(\Delta_1\sqrt{J}) \tag{A.2}$$

with respect to the unprimed reference system. The third equality in eq. (A.1) uses a non-trivial property of the odd Laplacian (3.2). In the infinitesimal case $\delta\Gamma^A = X^A$, the expression for the Jacobian J reduces to a divergence $\ln J \approx {\rm div}_1 X$, and one calculates

$$\delta\nu^{(0)}_{\rho} \;=\; \nu'^{(0)}_{\rho'} - \nu^{(0)}_{\rho} \;=\; -\,\nu^{(0)}_{J} \;=\; -\,\Delta_1(\ln\sqrt{J}) - \frac{1}{2}(\ln\sqrt{J},\ln\sqrt{J}) \;\approx\; -\frac{1}{2}\Delta_1{\rm div}_1 X \;, \qquad \text{(A.3)}$$

which is eq. (5.6).

B Proof of eq. (5.7)

The infinitesimal variation of $\nu^{(1)}$ yields 4 contributions to linear order in the variation $\delta\Gamma^A = X^A$,

$$\delta\nu^{(1)} \;=\; -\,\delta\nu^{(1)}_{I} - \delta\nu^{(1)}_{II} + \delta\nu^{(1)}_{III} + \delta\nu^{(1)}_{IV} \;. \qquad \text{(B.1)}$$

They are

$$\delta\nu^{(1)}_{I} \;:=\; (-1)^{\epsilon_A}\Big(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}X^C\Big)\Big(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^C}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}E^{AB}\Big) \;, \qquad \text{(B.2)}$$

$$\delta\nu^{(1)}_{II} \;:=\; (-1)^{\epsilon_A}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\left(\Big(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}X^C\Big)\Big(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^C}E^{AB}\Big)\right) \;=\; \delta\nu^{(1)}_{I} + \delta\nu^{(1)}_{V} \;, \qquad \text{(B.3)}$$

$$\delta\nu^{(1)}_{III} \;:=\; (-1)^{\epsilon_A}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}\left(\Big(X^A\frac{\overleftarrow{\partial^r}}{\partial\Gamma^C}\Big)E^{CB}\right) \;=\; \delta\nu^{(1)}_{IV} \;, \qquad \text{(B.4)}$$

$$\delta\nu^{(1)}_{IV} \;:=\; (-1)^{\epsilon_A}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}\left(E^{AC}\Big(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^C}X^B\Big)\right) \;=\; \delta\nu^{(1)}_{I} + \delta\nu^{(1)}_{V} + \delta\nu^{(1)}_{VI} \;, \qquad \text{(B.5)}$$

$$\delta\nu^{(1)}_{V} \;:=\; (-1)^{\epsilon_A}\Big(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^C}E^{AB}\Big)\Big(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}X^C\Big) \;, \qquad \text{(B.6)}$$

$$\delta\nu^{(1)}_{VI} \;:=\; (-1)^{\epsilon_A}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}\left(E^{AC}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^C}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}X^B(-1)^{\epsilon_B}\right) \;=\; 2\Delta_1{\rm div}_1 X \;, \qquad \text{(B.7)}$$

where we have noted various relations among the contributions. Altogether, the infinitesimal variation of $\nu^{(1)}$ becomes

$$\delta\nu^{(1)} \;=\; \delta\nu^{(1)}_{V} + 2\delta\nu^{(1)}_{VI} \;, \qquad \text{(B.8)}$$

which is eq. (5.7).

C Proof of eq. (5.8)

The infinitesimal variation of

$$\nu^{(2)} \;:=\; (-1)^{\epsilon_A\epsilon_C}\Big(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^D}E^{AB}\Big)E_{BC}\Big(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}E^{CD}\Big) \qquad \text{(C.1)}$$

yields 8 contributions to linear order in the variation $\delta\Gamma^A = X^A$, which may be organized as 2×4 terms

$$\delta\nu^{(2)} \;=\; 2(-\delta\nu^{(2)}_{I} - \delta\nu^{(2)}_{II} + \delta\nu^{(2)}_{III} + \delta\nu^{(2)}_{IV}) \;, \qquad \text{(C.2)}$$

due to a $(A,B) \leftrightarrow (D,C)$ symmetry in eq. (C.1). They are

$$\delta\nu_{I}^{(2)} := (-1)^{\epsilon_A\epsilon_C}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^D}E^{AB})E_{BF}(X^F\frac{\overleftarrow{\partial^r}}{\partial\Gamma^C})(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}E^{CD}) , \tag{C.3}$$

$$\delta\nu_{II}^{(2)} := (-1)^{\epsilon_A\epsilon_C}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^D}E^{AB})E_{BC}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}X^F)(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^F}E^{CD}) , \tag{C.4}$$

$$\delta\nu_{III}^{(2)} := (-1)^{\epsilon_A\epsilon_C}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^D}E^{AB})E_{BC}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}\left((X^C\frac{\overleftarrow{\partial^r}}{\partial\Gamma^F})E^{FD}\right) = \delta\nu_{I}^{(2)}+\delta\nu_{V}^{(2)} , \tag{C.5}$$

$$\delta\nu_{IV}^{(2)} := (-1)^{\epsilon_A\epsilon_C}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^D}E^{AB})E_{BC}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}\left(E^{CF}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^F}X^D)\right) = \delta\nu_{II}^{(2)}+\delta\nu_{VI}^{(2)} , \tag{C.6}$$

$$\delta\nu_{V}^{(2)} := (-1)^{\epsilon_A\epsilon_C}E^{FD}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^D}E^{AB})E_{BC}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}X^C\frac{\overleftarrow{\partial^r}}{\partial\Gamma^F}) = -\delta\nu_{V}^{(2)}+\delta\nu_{VI}^{(2)} , \tag{C.7}$$

$$\delta\nu_{VI}^{(2)} := (-1)^{\epsilon_A}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^C}E^{AB})(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}X^C) , \tag{C.8}$$

where we have noted various relations among the contributions. The Jacobi identity (2.8) for E^{AB} is used in the second equality of eq. (C.7). Altogether, the infinitesimal variation of $\nu^{(2)}$ becomes

$$\delta\nu^{(2)} = 3\delta\nu_{VI}^{(2)} , \tag{C.9}$$

which is eq. (5.8).

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Paper II

Semidensities,

Second-Class Constraints

and Conversion

in Anti-Poisson Geometry

BY

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J. Math. Phys. **49** (2008) 043516

arXiv:0705.3440

Semidensities, Second-Class Constraints and Conversion in Anti-Poisson Geometry

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February 24, 2013

Abstract

We consider Khudaverdian's geometric version of a Batalin-Vilkovisky (BV) operator Δ_E in the case of a degenerate anti-Poisson manifold. The characteristic feature of such an operator (aside from being a Grassmann-odd, nilpotent, second-order differential operator) is that it sends semidensities to semidensities. We find a local formula for the Δ_E operator in arbitrary coordinates. As an important application of this setup, we consider the Dirac antibracket on an antisymplectic manifold with antisymplectic second-class constraints. We show that the entire Dirac construction, including the corresponding Dirac BV operator Δ_{E_D}, exactly follows from conversion of the antisymplectic second-class constraints into first-class constraints on an extended manifold.

MCS number(s): 53A55; 58A50; 58C50; 81T70.
Keywords: Batalin-Vilkovisky Field-Antifield Formalism; Odd Laplacian; Anti-Poisson Geometry; Semidensity; Second-Class Constraints; Conversion.

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Contents

1 Introduction

Consider an antisymplectic manifold $(M;E)$ with coordinates Γ^A. Such structure was first used by Batalin and Vilkovisky to quantize Lagrangian gauge theories [1, 2, 3]. In general, antisymplectic geometry has many of the characteristic features of ordinary symplectic geometry, *e.g.* the Jacobi identity and the Darboux Theorem, but there are also important differences: There are no canonical volume form and no Liouville Theorem in antisymplectic geometry [4]. In the covariant Batalin-Vilkovisky (BV) formalism [5, 6] from around 1992 one is (among other things) instructed to make separate choices of a measure density $\rho = \rho(\Gamma)$ and a quantum action $W_{\rho} = W_{\rho}(\Gamma)$. However, the division into measure and action part is to a large extent an arbitrary division, *i.e.* it is always possible to shift parts of the measure ρ into the action W_{ρ} and vice versa. It is only a particular combination of these two quantities, namely the Boltzmann semidensity

$$\exp[\frac{i}{\hbar}W_E] \;\equiv\; \sqrt{\rho}\exp[\frac{i}{\hbar}W_{\rho}] \tag{1.1}$$

that enters the physical partition function $\mathcal{Z}$. For instance, if there exist global Darboux coordinates $\Gamma^A = \{\phi^{\alpha};\phi^*_{\alpha}\}$, the partition function reads

$$\mathcal{Z} \;=\; \int [d\phi]\, \exp[\frac{i}{\hbar}W_E]\Big|_{\phi^* = \frac{\partial\psi}{\partial\phi}} \;, \tag{1.2}$$

where $\psi = \psi(\phi)$ is the gauge fermion. (More generally, the partition function $\mathcal{Z}$ is described by the so-called W-X formalism [7, 8].) The field-antifield formalism was reformulated in Ref. [9] entirely in the minimal language of semidensities, which skips ρ altogether. According to this minimal approach, the Boltzmann semidensity $\exp[\frac{i}{\hbar}W_E]$ should satisfy the Quantum Master Equation

$$\Delta_E \exp[\frac{i}{\hbar}W_E] \;=\; 0 \tag{1.3}$$

to ensure independence of gauge-fixing. Here Δ_E is Khudaverdian's BV operator, which takes semidensities to semidensities, cf. Ref. [8, 10, 11, 12, 13] and Definition 2.3 below. Of course, the density ρ may always be re-introduced to compare with the 1992 formulation. In doing so, for an arbitrary choice of ρ,

1. the Boltzmann semidensity $\exp[\frac{i}{\hbar}W_E]$ descend to a Boltzmann scalar $\exp[\frac{i}{\hbar}W_{\rho}] = \exp[\frac{i}{\hbar}W_E]/\sqrt{\rho}$,

2. the Δ_E operator descend to a (not necessarily nilpotent) odd Laplacian Δ_{ρ}, which takes scalars to scalars, cf. Definition 2.2 below; and

3. the Quantum Master Eq. (1.3) descend to the Modified Quantum Master Equation

$$(\Delta_{\rho} + \nu_{\rho}) \exp[\frac{i}{\hbar}W_{\rho}] \;=\; 0 \;, \tag{1.4}$$

   where ν_{ρ} is an odd scalar, cf. Definition 2.8 below.

We emphasize that this construction works for any ρ. However, to arrive at the 1992 formulation [5, 6], which has $\nu_{\rho} = 0$ and a nilpotent odd Laplacian $\Delta_{\rho}^2 = 0$, one should impose conditions on ρ.

The paper is organized as follows. Anti-Poisson geometry is reviewed in Section 2. The notions of compatible two-form fields and bi-Darboux coordinates are introduced in Subsection 2.1. A new Theorem 2.1 provides necessary and sufficient conditions for the existence of bi-Darboux coordinates. The definition of the Δ_E operator for a degenerate anti-Poisson structure E is given using both Darboux and general coordinates in Subsection 2.3 and 2.4, respectively. The Δ_E formula in general coordinates does require the existence of a compatible two-form fields, however, it does not matter which compatible two-form field that is used (in case there is more than one choice), cf. Lemma 2.7. All information about how the Δ_E operator acts on semidensities can be packed into a Grassmann-odd scalar quantity ν_ρ, which already appeared in eq. (1.4) above. The odd scalar ν_ρ is important, because in practice it is easier to handle a scalar object rather than the full second-order differential operator Δ_E, and hence many of the ensuring arguments is performed using ν_ρ. The Dirac antibracket is an important application of the geometric setup from Section 2, since it always admits a compatible two-form field. Antisymplectic second-class constraints and the Dirac antibracket [6, 8, 14] are reviewed in Subsection 3.1. A Proposition 3.1 in Subsection 3.2 provides a useful formula for the corresponding Dirac odd scalar ν_{ρ,E_D}. Subsection 3.4 discusses the stability of the Dirac construction under reparameterizations of the second-class constraints. In Section 4 the Dirac construction is derived via conversion [15, 16, 17, 18, 19] of the antisymplectic second-class constraints into first-class constraints on an extended manifold. As an application of the construction to Batalin-Vilkovisky quantization, the corresponding Dirac and extended partition functions are provided in Subsections 3.6 and 4.7, respectively. Finally, Section 5 contains our conclusions.

General remark about notation. We have two types of grading: A Grassmann grading ε and an exterior form degree p. The sign conventions are such that two exterior forms ξ and η, of Grassmann parity ε_ξ, ε_η and exterior form degree p_ξ, p_η, respectively, commute in the following graded sense

$$\eta \wedge \xi \;=\; (-1)^{\varepsilon_\xi\varepsilon_\eta + p_\xi p_\eta}\xi \wedge \eta \tag{1.5}$$

inside the exterior algebra. We will often not write the exterior wedges "$\wedge$" explicitly.

2 Anti-Poisson Geometry

2.1 Antibracket and Compatible Two-Form

We consider an anti-Poisson manifold $(M;E^{AB})$ with a (possibly degenerate) antibracket

$$(F,G) \;=\; (F\overleftarrow{\partial^r_A})E^{AB}(\overrightarrow{\partial^l_B}G) \;=\; -(-1)^{(\varepsilon_F+1)(\varepsilon_G+1)}(G,F)\;, \qquad \overrightarrow{\partial^l_A} \;\equiv\; \frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}\;, \tag{2.1}$$

Here the Γ^A's denote local coordinates of Grassmann parity $\varepsilon_A \equiv \varepsilon(\Gamma^A)$, and $E^{AB}=E^{AB}(\Gamma)$ is the local matrix representation of the anti-Poisson structure E. The Jacobi identity

$$\sum_{\text{cycl. } F,G,H} (-1)^{(\varepsilon_F+1)(\varepsilon_H+1)}(F,(G,H)) \;=\; 0 \tag{2.2}$$

reads in local coordinates

$$\sum_{\text{cycl. } A,B,C} (-1)^{(\varepsilon_A+1)(\varepsilon_C+1)}E^{AD}(\overrightarrow{\partial^l_D}E^{BC}) \;=\; 0\;. \tag{2.3}$$

The main new feature (as compared to Ref. [9]) is that the anti-Poisson structure E^{AB} could be degenerate. There is an anti-Poisson analogue of Darboux's Theorem that states that locally, if the rank of E^{AB} is constant, there exist Darboux coordinates $\Gamma^A=\{\phi^\alpha;\phi^*_\alpha;\Theta^a\}$, such that the only non-vanishing antibrackets between the coordinates are $(\phi^\alpha,\phi^*_\beta)=\delta^\alpha_\beta=-(\phi^*_\beta,\phi^\alpha)$. In other words, the Jacobi identity is the integrability condition for the Darboux coordinates. The variables ϕ^α, ϕ^*_α and Θ^a are called *fields*, *antifields* and *Casimirs*, respectively.

We shall assume that the anti-Poisson manifold $(M;E^{AB})$ admits a globally defined odd two-form field E_{AB} with lower indices that is *compatible* with the anti-Poisson structure E^{AB} in the sense that

$$E^{AB}E_{BC}E^{CD} \;=\; E^{AD}\;,$$

$$E_{AB}E^{BC}E_{CD} \;=\; E_{AD} \; . \tag{2.4}$$

As always, the matrices E^{AB} and E_{AB} are assumed to have the Grassmann gradings

$$\varepsilon(E^{AB}) \;=\; \varepsilon_A + \varepsilon_B + 1 \;=\; \varepsilon(E_{AB}) \; , \tag{2.5}$$

and the skew-symmetries

$$\begin{aligned} E^{BA} &= -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} E^{AB} \; , \\ E_{BA} &= -(-1)^{\varepsilon_A\varepsilon_B} E_{AB} \; . \end{aligned} \tag{2.6}$$

The odd two-form field can be written as

$$E \;=\; \frac{1}{2} d\Gamma^A \; E_{AB} \; d\Gamma^B \;=\; -\frac{1}{2} E_{AB} \; d\Gamma^B \; d\Gamma^A \; . \tag{2.7}$$

The two-form field E_{AB} would be closed if

$$dE \;=\; 0 \; , \tag{2.8}$$

or equivalently, with all the indices written out, if

$$\sum_{\text{cycl. } A,B,C} (-1)^{\varepsilon_A\varepsilon_C} (\stackrel{\longrightarrow}{\partial^l_A} E_{BC}) \;=\; 0 \; . \tag{2.9}$$

A closed degenerate two-form is called a pre-antisymplectic structure. In the non-degenerate case, the matrix E_{AB} from eq. (2.4) would be a closed antisymplectic two-form field and the inverse of the anti-Poisson structure E^{AB}. In the degenerate case, there is in general *not a unique* matrix E_{AB} fulfilling eqs. (2.4), (2.5) and (2.6), and there is *no* reason for it to be closed. In Darboux coordinates $\Gamma^A = \{\phi^\alpha; \phi^*_\alpha; \Theta^a\}$, there is still a freedom in a compatible two-form

$$E \;=\; d\phi^*_\alpha \wedge d\phi^\alpha + d\Theta^a M_{a\alpha} \wedge d\phi^\alpha + d\phi^*_\alpha N^\alpha{}_a \wedge d\Theta^a + d\Theta^a M_{a\alpha} N^\alpha{}_b \wedge d\Theta^b \tag{2.10}$$

given by two arbitrary matrices $M_{a\alpha} = M_{a\alpha}(\Gamma)$ and $N^\alpha{}_a = N^\alpha{}_a(\Gamma)$. A Darboux coordinate system $\Gamma^A = \{\phi^\alpha; \phi^*_\alpha; \Theta^a\}$ is called a *bi-Darboux* coordinate system, if the two-form is just $E = d\phi^*_\alpha \wedge d\phi^\alpha$, *i.e.* if both the matrices $M_{a\alpha} = 0$ and $N^\alpha{}_a = 0$ in eq. (2.10) are equal to zero. In short, the Γ^A's are bi-Darboux coordinates, if both matrices E^{AB} and E_{AB} with upper and lower indices are on standard form.

Theorem 2.1 *Given an anti-Poisson manifold $(M; E^{AB})$ with a compatible two-form field E_{AB}. Then there locally exist bi-Darboux coordinates if and only if the two-form field E_{AB} is closed.*

There is a similar Bi-Darboux Theorem for even Poisson structures. A proof of Theorem 2.1 is given in Appendix A. One can define a projection as

$$P^A{}_C \;\equiv\; E^{AB} E_{BC} \; , \tag{2.11}$$

or equivalently,

$$P_A{}^C \;\equiv\; E_{AB} E^{BC} \;=\; (-1)^{\varepsilon_A(\varepsilon_C+1)} P^C{}_A \; . \tag{2.12}$$

It follows from property (2.4) that

$$P^A{}_B P^B{}_C \;=\; P^A{}_C \; . \tag{2.13}$$

In the non-degenerate case $P^A{}_B = \delta^A_B = P_B{}^A$.

2.2 Odd Laplacian Δ_ρ on Scalars

Recall that a scalar function $F=F(\Gamma)$, a density $\rho=\rho(\Gamma)$ and a semidensity $\sigma=\sigma(\Gamma)$ are by definition quantities that transform as

$$F \longrightarrow F' = F , \qquad \rho \longrightarrow \rho' = \frac{\rho}{J} , \qquad \sigma \longrightarrow \sigma' = \frac{\sigma}{\sqrt{J}} , \tag{2.14}$$

respectively, under general coordinate transformations $\Gamma^A \to \Gamma'^A$, where $J \equiv \mathrm{sdet}\frac{\partial\Gamma'^A}{\partial\Gamma^B}$ denotes the Jacobian. We shall ignore the global issues of orientation and choice of square root. Also we assume that densities ρ are invertible.

Definition 2.2 *Given a choice of a density ρ, the* **odd Laplacian** Δ_ρ *is defined as [6]*

$$\Delta_\rho \equiv \frac{(-1)^{\varepsilon_A}}{2\rho}\overrightarrow{\partial^l_A}\rho E^{AB}\overrightarrow{\partial^l_B} . \tag{2.15}$$

This Grassmann-odd, second-order operator takes scalar functions to scalar functions. In situations with more than one anti-Poisson structure E^{AB}, we shall sometimes use the slightly longer notation $\Delta_\rho \equiv \Delta_{\rho,E}$ to acknowledge that it depends on two inputs: ρ and E^{AB}. The odd Laplacian Δ_ρ "differentiates" the antibracket $(\cdot,\cdot)$, *i.e.* the following Leibniz-type rule holds

$$\Delta_\rho(F,G) = (\Delta_\rho F,G) + (-1)^{(\varepsilon_F+1)}(F,\Delta_\rho G) . \tag{2.16}$$

For further information on this important operator, see Ref. [8, 9] and Subsection 2.5 below.

2.3 The Δ_E Operator on Semidensities

There is an another important Grassmann-odd, nilpotent, second-order operator Δ_E that depends only on the anti-Poisson structure E^{AB}. Contrary to the odd Laplacian $\Delta_\rho \equiv \Delta_{\rho,E}$ of last Subsection 2.2, the Δ_E operator does *not* rely on a choice of density ρ. The caveat is that while the odd Laplacian Δ_ρ takes scalars to scalars, the Δ_E operator takes semidensities to semidensities of opposite Grassmann parity. Equivalently, the Δ_E operator transforms as

$$\Delta_E \longrightarrow \Delta'_E = \frac{1}{\sqrt{J}}\Delta_E\sqrt{J} \tag{2.17}$$

under general coordinate transformations $\Gamma^A \to \Gamma'^A$, cf. eq. (2.14). It is defined as follows:

Definition 2.3 *Let there be given an anti-Poisson manifold $(M;E)$. In Darboux coordinates Γ^A, the* Δ_E **operator** *is defined on a semidensity σ as [8, 10, 11, 12, 13]*

$$(\Delta_E\sigma) \equiv (\Delta_1\sigma) , \tag{2.18}$$

where Δ_1 denotes the expression (2.15) for the odd Laplacian $\Delta_{\rho=1}$ with ρ replaced by 1.

It is implicitly understood in eq. (2.18) that the formula for the Δ_1 operator (2.15) and the semidensity σ both refer to the same Darboux coordinates Γ^A. The parentheses in eq. (2.18) indicate that the equation should be understood as an equality among semidensities (in the sense of zeroth-order differential operators) rather than an identity among differential operators. The Definition 2.3 does not depend on the Darboux coordinate system being used, due to the following Lemma 2.4:

Lemma 2.4 *When using the Definition 2.3, the* $(\Delta_E\sigma)$ *transforms as a semidensity under (anti-canonical) transformations between sets of Darboux coordinates.*

Thus the Δ_E operator is a well-defined operator on an open cover of Darboux neighborhoods. Within this cover, the Δ_E is indirectly defined in non-Darboux coordinates by use of the transformation property (2.17). Lemma 2.4 was first proven in the non-degenerate case in Ref. [13] and in the degenerate case in Ref. [8]. We shall also give an independent proof in the next Subsection 2.4, cf. Lemma 2.6 below. In some cases the Δ_E operator may be extended to singular points (*i.e.* points where the rank of the anti-Poisson tensor E^{AB} jumps) by continuity.

Working in Darboux coordinates, it is obvious that the Δ_E operator super-commutes with itself, because the Γ^A-derivatives have no Γ^A's to act on when E^{AB} is on Darboux form. Therefore Δ_E is nilpotent,

$$\Delta_E^2 \;=\; \frac{1}{2}[\Delta_E, \Delta_E] \;=\; 0 \; . \tag{2.19}$$

Same sort of reasoning shows that $\Delta_E = \Delta_E^T$ is symmetric.

2.4 The Δ_E Operator in General Coordinates

We now give a definition of the Δ_E operator that does not refer to Darboux coordinates.

Definition 2.5 *Given an anti-Poisson manifold* $(M; E^{AB})$ *that admits a compatible two-form field* E_{AB}*. In arbitrary coordinates* Γ^A*, the* Δ_E *operator is defined as*

$$(\Delta_E\sigma) \;\equiv\; (\Delta_1\sigma) + \left(\frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{8} - \frac{\nu^{(3)}}{24} + \frac{\nu^{(4)}}{24} + \frac{\nu^{(5)}}{12}\right)\sigma \; , \tag{2.20}$$

where

$$\nu^{(1)} \;\equiv\; (-1)^{\varepsilon_A}(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A}E^{AB}) \; , \tag{2.21}$$

$$\nu^{(2)} \;\equiv\; (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_D}E^{AB})E_{BC}(\overrightarrow{\partial^l_A}E^{CD}) \; , \tag{2.22}$$

$$\nu^{(3)} \;\equiv\; (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}E^{BA}) \; , \tag{2.23}$$

$$\nu^{(4)} \;\equiv\; (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}E^{BF})P_F{}^A \; , \tag{2.24}$$

$$\begin{aligned}\nu^{(5)} \;&\equiv\; (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_D}E^{AB})E_{BC}(\overrightarrow{\partial^l_A}E^{CF})P_F{}^D \\ &=\; (-1)^{(\varepsilon_A+1)\varepsilon_B}E^{AD}(\overrightarrow{\partial^l_D}E^{BC})(\overrightarrow{\partial^l_C}E_{AF})P^F{}_B \; .\end{aligned} \tag{2.25}$$

Notice that in Darboux coordinates, where E^{AB} is constant, *i.e.* independent of the coordinates Γ^A, the last five terms $\nu^{(1)}$, $\nu^{(2)}$, $\nu^{(3)}$, $\nu^{(4)}$ and $\nu^{(5)}$ become zero. Hence the new Definition 2.5 agrees in Darboux coordinates with the previous Definition 2.3. The benefit of the new Definition 2.5 is that one now have an explicit formula for Δ_E in an arbitrary coordinate system. The full justification of Definition 2.5 is provided by the following Lemma 2.6 and Lemma 2.7.

Lemma 2.6 *When using the new Definition 2.5, the* $(\Delta_E\sigma)$ *transforms as a semidensity under general coordinate transformations.*

Lemma 2.7 *When using the new Definition 2.5, the* $(\Delta_E\sigma)$ *does not depend on the compatible two-form field* E_{AB} *used.*

The explicit formula (2.20) and Lemma 2.6 are the main results of Section 2.

PROOF OF LEMMA 2.7: The two-form field E_{AB} enters only the Definition 2.5 via $\nu^{(2)}$, $\nu^{(3)}$, $\nu^{(4)}$ and $\nu^{(5)}$. Assuming the Lemma 2.6, *i.e.* that the behavior (2.17) under general coordinate transformations has already been established, one may, in particular, go to Darboux coordinates, where $\nu^{(2)}$, $\nu^{(3)}$, $\nu^{(4)}$ and $\nu^{(5)}$ vanish identically.

□

To prove Lemma 2.6 we shall first reformulate it as an equivalent Lemma 2.9, cf. below. We shall also only explicitly consider the case where σ is invertible to simplify the presentation. (The non-invertible case is fundamentally no different.) In the invertible case, we customarily write the semidensity $\sigma = \sqrt{\rho}$ as a square root of a density ρ, and define a Grassmann-odd quantity ν_ρ as follows.

Definition 2.8 *The* **odd scalar** ν_ρ *is defined as*

$$\nu_\rho \;\equiv\; \frac{1}{\sqrt{\rho}}(\Delta_E\sqrt{\rho}) \;=\; \nu_\rho^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{8} - \frac{\nu^{(3)}}{24} + \frac{\nu^{(4)}}{24} + \frac{\nu^{(5)}}{12} \;, \tag{2.26}$$

where $\nu^{(1)}$, $\nu^{(2)}$, $\nu^{(3)}$, $\nu^{(4)}$, $\nu^{(5)}$ *are given in eqs. (2.21)–(2.25), and the quantity* $\nu_\rho^{(0)}$ *is given as*

$$\nu_\rho^{(0)} \;\equiv\; \frac{1}{\sqrt{\rho}}(\Delta_1\sqrt{\rho}) \;. \tag{2.27}$$

In situations with more than one anti-Poisson structure E^{AB}, we shall sometimes use the slightly longer notation $\nu_\rho \equiv \nu_{\rho,E}$. By dividing both sides of the definition (2.20) with the semidensity σ, one may reformulate the content of Lemma 2.6 as:

Lemma 2.9 *The Grassmann-odd quantity* ν_ρ *is a scalar, i.e. it does not depend on the coordinate system.*

We shall give two independent proofs of this important Lemma 2.9; one relying on Darboux Theorem and the other using infinitesimal coordinate transformations.

PROOF OF LEMMA 2.9 USING A DARBOUX COORDINATE PATCH: It is enough to consider how ν_ρ behaves on coordinate transformations $\Gamma_0^A \to \Gamma^A$ between Darboux coordinates Γ_0^A and general coordinates Γ^A. (An arbitrary coordinate transformation between two general coordinate patches can always be split into two successive coordinate transformations of the above kind by inserting a third Darboux coordinate patch in between.) The idea is now to first consider the expression (2.26) for ν_ρ in the Γ^A coordinate system, and decompose it in building blocks that refer to the Darboux coordinates Γ_0^A, *e.g.*

$$E^{AD} \;=\; (\Gamma^A \frac{\overleftarrow{\partial^r}}{\partial\Gamma_0^B}) E_0^{BC} (\frac{\overrightarrow{\partial^l}}{\partial\Gamma_0^C}\Gamma^D) \;, \qquad E_{AD} \;=\; (\frac{\overrightarrow{\partial^l}}{\partial\Gamma^a}\Gamma_0^B) \overset{0}{E}_{BC} (\Gamma_0^C \frac{\overleftarrow{\partial^r}}{\partial\Gamma^D}) \;, \qquad \rho \;=\; \frac{\rho_0}{J} \;. \tag{2.28}$$

Here $J \equiv \mathrm{sdet}(\partial\Gamma^{A}/\partial\Gamma_{0}^{B})$ denotes the Jacobian of the coordinate transformations $\Gamma_{0}^{A} \to \Gamma^{A}$. Recall that the two-form field E_{BC}^{0} is not necessarily constant in the Darboux coordinates Γ_{0}^{A}, cf eq. (2.10). By straightforward calculation, one gets

$$\nu_{\rho}^{(0)} \equiv \frac{1}{\sqrt{\rho}}(\Delta_{1,E}\sqrt{\rho}) = \frac{1}{\sqrt{\rho}}(\Delta_{J,E_{0}}\sqrt{\rho}) = \frac{1}{\sqrt{\rho_{0}}}(\Delta_{1,E_{0}}\sqrt{\rho_{0}}) - \frac{1}{\sqrt{J}}(\Delta_{1,E_{0}}\sqrt{J}) , \quad (2.29)$$

$$\nu^{(1)} = \frac{8}{\sqrt{J}}(\Delta_{1,E_{0}}\sqrt{J}) - (-1)^{\varepsilon_{B}}(\frac{\overrightarrow{\partial^{l}}}{\partial\Gamma_{0}^{A}}\Gamma^{B}, \frac{\overrightarrow{\partial^{l}}}{\partial\Gamma^{B}}\Gamma_{0}^{A}) , \quad (2.30)$$

$$\nu^{(2)} = -(-1)^{\varepsilon_{B}}(\frac{\overrightarrow{\partial^{l}}}{\partial\Gamma_{0}^{A}}\Gamma^{B}, \frac{\overrightarrow{\partial^{l}}}{\partial\Gamma^{B}}\Gamma_{0}^{A}) - 2(-1)^{\varepsilon_{B}}(\frac{\overrightarrow{\partial^{l}}}{\partial\Gamma_{0}^{A}}\Gamma^{B}, \frac{\overrightarrow{\partial^{l}}}{\partial\Gamma^{B}}\Gamma_{0}^{C})P_{C}^{0,A} , \quad (2.31)$$

$$\begin{aligned}\nu^{(3)} = {} & 3(-1)^{\varepsilon_{B}}(\frac{\overrightarrow{\partial^{l}}}{\partial\Gamma_{0}^{A}}\Gamma^{B}, \frac{\overrightarrow{\partial^{l}}}{\partial\Gamma^{B}}\Gamma_{0}^{C})P_{C}^{0,A} \\ & -(-1)^{(\varepsilon_{A}+1)(\varepsilon_{C}+1)}(\Gamma_{0}^{A}\frac{\overleftarrow{\partial^{r}}}{\partial\Gamma^{B}})(\Gamma^{B}\frac{\overleftarrow{\partial^{r}}}{\partial\Gamma_{0}^{C}}, E_{AD}^{0})E_{0}^{DC} , \end{aligned} \quad (2.32)$$

$$\begin{aligned}\nu^{(4)} = {} & (-1)^{\varepsilon_{B}}(\frac{\overrightarrow{\partial^{l}}}{\partial\Gamma_{0}^{A}}\Gamma^{B}, \frac{\overrightarrow{\partial^{l}}}{\partial\Gamma^{B}}\Gamma_{0}^{C})P_{C}^{0,A} + 2(-1)^{\varepsilon_{A}\varepsilon_{C}}P_{A}^{0,B}(\frac{\overrightarrow{\partial^{l}}}{\partial\Gamma_{0}^{B}}\Gamma^{C}, \Gamma_{0}^{A}\frac{\overleftarrow{\partial^{r}}}{\partial\Gamma^{D}})P^{D}{}_{C} \\ & -(-1)^{(\varepsilon_{A}+1)(\varepsilon_{C}+1)}(\Gamma_{0}^{A}\frac{\overleftarrow{\partial^{r}}}{\partial\Gamma^{B}})(\Gamma^{B}\frac{\overleftarrow{\partial^{r}}}{\partial\Gamma_{0}^{C}}, E_{AD}^{0})E_{0}^{DC} , \end{aligned} \quad (2.33)$$

$$\nu^{(5)} = -2(-1)^{\varepsilon_{B}}(\frac{\overrightarrow{\partial^{l}}}{\partial\Gamma_{0}^{A}}\Gamma^{B}, \frac{\overrightarrow{\partial^{l}}}{\partial\Gamma^{B}}\Gamma_{0}^{C})P_{C}^{0,A} - (-1)^{\varepsilon_{A}\varepsilon_{C}}P_{A}^{0,B}(\frac{\overrightarrow{\partial^{l}}}{\partial\Gamma_{0}^{B}}\Gamma^{C}, \Gamma_{0}^{A}\frac{\overleftarrow{\partial^{r}}}{\partial\Gamma^{D}})P^{D}{}_{C} . \quad (2.34)$$

The last equality in eq. (2.29) is a non-trivial property of the odd Laplacian. It is now easy to check that all but one of the above terms on the right-hand sides of eqs. (2.29)–(2.34) cancel in the pertinent linear combination (2.26), *i.e.*

$$\nu_{\rho} = \nu_{\rho}^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{8} - \frac{\nu^{(3)}}{24} + \frac{\nu^{(4)}}{24} + \frac{\nu^{(5)}}{12} = \frac{1}{\sqrt{\rho_{0}}}(\Delta_{1,E_{0}}\sqrt{\rho_{0}}) . \quad (2.35)$$

The surviving term, on the other hand, is just the definition for ν_{ρ} in the Darboux coordinates Γ_{0}^{A}.

□

Proof of Lemma 2.9 using infinitesimal coordinate transformations: Under an arbitrary infinitesimal coordinate transformation $\delta\Gamma^{A} = X^{A}$, one calculates

$$\delta\nu_{\rho}^{(0)} = -\frac{1}{2}\Delta_{1}\mathrm{div}_{1}X , \quad (2.36)$$

$$\delta\nu^{(1)} = 4\Delta_{1}\mathrm{div}_{1}X + (-1)^{\varepsilon_{A}}(\overrightarrow{\partial^{l}_{C}}E^{AB})(\overrightarrow{\partial^{l}_{B}}\overrightarrow{\partial^{l}_{A}}X^{C}) , \quad (2.37)$$

$$\delta\nu^{(2)} = (-1)^{\varepsilon_{A}}(\overrightarrow{\partial^{l}_{D}}E^{AB})\left(2P_{B}{}^{C}(\overrightarrow{\partial^{l}_{C}}\overrightarrow{\partial^{l}_{A}}X^{D}) + (\overrightarrow{\partial^{l}_{B}}\overrightarrow{\partial^{l}_{A}}X^{C})P_{C}{}^{D}\right) , \quad (2.38)$$

$$\begin{aligned}\delta\nu^{(3)} = {} & (-1)^{\varepsilon_{B}}(\overrightarrow{\partial^{l}_{A}}E_{BC})E^{CD}\left((\overrightarrow{\partial^{l}_{D}}X^{B}\overleftarrow{\partial^{r}_{F}})E^{FA} - (-1)^{(\varepsilon_{A}+1)(\varepsilon_{B}+1)}(\overrightarrow{\partial^{l}_{D}}X^{A}\overleftarrow{\partial^{r}_{F}})E^{FB}\right) \\ & -\frac{3}{2}(-1)^{\varepsilon_{A}}P_{C}{}^{D}(\overrightarrow{\partial^{l}_{D}}E^{AB})(\overrightarrow{\partial^{l}_{B}}\overrightarrow{\partial^{l}_{A}}X^{C}) , \end{aligned} \quad (2.39)$$

$$\delta\nu^{(4)} = -2(-1)^{\varepsilon_{B}}(\overrightarrow{\partial^{l}_{A}}\overrightarrow{\partial^{l}_{B}}X^{C})P_{C}{}^{D}(\overrightarrow{\partial^{l}_{D}}E^{BF})P_{F}{}^{A}$$

$$
\begin{aligned}
& +(-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}X^B\overleftarrow{\partial^r_F})E^{FA} \\
& +(-1)^{(\varepsilon_B+1)\varepsilon_F}P_F{}^A(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}X^F\overleftarrow{\partial^r_G})E^{GB} \\
& +\frac{1}{2}(-1)^{\varepsilon_A}P_C{}^D(\overrightarrow{\partial^l_D}E^{AB})(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A}X^C)\ , \qquad (2.40)
\end{aligned}
$$

$$
\begin{aligned}
\delta\nu^{(5)} \ = \ & -(-1)^{\varepsilon_A(\varepsilon_B+1)}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}X^A\overleftarrow{\partial^r_F})E^{FB} \\
& +2(-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}\overrightarrow{\partial^l_B}X^C)P_C{}^D(\overrightarrow{\partial^l_D}E^{BF})P_F{}^A\ . \qquad (2.41)
\end{aligned}
$$

A proof of eqs. (2.36) and (2.37) can be found in Ref. [9], and eqs. (2.38)–(2.41) are proven in Appendix B. One may verify that while the six constituents $\nu_\rho^{(0)}$, $\nu^{(1)}$, $\nu^{(2)}$, $\nu^{(3)}$, $\nu^{(4)}$ and $\nu^{(5)}$ separately have non-trivial transformation properties, the linear combination ν_ρ in eq. (2.26) is indeed a scalar.

□

The new Definition 2.5 is clearly symmetric $\Delta_E = \Delta_E^T$. To check explicitly in general coordinates that Δ_E is nilpotent is a straightforward (but admittedly tedious) exercise. However, since we have just proven that Δ_E behaves covariantly under general coordinate transformations, our previous proof of nilpotency from last Subsection 2.3 using Darboux coordinates suffices. To summarize:

Theorem 2.10 *The Δ_E operator (2.20) is nilpotent (2.19) if and only if the antibracket (2.1) satisfies the Jacobi identity (2.3).*

In the rest of the paper we will always assume that the Jacobi identity (2.3) is satisfied, and hence that the Δ_E operator (2.20) is nilpotent.

2.5 Nilpotency Condition for the odd Laplacian Δ_ρ

At this point it is instructive to recall the nilpotency condition for the odd Laplacian Δ_ρ, although we shall not assume that it is satisfied. It follows from the Jacobi identity (2.3) alone, that Δ_ρ^2 is a linear derivation, *i.e.* a first-order differential operator. The interplay between the two second-order differential operators Δ_E and Δ_ρ is perhaps best summarized by the following operator identity:

$$
\Delta_\rho + \nu_\rho \ = \ \frac{1}{\sqrt{\rho}}\Delta_E\sqrt{\rho}\ , \qquad (2.42)
$$

cf. eq. (5.9) of Ref. [9]. In words: Apart from the ν_ρ term the odd Laplacian Δ_ρ is the Δ_E operator dressed with a $\sqrt{\rho}$ factor. From this operator identity (2.42) and the nilpotency (2.19) of the Δ_E operator, one derives the explicit form of the linear derivation:

$$
\Delta_\rho^2 \ = \ (\nu_\rho,\ \cdot\ )\ . \qquad (2.43)
$$

Therefore the nilpotency condition for Δ_ρ reads [8, 11]

$$
\Delta_\rho^2 \ = \ 0 \qquad \Leftrightarrow \qquad \nu_\rho \text{ is a Casimir.} \qquad (2.44)
$$

Let us also mention for later that if one acts with the operator identity (2.42) on a scalar function $\sqrt{F}$, one gets

$$
\nu_{\rho F} \ = \ \nu_\rho + \frac{1}{\sqrt{F}}(\Delta_\rho\sqrt{F})\ . \qquad (2.45)
$$

2.6 Alternative Expressions

It is convenient to introduce

$$\begin{aligned}
\nu^{(23)} &\equiv \nu^{(2)}+\nu^{(3)} = (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}P_B{}^C)(\overrightarrow{\partial^l_C}E^{BA}) , && (2.46)\\
\nu^{(35)} &\equiv \nu^{(3)}+\nu^{(5)} = (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}P_B{}^C)P_C{}^D(\overrightarrow{\partial^l_D}E^{BA}) \\
&= (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}P^B{}_C)E^{CD}(\overrightarrow{\partial^l_D}P_B{}^A) , && (2.47)\\
\nu^{(45)} &\equiv \nu^{(4)}+\nu^{(5)} = (-1)^{\varepsilon_B}P_A{}^D(\overrightarrow{\partial^l_D}P_B{}^C)(\overrightarrow{\partial^l_C}E^{BA}) , && (2.48)\\
\nu^{(23)}_{(45)} &\equiv \nu^{(23)}-\nu^{(45)} = (-1)^{\varepsilon_B(\varepsilon_D+1)}(\overrightarrow{\partial^l_A}P^B{}_C)E^{CD}(\overrightarrow{\partial^l_B}P_D{}^A) . && (2.49)
\end{aligned}$$

Then the Δ_E operator (2.20) may be re-written as

$$\begin{aligned}
(\Delta_E\sigma) &= (\Delta_1\sigma)+\left(\frac{\nu^{(1)}}{8}-\frac{\nu^{(2)}}{24}-\frac{\nu^{(23)}}{12}+\frac{\nu^{(35)}+\nu^{(45)}}{24}\right)\sigma , && (2.50)\\
&= (\Delta_1\sigma)+\left(\frac{\nu^{(1)}}{8}-\frac{\nu^{(2)}+\nu^{(23)}-\nu^{(35)}+\nu^{(23)}_{(45)}}{24}\right)\sigma . && (2.51)
\end{aligned}$$

In the closed case (2.8) one may show that

$$\nu^{(35)}+\nu^{(45)} = 0 , \qquad (2.52)$$

so that the Δ_E operator (2.50) simplifies to

$$(\Delta_E\sigma) = (\Delta_1\sigma)+\left(\frac{\nu^{(1)}}{8}-\frac{\nu^{(2)}}{8}-\frac{\nu^{(3)}}{12}\right)\sigma . \qquad (2.53)$$

In the non-degenerate case, which is automatically closed, one also has

$$\nu^{(23)} = 0 , \qquad (2.54)$$

so that the Δ_E operator (2.50) simplifies even further to

$$(\Delta_E\sigma) = (\Delta_1\sigma)+\left(\frac{\nu^{(1)}}{8}-\frac{\nu^{(2)}}{24}\right)\sigma , \qquad (2.55)$$

in agreement with eq. (5.1) in Ref. [9].

3 Second-Class Constraints

3.1 Review of Dirac Antibracket

One of the most important examples of degenerate anti-Poisson structures is provided by the Dirac antibracket [6, 8, 14]. Consider a manifold $(M;E)$ with a non-degenerate anti-Poisson structure E^{AB} (called an antisymplectic phase space), and let a submanifold $\tilde{M}\equiv\{\Gamma\in M|\Theta(\Gamma)=0\}$ be the zero-locus of a set of constraints $\Theta^a\!=\!\Theta^a(\Gamma)$ with Grassmann parity $\varepsilon(\Theta^a)\!=\!\varepsilon_a$. (In this Subsection, the defining set of constraints is kept fixed for simplicity. We will consider reparametrizations of the constraints in Subsection 3.4.) Assume that the Θ^a constraints are second-class in the antibracket sense, *i.e.* the antibracket matrix

$$E^{ab} \equiv (\Theta^a, \Theta^b) \tag{3.1}$$

of the Θ^a constraints has by definition an inverse matrix E_{ab},

$$E_{ab}E^{bc} = \delta_a^c . \tag{3.2}$$

The Dirac antibracket is defined completely analogous to the usual Dirac bracket for even Poisson brackets [6],

$$(F,G)_D \equiv (F,G) - (F,\Theta^a)E_{ab}(\Theta^b,G) , \tag{3.3}$$

or in coordinates,

$$E^{AB}_{(D)} \equiv E^{AB} - (\Gamma^A,\Theta^a)E_{ab}(\Theta^b,\Gamma^B) . \tag{3.4}$$

The Dirac antibracket satisfies a strong Jacobi identity

$$\sum_{F,G,H \text{ cycl.}} (-1)^{(\varepsilon_F+1)(\varepsilon_H+1)}((F,G)_D,H)_D = 0 . \tag{3.5}$$

The adjective "strong" stresses the fact that the Jacobi identity holds off-shell with respect to the second-class constraints Θ^a, *i.e.* everywhere in the phase space M. There is a canonical Dirac two-form given by

$$E^D \equiv E - \frac{1}{2}d\Theta^a E_{ab}\wedge d\Theta^b , \tag{3.6}$$

or in local coordinates

$$E^{(D)}_{AB} \equiv E_{AB} - (\overrightarrow{\partial^l_A}\Theta^a)E_{ab}(\Theta^b\overleftarrow{\partial^r_B}) . \tag{3.7}$$

The two-form field $E^{(D)}_{AB}$ is compatible with the Dirac bracket, *i.e.* it satisfies the property (2.4), but it is *not* necessarily closed. Local coordinates $\Gamma^A = \{\gamma^A;\Theta^a\}$, where the second-class constraints Θ^a are part of the coordinates, are called *unitarizing* coordinates. In the physics terminology, the second-class constraints Θ^a represent unphysical degrees of freedom, which can be eliminated from the system, *i.e.* put to zero, to reveal a reduced submanifold $\tilde{M}$, whose coordinates γ^A constitute the true physical degrees of freedom. *Notation*: We use capital roman letters A, B, C, ... from the beginning of the alphabet as upper index for both the full and the reduced variables Γ^A and γ^A, respectively. A tilde "$\sim$" over an object will denote the corresponding reduced object.

Unitarizing coordinates $\Gamma^A = \{\gamma^A;\Theta^a\}$, where the second-class variables Θ^a and the physical variables γ^A are perpendicular to each other in the antibracket sense

$$(\gamma^A,\Theta^a) = 0 , \tag{3.8}$$

are called *transversal* coordinates. One may prove that transversal coordinate systems exist locally, although one might have to reparametrize the Θ^a constraints in order to get to them, cf. Subsection 3.4 below.

3.2 The Dirac Operators Δ_{E_D} and Δ_{ρ,E_D}

The next step is to build Khudaverdian's BV operator Δ_{E_D} for the degenerate Dirac antibracket structure (3.3), and, if a density ρ is available, the odd Laplacian Δ_{ρ,E_D}. In other words, one should substitute $E \to E_D$ everywhere in the previous Section 2. Some facts about the Δ_{E_D} operator are immediately clear. First of all, it is covariant under general coordinate transformations, cf. Subsection 2.4. Furthermore, it is strongly nilpotent

$$\Delta_{E_D}^2 \;=\; 0\;, \tag{3.9}$$

due to the strong Jacobi identity (3.5) and Theorem 2.10. The following Proposition 3.1 expresses the Dirac odd scalar ν_{ρ,E_D} in terms of the non-degenerate antisymplectic structure and the second-class constraints Θ^a.

Proposition 3.1 *The* **Dirac odd scalar** ν_{ρ,E_D} *is given by*

$$\nu_{\rho,E_D} \;=\; \nu_\rho - \frac{\nu_{\rho,D}^{(6)}}{2} - \frac{\nu_{\rho,D}^{(7)}}{2} - \frac{\nu_D^{(8)}}{8} + \frac{\nu_D^{(9)}}{24}\;, \tag{3.10}$$

where $\nu_\rho \equiv \nu_{\rho,E}$ *is the odd scalar for the non-degenerate antisymplectic structure* E*, and*

$$\nu_{\rho,D}^{(6)} \;\equiv\; (\Delta_\rho\Theta^a)E_{ab}(\Delta_\rho\Theta^b)(-1)^{\varepsilon_b}\;, \tag{3.11}$$

$$\nu_{\rho,D}^{(7)} \;\equiv\; (-1)^{\varepsilon_a+\varepsilon_b}(\Theta^a, E_{ab}(\Delta_\rho\Theta^b)) \;=\; (\Theta^a, (\Delta_\rho\Theta^b)E_{ba})\;, \tag{3.12}$$

$$\nu_D^{(8)} \;\equiv\; (-1)^{\varepsilon_b}(\Theta^a, (\Theta^b, E_{ba}))\;, \tag{3.13}$$

$$\begin{aligned}\nu_D^{(9)} \;&\equiv\; (-1)^{(\varepsilon_a+1)(\varepsilon_d+1)}(\Theta^d, E_{ab})E^{bc}(E_{cd}, \Theta^a)\\ &=\; -(-1)^{\varepsilon_b}(\Theta^a, E^{bc})E_{cd}(\Theta^d, E_{ba})\;.\end{aligned} \tag{3.14}$$

Proof of Proposition 3.1: Since both sides of eq. (3.10) are scalars under general coordinate transformations, it is sufficient to work in Darboux coordinates for the non-degenerate E^{AB} structure. By straightforward calculation, one gets

$$\nu_{\rho,D}^{(0)} \;=\; \nu_\rho^{(0)} - (\Delta_1\Theta^a)E_{ab}(\Theta^b, \ln\sqrt{\rho}) - \frac{(-1)^{\varepsilon_a}}{2\sqrt{\rho}}(\Theta^a, E_{ab}(\Theta^b, \sqrt{\rho}))\;, \tag{3.15}$$

$$\begin{aligned}\nu_D^{(1)} \;&=\; -4(\Delta_1\Theta^a)E_{ab}(\Delta_1\Theta^b)(-1)^{\varepsilon_b} - 4(-1)^{\varepsilon_a+\varepsilon_b}(\Theta^a, E_{ab}(\Delta_1\Theta^b))\\ &\quad -\nu_D^{(8)} - (-1)^{\varepsilon_a}(\overrightarrow{\partial^l_A}\Theta^a,\; E_{ab})(\Theta^b, \Gamma^A)\;,\end{aligned} \tag{3.16}$$

$$\begin{aligned}\nu_D^{(2)} \;&=\; (-1)^{\varepsilon_b\varepsilon_c}E_{ca}(\Theta^a,\; \Theta^b\overleftarrow{\partial^r_B})E_{(D)}^{BC}(\overrightarrow{\partial^l_C}\Theta^c,\; \Theta^d)E_{db}\\ &=\; -(-1)^{\varepsilon_a}(\Theta^b,\; \Theta^a\overleftarrow{\partial^r_A})(\Gamma^A, E_{ab})_D \;=\; -(-1)^{\varepsilon_a}(\Theta^b,\; \Theta^a\overleftarrow{\partial^r_A})(\Gamma^A, E_{ab}) - \frac{\nu_D^{(9)}}{3}\;,\end{aligned} \tag{3.17}$$

$$\nu_D^{(3)} \;=\; 0\;, \tag{3.18}$$

$$\nu_D^{(4)} \;=\; 0\;, \tag{3.19}$$

$$\nu_D^{(5)} \;=\; 0\;. \tag{3.20}$$

The pertinent linear combination (2.26) of eqs. (3.15)–(3.20) yields the eq. (3.10).

□

3.3 Annihilation Relations

The fact that the Θ^a constraints are null-directions for the Dirac construction is reflected slightly differently in 1) the Dirac antibracket $(\cdot,\cdot)_D$, 2) the Dirac odd Laplacian Δ_{ρ,E_D}, and 3) the Δ_{E_D}

operator. Explicitly, for a scalar function F, a density ρ and a semidensity σ, one has

$$
\begin{aligned}
(F, \Theta^a)_D &= 0 , &(3.21)\\
(\Delta_{\rho, E_D} \Theta^a) &= \frac{(-1)^{\varepsilon_A}}{2\rho} \overrightarrow{\partial}^l_A \rho (\Gamma^A, \Theta^a)_D \;=\; 0 , &(3.22)\\
[\overrightarrow{\Delta}_{E_D}, \Theta^a]\sigma &= [\overrightarrow{\Delta}_{1,E_D}, \Theta^a]\sigma \;=\; (\Delta_{1,E_D} \Theta^a)\sigma + (-1)^{\varepsilon_a} (\Theta^a, \sigma)_D \;=\; 0 , &(3.23)
\end{aligned}
$$

respectively. Eqs. (3.21)–(3.23) generalize to

$$
\begin{aligned}
(F, f(\Theta))_D &= 0 , &(3.24)\\
(\Delta_{\rho, E_D} f(\Theta)) &= 0 , &(3.25)\\
[\overrightarrow{\Delta}_{E_D}, f(\Theta)]\sigma &= 0 , &(3.26)
\end{aligned}
$$

for an arbitrary function $f(\Theta)$ of the constraints Θ^a. (In other words: f is here assumed not to depend on the physical variables γ^A.) Note however, that if Θ^a is not among the defining set of constraints, but only a linear combination of those (*i.e.* the coefficients in the linear combination could involve the physical variables γ^A), the last equality in each of the above eqs. (3.21)–(3.26) becomes weak, *i.e.* there could be off-shell contributions, cf. next Subsection 3.4 and Ref. [8].

3.4 Reparametrization of Second-Class Constraints

A general and tricky feature of the Dirac construction, is, that it *changes* if one uses another defining set of second-class constraints

$$\Theta^a \qquad \longrightarrow \qquad \Theta'^a \;=\; \Lambda^a{}_b(\Gamma)\, \Theta^b . \tag{3.27}$$

However, the dependence is so soft that physics, which lives on-shell, is not affected [8]. We shall here clarify in exactly what sense the Δ_{E_D} operator remains invariant on-shell under reparametrization of the constraints.

To warm up, let us recall that the Dirac antibrackets $(F, G)_D$ and $(F, G)'_D$, defined using the primed and unprimed constraints Θ'^a and Θ^a, respectively, are the same on-shell

$$(F, G)'_D \;\approx\; (F, G)_D . \tag{3.28}$$

Here the symbol "≈" is the Dirac weak equivalence symbol, which denotes equivalence modulo terms of order $\mathcal{O}(\Theta)$. More generally,

$$F' \approx F \;\wedge\; G' \approx G \qquad \Rightarrow \qquad (F', G')'_D \;\approx\; (F, G)_D . \tag{3.29}$$

Hence the reduced bracket

$$(\tilde{F}, \tilde{G})_\sim \;\equiv\; (F, G)_D|_{\Theta=0} , \tag{3.30}$$

is independent of both the choice of constraints Θ^a and the representatives $F = F(\Gamma)$, $G = G(\Gamma)$ on M. Here $\tilde{F} \equiv F|_{\Theta=0} = \tilde{F}(\gamma)$ and $\tilde{G} \equiv G|_{\Theta=0} = \tilde{G}(\gamma)$ are functions on the physical submanifold $\tilde{M}$.

On the other hand, to have a well-defined notion of reduced densities and semidensities on the physical submanifold $\tilde{M}$, it is necessary to let the densities and semidensities transform as

$$\rho' \;\approx\; \rho\, \Lambda , \qquad \sigma' \;\approx\; \sigma \sqrt{\Lambda} , \tag{3.31}$$

under reparametrization of the defining set of constraints $\Theta^a \to \Theta'^a = \Lambda^a{}_b\,\Theta^b$. Here

$$\Lambda \;\equiv\; \mathrm{sdet}(\Lambda^a{}_b) \tag{3.32}$$

denotes the superdeterminant of the reparametrization matrix $\Lambda^a{}_b = \Lambda^a{}_b(\Gamma)$. The reduction

$$\tilde{\rho} \;\equiv\; \rho|_{\Theta=0}\;, \qquad \tilde{\sigma} \;\equiv\; \sigma|_{\Theta=0}\;, \tag{3.33}$$

is then by definition performed in a unitarizing coordinate system $\Gamma^A = \{\gamma^A;\Theta^a\}$, where it is implicitly understood that the Θ^a coordinates coincide with the defining set of constraints. Similar to the Dirac antibracket $(\cdot,\cdot)_D$, we imagine that the densities and semidensities refer to an internal defining set of Θ^a constraints. If one chooses another defining set of constraints Θ'^a, and an accompanying unitarizing coordinate system $\Gamma'^A = \{\gamma'^A;\Theta'^a\}$, the superdeterminant factor Λ in the reparametrization rule (3.31) is designed to cancel the Jacobian factor J from the coordinate transformation (2.14) on-shell, so that the reduced definition (3.33) stays the same.

Similarly, it is necessary that the Δ_{E_D} operator, which takes semidensities to semidensities, transforms as

$$\Delta'_{E_D} \;\approx\; \sqrt{\Lambda}\;\Delta_{E_D}\frac{1}{\sqrt{\Lambda}} \tag{3.34}$$

as an operator identity. Stated more precisely, the odd scalar ν_{ρ,E_D} from Definition 2.8 should be invariant on-shell

$$\nu_{\rho',E'_D} \;\approx\; \nu_{\rho,E_D} \tag{3.35}$$

under reparametrization of the constraints. This is the core issue at stake. To prove that it indeed holds, first note that it is enough to check the claim (3.35) if the set of unprimed constraints Θ^a happens to belong to a set of transversal coordinates $\Gamma^A = \{\gamma^A;\Theta^a\}$. (If this is not the case, one can always locally find a transversal coordinate system, and split the above reparamerization into two successive reparamerizations that both involve the transversal coordinates.) Transversal coordinates will simplify considerably the ensuing calculations. In general, the on-shell change of ν_{ρ,E_D} depends on how the Dirac antibracket $(\cdot,\cdot)_D$ changes up to the second order in Θ^a, cf. eq. (4.39) in Ref. [8]. Explicitly, one may show that the quantities $\nu^{(0)}_{\rho,D}$, $\nu^{(1)}_D$, $\nu^{(2)}_D$, $\nu^{(3)}_D$, $\nu^{(4)}_D$ and $\nu^{(5)}_D$, defined in eqs. (2.21)–(2.25) and (2.27), transform as

$$\nu'^{(0)}_{\rho,D} \;\equiv\; \frac{1}{\sqrt{\rho'}}(\Delta_{1,E'_D}\sqrt{\rho'}) \;\approx\; \frac{1}{\sqrt{\Lambda\rho}}(\Delta_{\frac{1}{\Lambda},E_D}\sqrt{\Lambda\rho}) \;=\; \nu^{(0)}_{\rho,D} - \sqrt{\Lambda}(\Delta_{1,E_D}\frac{1}{\sqrt{\Lambda}})\;, \tag{3.36}$$

$$\nu'^{(1)}_D \;\approx\; \nu^{(1)}_D + 8\sqrt{\Lambda}(\Delta_{1,E_D}\frac{1}{\sqrt{\Lambda}}) - (-1)^{\varepsilon_b}(\frac{\overrightarrow{\partial^l}}{\partial\Theta'^a}\Theta^b,\;\frac{\overrightarrow{\partial^l}}{\partial\Theta^b}\Theta'^a)_D\;, \tag{3.37}$$

$$\nu'^{(2)}_D \;\approx\; \nu^{(2)}_D - (-1)^{\varepsilon_b}(\frac{\overrightarrow{\partial^l}}{\partial\Theta'^a}\Theta^b,\;\frac{\overrightarrow{\partial^l}}{\partial\Theta^b}\Theta'^a)_D\;, \tag{3.38}$$

$$\nu'^{(3)}_D \;\approx\; \nu^{(3)}_D\;, \tag{3.39}$$

$$\nu'^{(4)}_D \;\approx\; \nu^{(4)}_D\;, \tag{3.40}$$

$$\nu'^{(5)}_D \;\approx\; \nu^{(5)}_D\;. \tag{3.41}$$

The last equality in eq. (3.36) is a non-trivial property of the odd Laplacian. It is now easy to see that the relevant linear combination ν_{ρ,E_D} of $\nu^{(0)}_{\rho,D}$, $\nu^{(1)}_D$, $\nu^{(2)}_D$, $\nu^{(3)}_D$, $\nu^{(4)}_D$ and $\nu^{(5)}_D$ is invariant on-shell.

3.5 Nilpotency Condition for the odd Dirac Laplacian Δ_{ρ,E_D}

One of the surprising conclusions of Ref. [8] was that one cannot maintain a strong nilpotency of the Dirac odd Laplacian Δ_{ρ,E_D} under reparametrization of the second-class constraints. This is consistent with our new results. Using the terminology of last Subsection 3.4, one would say that the effect is caused by the off-shell variations of the odd scalar ν_{ρ,E_D} and the Dirac antibracket $(\cdot,\cdot)_D$, cf. the following calculation:

$$\Delta^2_{\rho',E'_D} \;=\; (\nu_{\rho',E'_D},\;\cdot\;)'_D \;\approx\; (\nu_{\rho,E_D},\;\cdot\;)_D \;=\; \Delta^2_{\rho,E_D} \;. \tag{3.42}$$

Here use is made of eqs. (2.43), (3.29) and (3.35). This should be compared to the situation with the Δ_{E_D} operator where the strong nilpotency (3.9) is manifest from the onset, regardless of which defining set of Θ^a constraints is used.

3.6 Dirac Partition Function

As an application of the Δ_{E_D} operator, it is interesting to consider the first-level Dirac partition function in the $\lambda^*_\alpha=0$ gauge. A review of the first-level formalism can be found in Ref. [8]. The partition function reads

$$\mathcal{Z}_D \;=\; \int [d\Gamma][d\lambda]\,\exp[\frac{i}{\hbar}(W_{E_D}+X_{E_D})]\Big|_{\lambda^*=0}\prod_a \delta(\Theta^a)\;, \tag{3.43}$$

where $W_{E_D}=W_{E_D}(\Gamma)$ and $X_{E_D}=X_{E_D}(\Gamma;\lambda,\lambda^*)$ satisfy the Quantum Master Equations

$$\Delta_{E_D}\exp[\frac{i}{\hbar}W_{E_D}] \;=\; 0\;, \tag{3.44}$$

$$((-1)^{\varepsilon_\alpha}\frac{\overrightarrow{\partial^l}}{\partial\lambda^\alpha}\frac{\overrightarrow{\partial^l}}{\partial\lambda^*_\alpha}+\Delta_{E_D})\exp[\frac{i}{\hbar}X_{E_D}] \;=\; 0\;. \tag{3.45}$$

The formula (3.43) for the Dirac partition function $\mathcal{Z}_D$ differs from the original formula [8, 14] by not depending on a ρ. Instead, the partition function $\mathcal{Z}_D$ is invariant under general coordinate transformations and under reparametrization of the Θ^a constraints because the Boltzmann semidensities $\exp[\frac{i}{\hbar}W_{E_D}]$ and $\exp[\frac{i}{\hbar}X_{E_D}]$ transform according to (2.14) and (3.31). Given an arbitrary density ρ, it is possible to introduce Boltzmann scalars

$$\exp[\frac{i}{\hbar}W_\rho] \;\equiv\; \exp[\frac{i}{\hbar}W_{E_D}]/\sqrt{\rho}\;, \tag{3.46}$$

$$\exp[\frac{i}{\hbar}X_\rho] \;\equiv\; \exp[\frac{i}{\hbar}X_{E_D}]/\sqrt{\rho}\;, \tag{3.47}$$

which satisfy corresponding Modified Quantum Master Equations similar to eq. (1.4).

4 Conversion of Second-Class into First-Class

Originally, the conversion of second-class constraints into first-class constraints was developed for even Poisson geometry [15, 16, 17, 18]. Later it was adapted to anti-Poisson geometry in Ref. [19], more precisely to the Dirac antibracket $(\cdot,\cdot)_D$ and odd Laplacian Δ_{ρ,E_D}. In this Section 4 we develop the anti-Poisson conversion method further and show that the Dirac Δ_{E_D} operator from last Section 3 can also be derived via conversion.

4.1 Extended Manifold $M_{\rm ext}$

As in Section 3 the starting point is a general non-degenerate antisymplectic manifold $(M;E)$ with a set of globally defined second-class constraints $\Theta^a = \Theta^a(\Gamma)$, which have Grassmann parity $\varepsilon(\Theta^a)\!=\!\varepsilon_a$. We now consider a cartesian product $M_{\rm ext} \equiv M \times V$, where $(V;\omega)$ is a vector space with a constant and non-degenerate antisymplectic metric, and such that the dimension of V is equal to the number of Θ^a constraints. We will often identify M with $M \times \{0\} \subseteq M_{\rm ext}$. The extended manifold $M_{\rm ext}$ has antisymplectic structure $E_{\rm ext} \equiv E \oplus \omega$.

Assume that points (*i.e.* vectors) in the vector space V are described by a set of coordinates Φ_a with Grassmann parity $\varepsilon(\Phi_a)\!=\!\varepsilon_a\!+\!1$. For each set of local coordinates Γ^A for the manifold M, the extended manifold $M_{\rm ext}$ will have local coordinates $\Gamma^A_{\rm ext}\!\equiv\!\{\Gamma^A;\Phi_a\}$. *Notation*: We use capital roman letters A, B, C, ... from the beginning of the alphabet as upper index for both the original and the extended variables Γ^A and $\Gamma^A_{\rm ext}$, respectively. In detail, the extended antibracket $(\cdot,\cdot)_{\rm ext}$ on $M_{\rm ext}$ reads

$$(\Gamma^A,\Gamma^B)_{\rm ext} \;\equiv\; (\Gamma^A,\Gamma^B) \;=\; E^{AB} \;, \tag{4.1}$$

$$(\Gamma^A,\Phi_a)_{\rm ext} \;\equiv\; 0 \;, \tag{4.2}$$

$$(\Phi_a,\Phi_b)_{\rm ext} \;\equiv\; \omega_{ab} \;, \qquad\qquad \varepsilon(\omega_{ab}) \;=\; \varepsilon_a\!+\!\varepsilon_b\!+\!1 \;, \tag{4.3}$$

where, in particular, the antisymplectic matrix $\omega_{ab} = -(-1)^{\varepsilon_a\varepsilon_b}\omega_{ba}$ does not depend on Γ^A nor on Φ_a. In other words, up to a constant matrix, the Φ_a coordinates are global Darboux coordinates for the vector space V.

4.2 First-Class Constraints T^a

One next seeks Abelian first-class constraints $T^a\!=\!T^a(\Gamma;\Phi)$ such that

$$(T^a,T^b)_{\rm ext} \;=\; 0 \;, \qquad\qquad T^a|_{\Phi=0} \;=\; \Theta^a \;. \tag{4.4}$$

Eq. (4.4) is the defining relation for the conversion of second-class constraints Θ^a into first-class constraint T^a. The first-class constraints T^a are treated as power series expansions in the Φ_a variables

$$T^a \;=\; \Theta^a + \left\{\begin{array}{c} X_L^{ab}\Phi_b \\ \Phi_b X_R^{ba}\end{array}\right\} + \frac{1}{2}\left\{\begin{array}{c} Y_L^{abc}\Phi_c\Phi_b \\ \Phi_b Y_M^{bac}\Phi_c \\ \Phi_b\Phi_c Y_R^{cba}\end{array}\right\} + \frac{1}{6}Z_L^{abcd}\Phi_d\Phi_c\Phi_b + {\cal O}(\Phi^4) \;. \tag{4.5}$$

The expressions $X_L^{ab}\Phi_b \equiv \Phi_b X_R^{ba}$ and $Y_L^{abc}\Phi_c\Phi_b \equiv \Phi_b Y_M^{bac}\Phi_c \equiv \Phi_b\Phi_c Y_R^{cba}$ inside the curly brackets "{ }" of eq. (4.5) reflect various (equivalent) ways of ordering the Φ^a variables. The rules for shifting between the ordering prescriptions are

$$X_L^{ab} \;=\; (-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}X_R^{ba} \;, \tag{4.6}$$

$$(-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}Y_L^{bac} \;=\; Y_M^{abc} \;=\; (-1)^{(\varepsilon_b+1)(\varepsilon_c+1)}Y_R^{acb} \;. \tag{4.7}$$

One may show that a solution T^a to the system (4.4) exists, but that it is not unique. For instance, the condition on the $X^{ab}\!=\!X^{ab}(\Gamma)$ structure functions reads

$$E^{ad} \;\equiv\; (\Theta^a,\Theta^d) \;=\; -\,X_L^{ab}\omega_{bc}X_R^{cd} \;. \tag{4.8}$$

The matrices X_L^{ab} and X_R^{ab} are necessarily invertible with inverse matrices $X^L_{ab}\!=\!(-1)^{\varepsilon_a\varepsilon_b}X^R_{ab}$, since both $E^{ab}\!\equiv\!(\Theta^a,\Theta^b)$ and $\omega_{ab}\!\equiv\!(\Phi_a,\Phi_b)_{\rm ext}$ in eq. (4.8) are invertible. One may view X^{ab} as a Grassmann-odd vielbein between the curved second-class matrix E^{ab} and the flat metric ω_{ab}. At the next order in Φ^a, the condition on the $Y^{abc}\!=\!Y^{abc}(\Gamma)$ structure functions reads

$$(\Theta^a,X_R^{cb}) + X_L^{ad}\omega_{de}Y_R^{ecb} + (X_L^{ac},\Theta^b) + Y_L^{acd}\omega_{de}X_R^{eb} \;=\; 0 \;, \tag{4.9}$$

and so forth.

4.3 Gauge Invariance

The idea is now to view the first-class constraints T^{a} as generators of gauge symmetry and $\Phi_{a}=0$ as a particular gauge. We start by defining gauge-invariant observables on the extended manifold $M_{\rm ext}$.

Definition 4.1 *A scalar function* $\bar{F}=\bar{F}(\Gamma;\Phi)$*, a density* $\bar{\rho}=\bar{\rho}(\Gamma;\Phi)$ *or a semidensity* $\bar{\sigma}=\bar{\sigma}(\Gamma;\Phi)$ *on the extended manifold* $M_{\rm ext}$ *is called a* **gauge-invariant extension** *of a scalar function* $F=F(\Gamma)$*, a density* $\rho=\rho(\Gamma)$ *or a semidensity* $\sigma=\sigma(\Gamma)$ *on the original manifold* M*, if the following conditions are satisfied*

$$
\begin{aligned}
(\bar{F},T^{a})_{\rm ext} &= 0 , & \bar{F}|_{\Phi=0} &= F , & (4.10)\\
(\Delta_{\bar{\rho}}T^{a}) &= 0 , & \bar{\rho}|_{\Phi=0} &= \rho j , & (4.11)\\
[\overrightarrow{\Delta}_{E_{\rm ext}},T^{a}]\bar{\sigma} &= 0 , & \bar{\sigma}|_{\Phi=0} &= \sigma\sqrt{j} , & (4.12)
\end{aligned}
$$

respectively, where the j*-factor is defined in eq. (4.13) below.*

4.4 The j-Factor

The factor

$$
j \;\equiv\; \bar{\jmath}|_{\Phi=0} \;=\; {\rm sdet}(\omega_{ac}X_{R}^{cb}) \qquad (4.13)
$$

is defined as the $\Phi=0$ restriction of the superdeterminant

$$
\bar{\jmath} \;\equiv\; {\rm sdet}(\Phi_{a},T^{b})_{\rm ext} \;=\; \int[d\bar{C}][dC]\;\exp\left[\frac{i}{\hbar}\bar{C}^{a}(\Phi_{a},T^{b})_{\rm ext}C_{b}\right] , \qquad \varepsilon(\bar{C}^{a}) \;=\; \varepsilon_{a}{+}1 \;=\; \varepsilon(C_{a}) \;. \quad (4.14)
$$

The j-factor (4.13) is independent of the choice of X^{ab} structure functions because of eq. (4.8). It is a density for the vector space V such that the corresponding volume form $j[d\Phi]$ on V is independent of the choice of coordinates Φ_{a}. In this way the multiplication with j in eq. (4.11) transforms a density ρ on the manifold M into a density ρj for the extended manifold $M_{\rm ext}\equiv M\times V$. The j-factor is unique up to an overall constant and can be physically explained as a Faddeev-Popov determinant, see Subsection 4.7.

Below we shall overwhelmingly justify the j-factor in Definition 4.1, in particular, through the Conversion Theorem 4.2, but let us start by briefly mentioning a curious implication. Consider what happens to the set of vielbein solutions X_{L}^{ab} to eq. (4.8) under reparametrizations of the defining set of second-class constraints $\Theta^{a}\to\Theta'^{a}=\Lambda^{a}{}_{b}\,\Theta^{b}$. It is natural to expect that there exists a bijective map $X_{L}^{ab}\to X_{L}'^{ab}$ between the solutions such that

$$
X_{L}'^{ac} \;\approx\; \Lambda^{a}{}_{b}X_{L}^{bc} \;, \qquad (4.15)
$$

where "≈" denotes weak equivalence, cf. Subsection 3.4. According to such map, the j-factor would transform as

$$
j' \;\approx\; \Lambda\, j \;. \qquad (4.16)
$$

Recalling the transformation rule (3.31) for ρ, this implies that the density $\bar{\rho}|_{\Phi=0}=\rho j$ on $M_{\rm ext}$ changes with the *square* of Λ,

$$
\bar{\rho}'|_{\Phi=0} \;\approx\; \Lambda^{2}\,\bar{\rho}|_{\Phi=0} \;. \qquad (4.17)
$$

So while the j-factor does indeed cancel the effect of changing the Φ_{a} coordinates, it *doubles* the effect of changing the second-class constraints Θ^{a}! Nevertheless, this doubling phenomenon fits nicely with the rest of the conversion construction, cf. Subsection 4.7 below.

4.5 Discussion of Gauge Invariance

Let us now justify the conditions (4.10)–(4.12). The first condition (4.10) is simply the antisymplectic definition of gauge invariance. As an example of condition (4.10), note that a first-class constraint $T^a = \bar{\Theta}^a$ is a gauge-invariant extension of the corresponding second-class constraint Θ^a. The other two conditions (4.11) and (4.12) are a priori less obvious, but there are many reasons to impose them:

1. The three conditions (4.10)–(4.12) are covariant with respect to coordinate changes.
2. The conditions (4.10)–(4.12) are consistent with each others, say, if one considers a density $\rho' = \rho F$, or a semidensity $\sigma = \sqrt{\rho}$.
3. The conditions (4.10)–(4.12) are natural counterparts of the annihilation properties (3.21)–(3.23).
4. One may show that there exist unique gauge-invariant extensions $\bar{F}$, $\bar{\rho}$ and $\bar{\sigma}$ satisfying the condition (4.10), (4.11) and (4.12), respectively.
5. The extended antibracket $(\cdot,\cdot)_{\rm ext}$, the extended odd Laplacian $\Delta_{\bar{\rho}} \equiv \Delta_{\bar{\rho},E_{\rm ext}}$, the extended $\Delta_{E_{\rm ext}}$ operator and the extended odd scalar $\nu_{\bar{\rho}} \equiv \nu_{\bar{\rho},E_{\rm ext}}$ are compatibly with the gauge-invariance conditions (4.10)–(4.12), *i.e.*

$$
\begin{aligned}
(\bar{F}\bar{G},T^a)_{\rm ext} &= \bar{F}(\bar{G},T^a)_{\rm ext} + (-1)^{\varepsilon_F\varepsilon_G}\bar{G}(\bar{F},T^a)_{\rm ext} \;=\; 0 \;, && (4.18)\\
((\bar{F},\bar{G})_{\rm ext},T^a)_{\rm ext} &= (\bar{F},(\bar{G},T^a)_{\rm ext})_{\rm ext} + (-1)^{(\varepsilon_a+1)(\varepsilon_G+1)}((\bar{F},T^a)_{\rm ext},\bar{G})_{\rm ext} \;=\; 0 \;, && (4.19)\\
(\Delta_{\bar{\rho}}\bar{F},T^a)_{\rm ext} &= \Delta_{\bar{\rho}}(\bar{F},T^a)_{\rm ext} + (-1)^{\varepsilon_F}(\bar{F},\Delta_{\bar{\rho}}T^a)_{\rm ext} \;=\; 0 \;, && (4.20)\\
[\overrightarrow{\Delta}_{E_{\rm ext}},T^a](\Delta_{E_{\rm ext}}\bar{\sigma}) &= (\Delta_{E_{\rm ext}}T^a\Delta_{E_{\rm ext}}\bar{\sigma}) \;=\; (\Delta_{E_{\rm ext}}[T^a,\overrightarrow{\Delta}_{E_{\rm ext}}]\bar{\sigma}) \;=\; 0 \;, && (4.21)\\
(\nu_{\bar{\rho}},T^a)_{\rm ext} &= (\Delta^2_{\bar{\rho}}T^a) \;=\; 0 \;. && (4.22)
\end{aligned}
$$

   Here use is made of the ordinary Leibniz rule, the Jacobi identity (2.2), the BV Leibniz rule (2.16), the eq. (2.19) and the eq. (2.43), respectively.
6. The conditions (4.10)–(4.12) imply the Conversion Theorem 4.2 below.

4.6 The Conversion Map

The gauge-invariant extension map

$$
\mathcal{F}(M) \;\ni\; F \;\stackrel{\cong}{\longmapsto}\; \bar{F} \;\in\; \mathcal{F}(M_{\rm ext})_{\rm inv} \tag{4.23}
$$

(which is also known as the *conversion map*) is an isomorphism of functions on M to gauge-invariant function on $M_{\rm ext}$, cf. point 4 of the last Subsection 4.5. The inverse conversion map is simply the restriction to M,

$$
\mathcal{F}(M_{\rm ext})_{\rm inv} \;\ni\; \bar{F} \;\stackrel{\cong}{\longmapsto}\; \bar{F}\big|_{\Phi=0} \;\in\; \mathcal{F}(M) \;. \tag{4.24}
$$

The following Theorem 4.2 is the heart of the conversion method. It shows that the inverse conversion map transforms the extended model into the Dirac construction.

Theorem 4.2 *The restrictions to M of the extended antibracket $(\cdot,\cdot)_{\rm ext}$, the extended odd Laplacian $\Delta_{\bar{\rho}} \equiv \Delta_{\bar{\rho},E_{\rm ext}}$, the extended $\Delta_{E_{\rm ext}}$ operator and the extended odd scalar $\nu_{\bar{\rho}} \equiv \nu_{\bar{\rho},E_{\rm ext}}$ reproduce the* *corresponding Dirac constructions:*

$$\begin{aligned}
(\bar{F},\bar{G})_{\rm ext}\Big|_{\Phi=0} &= (F,G)_{D} \ , & (4.25)\\
(\Delta_{\bar{\rho},E_{\rm ext}}\bar{F})\Big|_{\Phi=0} &= (\Delta_{\rho,E_{D}}F) \ , & (4.26)\\
(\Delta_{E_{\rm ext}}\bar{\sigma})\Big|_{\Phi=0} &= \sqrt{j}(\Delta_{E_{D}}\sigma) \ , & (4.27)\\
\nu_{\bar{\rho},E_{\rm ext}}\Big|_{\Phi=0} &= \nu_{\rho,E_{D}} \ . & (4.28)
\end{aligned}$$

In principle, it is enough to prove eq. (4.28), since eq. (4.28) ⇔ eq. (4.27) ⇒ eq. (4.26) ⇒ eq. (4.25). Nevertheless, we shall give independent proofs of eqs. (4.25), (4.26) and (4.28) in Appendix C. The following Corollary 4.3 restates the conclusions of Conversion Theorem 4.2 using the forward conversion map.

Corollary 4.3

$$\begin{aligned}
(FG)^{-} &= \bar{F}\bar{G} \ , & (4.29)\\
((F,G)_{D})^{-} &= (\bar{F},\bar{G})_{\rm ext} \ , & (4.30)\\
(\Delta_{\rho,E_{D}}F)^{-} &= (\Delta_{\bar{\rho},E_{\rm ext}}\bar{F}) \ , & (4.31)\\
(\sqrt{j}\Delta_{E_{D}}\sigma)^{-} &= (\Delta_{E_{\rm ext}}\bar{\sigma}) \ , & (4.32)\\
(\nu_{\rho,E_{D}})^{-} &= \nu_{\bar{\rho},E_{\rm ext}} \ . & (4.33)
\end{aligned}$$

In particular, eqs. (4.25) and (4.30) show that the conversion map is an isomorphism in the sense of anti-Poisson algebras between the Dirac anti-Poisson algebra $(\mathcal{F}(M);(\cdot,\cdot)_{D})$ and the anti-Poisson algebra $(\mathcal{F}(M_{\rm ext})_{\rm inv};(\cdot,\cdot)_{\rm ext})$ of gauge-invariant functions on $M_{\rm ext}$.

4.7 Extended Partition Function

The first-level partition function in the $\lambda^{*}_{\alpha}=0$ gauge reads

$$\mathcal{Z}_{\rm ext} \;=\; \int[d\Gamma_{\rm ext}][d\lambda]\,\exp[\frac{i}{\hbar}(W_{E_{\rm ext}}+X_{E_{\rm ext}})]\Big|_{\lambda^{*}=0}\frac{1}{{\rm sdet}(\chi_{a},T^{b})_{\rm ext}}\prod_{c}\delta(T^{c})\prod_{d}\delta(\chi_{d}) \ , \qquad (4.34)$$

where $W_{E_{\rm ext}}=W_{E_{\rm ext}}(\Gamma_{\rm ext})$ and $X_{E_{\rm ext}}=X_{E_{\rm ext}}(\Gamma_{\rm ext};\lambda,\lambda^{*})$ satisfy the Quantum Master Equations

$$\begin{aligned}
\Delta_{E_{\rm ext}}\exp[\frac{i}{\hbar}W_{E_{\rm ext}}] &= 0 \ , & (4.35)\\
((-1)^{\varepsilon_{\alpha}}\frac{\overrightarrow{\partial^{l}}}{\partial\lambda^{\alpha}}\frac{\overrightarrow{\partial^{l}}}{\partial\lambda^{*}_{\alpha}}+\Delta_{E_{\rm ext}})\exp[\frac{i}{\hbar}X_{E_{\rm ext}}] &= 0 \ , & (4.36)
\end{aligned}$$

and they are gauge invariant in the sense of condition (4.12):

$$\begin{aligned}
[\overrightarrow{\Delta}_{E_{\rm ext}},T^{a}]\exp[\frac{i}{\hbar}W_{E_{\rm ext}}] &= 0 \ , \qquad & \exp[\tfrac{i}{\hbar}W_{E_{\rm ext}}]\Big|_{\Phi=0} &= \sqrt{j}\exp[\frac{i}{\hbar}W_{E_{D}}] \ , & (4.37)\\
[\overrightarrow{\Delta}_{E_{\rm ext}},T^{a}]\exp[\frac{i}{\hbar}X_{E_{\rm ext}}] &= 0 \ , \qquad & \exp[\tfrac{i}{\hbar}X_{E_{\rm ext}}]\Big|_{\Phi=0} &= \sqrt{j}\exp[\frac{i}{\hbar}X_{E_{D}}] \ . & (4.38)
\end{aligned}$$

Here the Boltzmann semidensities $\exp[\frac{i}{\hbar}W_{E_D}]$ and $\exp[\frac{i}{\hbar}X_{E_D}]$ satisfy the Quantum Master Equations (3.44) and (3.45), respectively. It is an important fact that in the gauge $\chi_a = \Phi_a$, the expression (4.34) for the extended partition function reduces to the Dirac partition function (3.43), *i.e.*

$$\mathcal{Z}_{\rm ext} \ = \ \mathcal{Z}_D \ . \tag{4.39}$$

Given a density $\rho = \rho(\Gamma)$ on M, and a density $\bar{\rho} = \bar{\rho}(\Gamma_{\rm ext})$ on $M_{\rm ext}$ that satisfies eq. (4.11), it is possible to introduce Boltzmann scalars

$$\exp[\frac{i}{\hbar}W_{\bar{\rho}}] \ \equiv \ \exp[\frac{i}{\hbar}W_{E_{\rm ext}}]/\sqrt{\bar{\rho}} \ , \tag{4.40}$$

$$\exp[\frac{i}{\hbar}X_{\bar{\rho}}] \ \equiv \ \exp[\frac{i}{\hbar}X_{E_{\rm ext}}]/\sqrt{\bar{\rho}} \ , \tag{4.41}$$

which satisfy corresponding Modified Quantum Master Equations similar to eq. (1.4). The Quantum Actions $W_{\bar{\rho}}$ and $X_{\bar{\rho}}$ defined this way are automatically gauge invariant

$$(W_{\bar{\rho}}, T^a)_{\rm ext} \ = \ 0 \ , \qquad W_{\bar{\rho}}\Big|_{\Phi=0} \ = \ W_{\rho} \ , \tag{4.42}$$

$$(X_{\bar{\rho}}, T^a)_{\rm ext} \ = \ 0 \ , \qquad X_{\bar{\rho}}\Big|_{\Phi=0} \ = \ X_{\rho} \ . \tag{4.43}$$

Here W_ρ and X_ρ are defined in eqs. (3.46) and (3.47), respectively.

5 Conclusions

We have shown for a general degenerate anti-Poisson manifold (under the relatively mild assumption of a compatible two-form field) how to define in arbitrary coordinates the Δ_E operator, which takes semidensities to semidensities, cf. Lemma 2.6. A large class of such degenerate antibrackets are provided by the Dirac antibracket construction. We have given a formula for the Dirac Δ_{E_D} operator, cf. Proposition 3.1, and shown in Subsection 3.4 that it is on-shell invariant under reparametrizations of the second-class constraints. Finally, we showed that the Dirac Δ_{E_D} operator also follows from the antisymplectic conversion scheme, cf. Conversion Theorem 4.2.

Let us conclude with the following remark. It is often pointed out that the antibracket $(\cdot,\cdot)$ is a descendant of the odd Laplacian Δ_ρ. It measures the failure of the odd Laplacian Δ_ρ to act as a linear derivation, *i.e.* to satisfy the Leibniz rule. It can be written as a double-commutator [7, 20, 21]

$$(F,G) \ = \ (-1)^{\varepsilon_F}[[\overrightarrow{\Delta}_\rho, F], G]1 \ . \tag{5.1}$$

In turn, the odd Laplacian Δ_ρ is a descendant of the Δ_E operator [8, 9]

$$(\Delta_\rho F) \ = \ \frac{1}{\sqrt{\rho}}[\overrightarrow{\Delta}_E, F]\sqrt{\rho} \ . \tag{5.2}$$

That is, one has schematically the following hierarchy:

$$\begin{array}{ccc} \Delta_E \text{ operator} & & \\ \Downarrow & & \\ \text{Odd Laplacian } \Delta_\rho & \Leftarrow & \text{Density } \rho \\ \Downarrow & & \\ \text{Antibracket } (\cdot,\cdot) & & \end{array} \tag{5.3}$$

Whereas the Δ_E operator is manifestly nilpotent, cf. Theorem 2.10, there is no fundamental reason to require the odd Laplacian Δ_ρ to be nilpotent. (Of course, if Δ_ρ is not nilpotent, the Boltzmann scalar $\exp[\frac{i}{\hbar}W_\rho]$ would in general have to satisfy a Modified Quantum Master Equation with a non-trivial ν_ρ term, cf. eq. (1.4). See also the recent preprint [22].) The Dirac odd Laplacian Δ_{ρ,E_D} offers more evidence that nilpotency of the odd Laplacian is not fundamental, at least not in its strong formulation, since in this case the nilpotency can only be maintained weakly under reparametrizations of the second-class constraints Θ^a, cf. Ref. [8] and Subsection 3.5.

ACKNOWLEDGEMENT: The Author thanks I.A. Batalin and P.H. Damgaard for discussions. The Author also thanks the Erwin Schrödinger Institute for warm hospitality. This work is supported by the Ministry of Education of the Czech Republic under the project MSM 0021622409.

A Proof of bi-Darboux Theorem 2.1

If there exists an atlas of bi-Darboux coordinates, the two-form $E = d\phi^*_\alpha \wedge d\phi^\alpha$ is obviously closed. Now consider the other direction. Assume that the two-form E is closed. Then there locally exists a pre-antisymplectic one-form potential ϑ such that

$$d\vartheta \;=\; E\ . \tag{A.1}$$

Independently one knows that locally there exist Darboux coordinates $\Gamma^A = \{\phi^\alpha; \phi^*_\alpha; \Theta^a\}$. Since the two-form E is assumed to be compatible with the anti-Poisson structure, it must be of the form (2.10). It is always possible to organize the pre-antisymplectic one-form potential as

$$\vartheta \;\sim\; \phi^*_\alpha d\phi^\alpha + \vartheta_A d\gamma^A + \vartheta'_a d\Theta^a\ , \tag{A.2}$$

where $\gamma^A = \{\phi^\alpha; \phi^*_\alpha\}$ collectively denotes the fields and the antifields without the Casimirs. The symbol "$\sim$" denotes equality modulo exact terms, whose precise expressions are irrelevant, since we are ultimately only interested in the two-form E. It follows from eqs. (2.10), (A.1) and (A.2) that

$$(\frac{\overrightarrow{\partial^l}}{\partial\gamma^A}\vartheta_B) \;=\; (-1)^{\varepsilon_A\varepsilon_B}(A \leftrightarrow B)\ , \tag{A.3}$$

and hence there locally exists a fermionic function Ψ' such that

$$\vartheta_A \;=\; (\frac{\overrightarrow{\partial^l}}{\partial\gamma^A}\Psi')\ . \tag{A.4}$$

Defining

$$\vartheta_a \;\equiv\; \vartheta'_a - (\frac{\overrightarrow{\partial^l}}{\partial\Theta^a}\Psi')\ , \tag{A.5}$$

the pre-antisymplectic one-form potential (A.2) reduces to

$$\vartheta \;\sim\; \phi^*_\alpha d\phi^\alpha + \vartheta_a d\Theta^a\ . \tag{A.6}$$

We would like to show that the second term $\vartheta_a d\Theta^a$ in eq. (A.6) vanishes under a suitable anticanonical transformation. Eqs. (A.1) and (A.6) imply that the matrices $M_{a\alpha}$ and $N^\alpha{}_a$ in eq. (2.10) are

$$-M_{a\alpha} \;=\; (\vartheta_a\frac{\overleftarrow{\partial^r}}{\partial\phi^\alpha}) \;=\; (\vartheta_a, \phi^*_\alpha)\ , \tag{A.7}$$

$$N^\alpha{}_a \;=\; (\frac{\overrightarrow{\partial^l}}{\partial\phi^*_\alpha}\vartheta_a) \;=\; (\phi^\alpha, \vartheta_a)\ , \tag{A.8}$$

and that the pre-antisymplectic potential components $\vartheta_a = \vartheta_a(\Gamma)$ satisfy a flatness condition:

$$F_{ab} \equiv (\frac{\overrightarrow{\partial^l}}{\partial\Theta^a}\vartheta_b) - (-1)^{\varepsilon_a\varepsilon_b}(\frac{\overrightarrow{\partial^l}}{\partial\Theta^b}\vartheta_a) + (\vartheta_a, \vartheta_b) = 0 . \tag{A.9}$$

Put more illuminating, the condition (A.9) implies that the vector fields

$$D_a \equiv \frac{\overrightarrow{\partial^l}}{\partial\Theta^a} + \mathrm{ad}\vartheta_a \tag{A.10}$$

commute

$$[D_a, D_b] = \mathrm{ad}F_{ab} = 0 . \tag{A.11}$$

Here the adjoint action "ad" refers to the antibracket $(\mathrm{ad}F)G \equiv (F, G)$, where F and G are functions. In other words, $\mathrm{ad}F$ denotes the Hamiltonian vector field with Hamiltonian F. The vector fields D_a are not Hamiltonian, although they do preserve the antibracket

$$D_a(F, G) = (D_a[F], G) + (-1)^{\varepsilon_a(\varepsilon_F+1)}(F, D_a[G]) , \tag{A.12}$$

i.e. they are generators of anticanonical transformations that do not leave the Casimirs invariant. It is an important fact that the D_a are covariant derivatives in the Casimir directions with a Lie algebra valued gauge potential $\mathrm{ad}\vartheta_a$. Here the Lie algebra is (a subalgebra of) the space $\Gamma(TM)$ of vector fields, equipped with the commutator $[\cdot, \cdot]$, *i.e.* the Lie bracket of vector fields. An infinitesimal variation $\delta\vartheta_a$ of the pre-antisymplectic potential components ϑ_a must satisfy

$$D_a[\delta\vartheta_b] = (-1)^{\varepsilon_a\varepsilon_b}(a \leftrightarrow b) \tag{A.13}$$

in order to respect the flatness condition (A.9). The last eq. (A.13) implies in turn, that the only allowed infinitesimal variations $\delta\vartheta_a$ are infinitesimal gauge transformations

$$\delta\vartheta_a = D_a[\delta\Psi] , \tag{A.14}$$

where $\delta\Psi$ is an infinitesimal fermionic gauge generator. The infinitesimal gauge transformation of the gauge potential $\mathrm{ad}\vartheta_a$ is

$$\mathrm{ad}(\delta\vartheta_a) = [D_a, \mathrm{ad}(\delta\Psi)] , \tag{A.15}$$

where use is made of eq. (A.12). Despite the appearance, the eq. (A.15) is exactly the standard formula $\delta A_\mu = D_\mu\varepsilon$ for infinitesimal non-Abelian gauge transformations. Any discrepancy is merely in notation, not in content. So one can take advantage of well-known facts about non-Abelian gauge theory and *e.g.* Wilson-lines. In particular, the infinitesimal transformations (A.14) and (A.15) generalize to finite gauge transformations. The field strength (or curvature) is zero, cf. eq. (A.11), so the gauge potential $\mathrm{ad}\vartheta_a$ is pure gauge. This means that there locally exists a gauge where the gauge potential vanishes identically,

$$\mathrm{ad}\vartheta_a = 0 . \tag{A.16}$$

An infinitesimal gauge transformation (A.14) may be implemented with the help of a Hamiltonian vector field $\mathrm{ad}(\delta\Psi)$ with infinitesimal Hamiltonian $\delta\Psi$. Using the active picture, the Lie derivative of the pre-antisymplectic one-form potential with respect to the Hamiltonian vector field $\mathrm{ad}(\delta\Psi)$ is

$$\mathcal{L}_{\mathrm{ad}(\delta\Psi)}\vartheta = [i_{\mathrm{ad}(\delta\Psi)}, d]\vartheta \sim i_{\mathrm{ad}(\delta\Psi)}E = (\delta\Psi\frac{\overleftarrow{\partial^r}}{\partial\gamma^A})d\gamma^a + (\delta\Psi, \vartheta_a)d\Theta^a \sim -D_a[\delta\Psi]d\Theta^a . \tag{A.17}$$

i.e. by flowing along the Hamiltonian vector field $\mathrm{ad}(\delta\Psi)$, one may mimic (minus) the infinitesimal gauge transformation (A.14). More generally, finite gauge transformations of ϑ_a are in one-to-one correspondence with anticanonical transformations that leave the Casimirs invariant. In particular, one may go to the trivial gauge (A.16) where the ϑ_a themselves are Casimirs. The flatness condition (A.9) then reduces to

$$(\frac{\overrightarrow{\partial^l}}{\partial\Theta^a}\vartheta_b) \;=\; (-1)^{\varepsilon_a\varepsilon_b}(a\leftrightarrow b)\;, \tag{A.18}$$

so there exists a fermionic Casimir function $\Psi=\Psi(\Theta)$ such that

$$\vartheta_a \;=\; (\frac{\overrightarrow{\partial^l}}{\partial\Theta^a}\Psi)\;, \tag{A.19}$$

and hence the second term in eq. (A.6) is just an exact term,

$$\vartheta_a d\Theta^a \;=\; d\Psi \;\sim\; 0\;. \tag{A.20}$$

This shows that there locally exists an anticanonical transformation that leaves the Casimirs invariant, such that the two-form reduces to $E = d\phi^*_\alpha\wedge d\phi^\alpha$.

□

B Details from the Proof of Lemma 2.9

B.1 Proof of eq. (2.38)

The infinitesimal variation of $\nu^{(2)}$ in eq. (2.22) yields 8 contributions to linear order in the variation $\delta\Gamma^A = X^A$, which may be organized as 2×4 terms

$$\delta\nu^{(2)} \;=\; 2(-\delta\nu^{(2)}_I - \delta\nu^{(2)}_{II} + \delta\nu^{(2)}_{III} + \delta\nu^{(2)}_{IV})\;, \tag{B.1}$$

due to a $(A,B)\leftrightarrow(D,C)$ symmetry in eq. (2.22). They are

$$\delta\nu^{(2)}_I \;\equiv\; (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_D}E^{AB})E_{BF}(X^F\overleftarrow{\partial^r_C})(\overrightarrow{\partial^l_A}E^{CD})\;, \tag{B.2}$$

$$\delta\nu^{(2)}_{II} \;\equiv\; (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_D}E^{AB})E_{BC}(\overrightarrow{\partial^l_A}X^F)(\overrightarrow{\partial^l_F}E^{CD})\;, \tag{B.3}$$

$$\delta\nu^{(2)}_{III} \;\equiv\; (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_D}E^{AB})E_{BC}\overrightarrow{\partial^l_A}\left((X^C\overleftarrow{\partial^r_F})E^{FD}\right) \;=\; \delta\nu^{(2)}_I + \delta\nu^{(2)}_V\;, \tag{B.4}$$

$$\delta\nu^{(2)}_{IV} \;\equiv\; (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_D}E^{AB})E_{BC}\overrightarrow{\partial^l_A}\left(E^{CF}(\overrightarrow{\partial^l_F}X^D)\right) \;=\; \delta\nu^{(2)}_{II} + \delta\nu^{(2)}_{VI}\;, \tag{B.5}$$

$$\delta\nu^{(2)}_V \;\equiv\; (-1)^{\varepsilon_A\varepsilon_C}E^{FD}(\overrightarrow{\partial^l_D}E^{AB})E_{BC}(\overrightarrow{\partial^l_A}X^C\overleftarrow{\partial^r_F}) \;=\; -\,\delta\nu^{(2)}_V + \delta\nu^{(2)}_{VII}\;, \tag{B.6}$$

$$\delta\nu^{(2)}_{VI} \;\equiv\; (-1)^{\varepsilon_A}(\overrightarrow{\partial^l_D}E^{AB})P_B{}^C(\overrightarrow{\partial^l_C}\overrightarrow{\partial^l_A}X^D)\;, \tag{B.7}$$

$$\delta\nu^{(2)}_{VII} \;\equiv\; (-1)^{\varepsilon_A}P_C{}^D(\overrightarrow{\partial^l_D}E^{AB})(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A}X^C)\;, \tag{B.8}$$

where we have noted various relations among the contributions. The Jacobi identity (2.3) for E^{AB} is used in the second equality of eq. (B.6). Altogether, the infinitesimal variation of $\nu^{(2)}$ becomes

$$\delta\nu^{(2)} \;=\; 2\delta\nu^{(2)}_{VI} + \delta\nu^{(2)}_{VII}\;, \tag{B.9}$$

which is eq. (2.38).

B.2 Proof of eq. (2.39)

The infinitesimal variation of $\nu^{(3)}$ in eq. (2.23) yields 6 contributions to linear order in the variation $\delta\Gamma^A = X^A$,

$$\delta\nu^{(3)} \;=\; \delta\nu^{(3)}_I + \delta\nu^{(3)}_{II} + \delta\nu^{(3)}_{III} - \delta\nu^{(3)}_{IV} - \delta\nu^{(3)}_V - \delta\nu^{(3)}_{VI} \; . \tag{B.10}$$

They are

$$\delta\nu^{(3)}_I \;\equiv\; (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})(X^C\overleftarrow{\partial^r_F})E^{FD}(\overrightarrow{\partial^l_D}E^{BA}) \; , \tag{B.11}$$

$$\delta\nu^{(3)}_{II} \;\equiv\; (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}\overrightarrow{\partial^l_D}\left((X^B\overleftarrow{\partial^r_F})E^{FA}\right) \;=\; \delta\nu^{(3)}_{VII} + \delta\nu^{(3)}_{VIII} \; , \tag{B.12}$$

$$\delta\nu^{(3)}_{III} \;\equiv\; (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}\overrightarrow{\partial^l_D}\left(E^{BF}(\overrightarrow{\partial^l_F}X^A)\right) \;=\; \delta\nu^{(3)}_{IV} + \delta\nu^{(3)}_{IX} \; , \tag{B.13}$$

$$\delta\nu^{(3)}_{IV} \;\equiv\; (-1)^{\varepsilon_B}E^{CD}(\overrightarrow{\partial^l_D}E^{BA})(\overrightarrow{\partial^l_A}X^F)(\overrightarrow{\partial^l_F}E_{BC}) \; , \tag{B.14}$$

$$\delta\nu^{(3)}_V \;\equiv\; (-1)^{\varepsilon_B}E^{CD}(\overrightarrow{\partial^l_D}E^{BA})\overrightarrow{\partial^l_A}\left((\overrightarrow{\partial^l_B}X^F)E_{FC}\right) \;=\; \delta\nu^{(3)}_{VII} + \delta\nu^{(3)}_X \; , \tag{B.15}$$

$$\delta\nu^{(3)}_{VI} \;\equiv\; (-1)^{\varepsilon_B}E^{CD}(\overrightarrow{\partial^l_D}E^{BA})\overrightarrow{\partial^l_A}\left(E_{BF}(X^F\overleftarrow{\partial^r_C})\right) \;=\; \delta\nu^{(3)}_I - \delta\nu^{(3)}_{XI} \; , \tag{B.16}$$

$$\begin{aligned}\delta\nu^{(3)}_{VII} \;&\equiv\; (-1)^{\varepsilon_A(\varepsilon_B+1)}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}E^{AF})(\overrightarrow{\partial^l_F}X^B) \\ &=\; -(-1)^{\varepsilon_B(\varepsilon_C+1)}E^{CD}(\overrightarrow{\partial^l_D}E^{BA})(\overrightarrow{\partial^l_A}E_{CF})(X^F\overleftarrow{\partial^r_B}) \; ,\end{aligned} \tag{B.17}$$

$$\delta\nu^{(3)}_{VIII} \;\equiv\; (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}X^B\overleftarrow{\partial^r_F})E^{FA} \; , \tag{B.18}$$

$$\delta\nu^{(3)}_{IX} \;\equiv\; (-1)^{\varepsilon_A(\varepsilon_B+1)}(\overrightarrow{\partial^l_A}E_{BC})E^{CD}(\overrightarrow{\partial^l_D}X^A\overleftarrow{\partial^r_F})E^{FB} \; , \tag{B.19}$$

$$\delta\nu^{(3)}_X \;\equiv\; (-1)^{\varepsilon_A}P_C{}^D(\overrightarrow{\partial^l_D}E^{AB})(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A}X^C) \; , \tag{B.20}$$

$$\delta\nu^{(3)}_{XI} \;\equiv\; (-1)^{\varepsilon_B(\varepsilon_C+1)}E^{CD}(\overrightarrow{\partial^l_D}E^{BA})(\overrightarrow{\partial^l_A}\overrightarrow{\partial^l_C}X^F)E_{FB} \;=\; -\,\delta\nu^{(3)}_X - \delta\nu^{(3)}_{XI} \; , \tag{B.21}$$

where we have noted various relations among the contributions. The Jacobi identity (2.3) for E^{AB} is used in the second equality of eq. (B.21). Altogether, the infinitesimal variation of $\nu^{(3)}$ becomes

$$\delta\nu^{(3)} \;=\; \delta\nu^{(3)}_{VIII} + \delta\nu^{(3)}_{IX} - \frac{3}{2}\delta\nu^{(3)}_X \; , \tag{B.22}$$

which is eq. (2.39).

B.3 Proof of eq. (2.40)

The infinitesimal variation of $\nu^{(4)}$ in eq. (2.24) yields 6 contributions to linear order in the variation $\delta\Gamma^A = X^A$,

$$\delta\nu^{(4)} \;=\; -\,\delta\nu^{(4)}_I - \delta\nu^{(4)}_{II} + \delta\nu^{(4)}_{III} + \delta\nu^{(4)}_{IV} + \delta\nu^{(4)}_V - \delta\nu^{(4)}_{VI} \; . \tag{B.23}$$

They are

$$\delta\nu^{(4)}_I \;\equiv\; (-1)^{\varepsilon_B}E^{CD}(\overrightarrow{\partial^l_D}E^{BF})P_F{}^A\overrightarrow{\partial^l_A}\left((\overrightarrow{\partial^l_B}X^G)E_{GC}\right) \;=\; -\,\delta\nu^{(4)}_{VII} + \delta\nu^{(4)}_{VIII} \; , \tag{B.24}$$

$$\delta\nu^{(4)}_{II} \;\equiv\; (-1)^{\varepsilon_B}E^{CD}(\overrightarrow{\partial^l_D}E^{BF})P_F{}^A\overrightarrow{\partial^l_A}\left(E_{BG}(X^G\overleftarrow{\partial^r_C})\right) \;=\; \delta\nu^{(4)}_{III} - \delta\nu^{(4)}_{IX} \; , \tag{B.25}$$

$$\delta\nu^{(4)}_{III} \equiv (-1)^{\varepsilon_B} P_F{}^A(\overrightarrow{\partial}^l_A E_{BC})(X^C \overleftarrow{\partial}^r_G)E^{GD}(\overrightarrow{\partial}^l_D E^{BF}) \ , \tag{B.26}$$

$$\delta\nu^{(4)}_{IV} \equiv (-1)^{\varepsilon_B} P_F{}^A(\overrightarrow{\partial}^l_A E_{BC})E^{CD}\overrightarrow{\partial}^l_D \left((X^B \overleftarrow{\partial}^r_G)E^{GF}\right) = \delta\nu^{(4)}_X + \delta\nu^{(4)}_{XI} \ , \tag{B.27}$$

$$\delta\nu^{(4)}_{V} \equiv (-1)^{\varepsilon_B} P_F{}^A(\overrightarrow{\partial}^l_A E_{BC})E^{CD}\overrightarrow{\partial}^l_D \left(E^{BG}(\overrightarrow{\partial}^l_G X^F)\right) = \delta\nu^{(4)}_{VI} + \delta\nu^{(4)}_{XII} \ , \tag{B.28}$$

$$\delta\nu^{(4)}_{VI} \equiv (-1)^{\varepsilon_B} P_G{}^A(\overrightarrow{\partial}^l_A E_{BC})E^{CD}(\overrightarrow{\partial}^l_D E^{BF})(\overrightarrow{\partial}^l_F X^G) \ , \tag{B.29}$$

$$\delta\nu^{(4)}_{VII} \equiv (-1)^{\varepsilon_B(\varepsilon_C+1)} E^{CD}(\overrightarrow{\partial}^l_D E^{BF})P_F{}^A(\overrightarrow{\partial}^l_A E_{CG})(X^G \overleftarrow{\partial}^r_B) \ , \tag{B.30}$$

$$\delta\nu^{(4)}_{VIII} \equiv (-1)^{\varepsilon_B}(\overrightarrow{\partial}^l_A \overrightarrow{\partial}^l_B X^C)P_C{}^D(\overrightarrow{\partial}^l_D E^{BF})P_F{}^A \ , \tag{B.31}$$

$$\delta\nu^{(4)}_{IX} \equiv (-1)^{\varepsilon_B(\varepsilon_C+1)} E^{CD}(\overrightarrow{\partial}^l_D E^{BF})P_F{}^A(\overrightarrow{\partial}^l_A \overrightarrow{\partial}^l_C X^G)E_{GB} = -\delta\nu^{(4)}_{VIII} - \delta\nu^{(4)}_{XIII} \ , \tag{B.32}$$

$$\delta\nu^{(4)}_{X} \equiv (-1)^{(\varepsilon_B+1)\varepsilon_F} P_F{}^A(\overrightarrow{\partial}^l_A E_{BC})E^{CD}(\overrightarrow{\partial}^l_D E^{FG})(\overrightarrow{\partial}^l_G X^B) = -\delta\nu^{(4)}_{VII} \ , \tag{B.33}$$

$$\delta\nu^{(4)}_{XI} \equiv (-1)^{\varepsilon_B}(\overrightarrow{\partial}^l_A E_{BC})E^{CD}(\overrightarrow{\partial}^l_D X^B \overleftarrow{\partial}^r_F)E^{FA} \ , \tag{B.34}$$

$$\delta\nu^{(4)}_{XII} \equiv (-1)^{(\varepsilon_B+1)\varepsilon_F} P_F{}^A(\overrightarrow{\partial}^l_A E_{BC})E^{CD}(\overrightarrow{\partial}^l_D X^F \overleftarrow{\partial}^r_G)E^{GB} \ , \tag{B.35}$$

$$\delta\nu^{(4)}_{XIII} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} E^{AD}(\overrightarrow{\partial}^l_D E^{BC})(\overrightarrow{\partial}^l_C \overrightarrow{\partial}^l_A X^G)E_{GB} = -\delta\nu^{(4)}_{XIII} - \delta\nu^{(4)}_{XIV} \ , \tag{B.36}$$

$$\delta\nu^{(4)}_{XIV} \equiv (-1)^{\varepsilon_A} P_C{}^D(\overrightarrow{\partial}^l_D E^{AB})(\overrightarrow{\partial}^l_B \overrightarrow{\partial}^l_A X^C) \ , \tag{B.37}$$

where we have noted various relations among the contributions. The Jacobi identity (2.3) for E^{AB} is used in the second equality of eqs. (B.32) and (B.36). Altogether, the infinitesimal variation of $\nu^{(4)}$ becomes

$$\delta\nu^{(4)} = -\delta\nu^{(4)}_{VIII} + \delta\nu^{(4)}_{IX} + \delta\nu^{(4)}_{XI} + \delta\nu^{(4)}_{XII} = -2\delta\nu^{(4)}_{VIII} + \delta\nu^{(4)}_{XI} + \delta\nu^{(4)}_{XII} + \frac{1}{2}\delta\nu^{(4)}_{XIV} \ , \tag{B.38}$$

which is eq. (2.39).

B.4 Proof of eq. (2.41)

The infinitesimal variation of $\nu^{(5)}$ in eq. (2.25) yields 8 contributions to linear order in the variation $\delta\Gamma^A = X^A$,

$$\delta\nu^{(5)} = \delta\nu^{(5)}_I + \delta\nu^{(5)}_{II} + \delta\nu^{(5)}_{III} - \delta\nu^{(5)}_{IV} - \delta\nu^{(5)}_{V} - \delta\nu^{(5)}_{VI} + \delta\nu^{(5)}_{VII} - \delta\nu^{(5)}_{VIII} \ . \tag{B.39}$$

They are

$$\delta\nu^{(5)}_{I} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B}(X^A \overleftarrow{\partial}^r_G)E^{GD}(\overrightarrow{\partial}^l_D E^{BC})(\overrightarrow{\partial}^l_C E_{AF})P^F{}_B \ , \tag{B.40}$$

$$\delta\nu^{(5)}_{II} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B}(\overrightarrow{\partial}^l_C E_{AF})P^F{}_B E^{AD}\overrightarrow{\partial}^l_D \left((X^B \overleftarrow{\partial}^r_G)E^{GC}\right) = \delta\nu^{(5)}_{VIII} + \delta\nu^{(5)}_{IX} \ , \tag{B.41}$$

$$\delta\nu^{(5)}_{III} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B}(\overrightarrow{\partial}^l_C E_{AF})P^F{}_B E^{AD}\overrightarrow{\partial}^l_D \left(E^{BG}(\overrightarrow{\partial}^l_G X^C)\right) = \delta\nu^{(5)}_{IV} - \delta\nu^{(5)}_{X} \ , \tag{B.42}$$

$$\delta\nu^{(5)}_{IV} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} E^{AD}(\overrightarrow{\partial}^l_D E^{BC})(\overrightarrow{\partial}^l_C X^G)(\overrightarrow{\partial}^l_G E_{AF})P^F{}_B \ , \tag{B.43}$$

$$\delta\nu^{(5)}_{V} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} P^F{}_B E^{AD}(\overrightarrow{\partial}^l_D E^{BC})\overrightarrow{\partial}^l_C \left((\overrightarrow{\partial}^l_A X^G)E_{GF}\right) = \delta\nu^{(5)}_{I} + \delta\nu^{(5)}_{XI} \ , \tag{B.44}$$

$$\delta\nu^{(5)}_{VI} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} P^F{}_B E^{AD}(\overrightarrow{\partial^l_D} E^{BC})\overrightarrow{\partial^l_C}\left(E_{AG}(X^G\overleftarrow{\partial^r_F})\right) = \delta\nu^{(5)}_{VII} - \delta\nu^{(5)}_{XII} , \quad (B.45)$$

$$\delta\nu^{(5)}_{VII} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} E^{AD}(\overrightarrow{\partial^l_D} E^{BC})(\overrightarrow{\partial^l_C} E_{AF})(X^F\overleftarrow{\partial^r_G})P^G{}_B , \quad (B.46)$$

$$\delta\nu^{(5)}_{VIII} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} E^{AD}(\overrightarrow{\partial^l_D} E^{BC})(\overrightarrow{\partial^l_C} E_{AF})P^F{}_G(X^G\overleftarrow{\partial^r_B}) , \quad (B.47)$$

$$\begin{aligned}\delta\nu^{(5)}_{IX} &\equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} E^{AD}(\overrightarrow{\partial^l_D} X^B\overleftarrow{\partial^r_G})E^{GC}(\overrightarrow{\partial^l_C} E_{AF})P^F{}_B \\ &= (-1)^{\varepsilon_B\varepsilon_F} P_F{}^D(\overrightarrow{\partial^l_D} X^B\overleftarrow{\partial^r_G})E^{GC}(\overrightarrow{\partial^l_C} E^{FA})E_{AB} = \delta\nu^{(5)}_{XI} + \delta\nu^{(5)}_{XII} , \quad (B.48)\end{aligned}$$

$$\delta\nu^{(5)}_{X} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_C}(\overrightarrow{\partial^l_C} E_{AB})E^{BF}(\overrightarrow{\partial^l_F} X^C\overleftarrow{\partial^r_G})E^{GA} , \quad (B.49)$$

$$\delta\nu^{(5)}_{XI} \equiv (-1)^{(\varepsilon_A+1)\varepsilon_B} E^{AD}(\overrightarrow{\partial^l_D} E^{BC})(\overrightarrow{\partial^l_C}\overrightarrow{\partial^l_A} X^G)E_{GB} , \quad (B.50)$$

$$\delta\nu^{(5)}_{XII} \equiv (-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A}\overrightarrow{\partial^l_B} X^C)P_C{}^D(\overrightarrow{\partial^l_D} E^{BF})P_F{}^A , \quad (B.51)$$

where we have noted various relations among the contributions. The Jacobi identity (2.3) for E^{AB} is used in the third equality of eq. (B.48). Altogether, the infinitesimal variation of $\nu^{(5)}$ becomes

$$\delta\nu^{(5)} = \delta\nu^{(5)}_{IX} - \delta\nu^{(5)}_{X} - \delta\nu^{(5)}_{XI} + \delta\nu^{(5)}_{XII} = -\delta\nu^{(5)}_{X} + 2\delta\nu^{(5)}_{XII} , \quad (B.52)$$

which is eq. (2.41).

C Proof of Conversion Theorem 4.2

C.1 The $\bar{\jmath}$ Superdeterminant

Even-though it is only the $\jmath$-factor (4.13) and not the whole $\bar{\jmath}$ superdeterminant (4.14) that enters the conversion map, it is nevertheless convenient to organize the discussion in terms of coefficient functions for (the logarithm of) the $\bar{\jmath}$ superdeterminant

$$\ln\bar{\jmath} \equiv \bar{n} = n + \left\{\begin{array}{c} n^a_L\Phi_a \\ \Phi_a n^a_R \end{array}\right\} + \frac{1}{2}n^{ab}_L\Phi_b\Phi_a + \mathcal{O}(\Phi^3) , \qquad n \equiv \ln\jmath . \quad (C.1)$$

By combining eqs. (4.5), (4.14) and (C.1), one finds the first-order coefficient functions n^a to be

$$\begin{aligned} n^a_L &= (-1)^{\varepsilon_b} X^R_{bc} Y^{cba}_M = (-1)^{\varepsilon_b+1} X^L_{bc} Y^{cba}_L \\ n^a_R &= Y^{abc}_M X^L_{cb}(-1)^{\varepsilon_b} = Y^{abc}_R X^R_{cb}(-1)^{\varepsilon_b+1} . \end{aligned} \quad (C.2)$$

The second-order coefficient functions read

$$n^{cd}_L = (-1)^{\varepsilon_b+1} X^L_{ba} Z^{abcd}_L + (-1)^{(\varepsilon_a+1)\varepsilon_c} X^R_{ab} Y^{bce}_R X^R_{ef} Y^{fad}_M . \quad (C.3)$$

In particular, the contracted second-order coefficient function is

$$(-1)^{\varepsilon_c+1} n^{cd}_L \omega_{dc} = z^{(1)} - y^{(2)} , \quad (C.4)$$

where we have introduced the following short-hand notation

$$y^{(2)} \equiv (-1)^{\varepsilon_a\varepsilon_f} X^R_{ab} Y^{bfc}_M \omega_{cd} Y^{dae}_M X^L_{ef} , \quad (C.5)$$

$$z^{(1)} \equiv (-1)^{\varepsilon_b+\varepsilon_c} X^L_{ba} Z^{abcd}_L \omega_{dc} . \quad (C.6)$$

Since there is not a unique choice of the structure functions X^{ab}, Y^{abc}, Z^{abcd}, etc, one must apply the T^a involution relation (4.4) to eliminate their appearances. We have to wait until Subsection C.5 to completely eliminate all Y^{abc} appearances, but we can do a first step in this direction. The quadratic Y^{abc} dependence inside the odd $y^{(2)}$ variable (C.5) can be related to a linear Y^{abc} dependence inside a new $y^{(1)}$ variable as follows

$$\begin{aligned}
0 &\stackrel{(4.4)}{=} \frac{1}{2}(-1)^{\varepsilon_a+1}Y_R^{bac}X_{cd}^R\frac{\overrightarrow{\partial^l}}{\partial\Phi_d}\, X_{ae}^L(T^e,T^f)_{\rm ext}X_{fb}^R\Big|_{\Phi=0} \\
&= \frac{1}{2}(-1)^{\varepsilon_c\varepsilon_e}X_{cd}^R\frac{\overrightarrow{\partial^l}}{\partial\Phi_d}\,(T^e,T^f)_{\rm ext}\Big|_{\Phi=0}X_{fb}^RY_M^{bca}X_{ae}^L \\
&= (-1)^{\varepsilon_b}Y_R^{cba}(X_{ab}^R,\Theta^d)X_{dc}^R+(-1)^{\varepsilon_a\varepsilon_f}X_{ab}^RY_M^{bfc}\omega_{cd}Y_M^{dae}X_{ef}^L \;=\; y^{(1)}+y^{(2)}\;, \qquad \text{(C.7)}
\end{aligned}$$

where

$$y^{(1)} \;\equiv\; (-1)^{\varepsilon_b}Y_R^{cba}(X_{ab}^R,\Theta^d)X_{dc}^R\;. \qquad \text{(C.8)}$$

The only way the Z_L^{abcd} structure functions enters the discussion is through the odd $z^{(1)}$ variable (C.6). It can be eliminated using the following equation

$$\begin{aligned}
0 &\stackrel{(4.4)}{=} X_{dc}^R\frac{\overrightarrow{\partial^l}}{\partial\Phi_c}(-1)^{\varepsilon_a}X_{ab}^R\frac{\overrightarrow{\partial^l}}{\partial\Phi_b}\,(T^a,T^d)_{\rm ext}\Big|_{\Phi=0} \\
&= (-1)^{\varepsilon_a}X_{ab}^R\frac{\overrightarrow{\partial^l}}{\partial\Phi_b}\,(T^a,T^d)_{\rm ext}\frac{\overleftarrow{\partial^r}}{\partial\Phi_c}X_{cd}^L(-1)^{\varepsilon_d}\Bigg|_{\Phi=0} \\
&= (n,n)+n_L^a\omega_{ab}n_R^b+2(-1)^{\varepsilon_b+\varepsilon_c}X_{ba}^L(Y_L^{abc},\Theta^d)X_{dc}^R+(-1)^{\varepsilon_a\varepsilon_d}X_{ab}^R(X_R^{bd},X_L^{ac})X_{cd}^L \\
&\quad +2(-1)^{\varepsilon_b+\varepsilon_c}X_{ba}^LZ_L^{abcd}\omega_{dc}+(-1)^{\varepsilon_a\varepsilon_f}X_{ab}^RY_M^{bfc}\omega_{cd}Y_M^{dae}X_{ef}^L \\
&= (n,n)+n_L^a\omega_{ab}n_R^b+2(n_R^a,\Theta^b)X_{ba}^R+(-1)^{\varepsilon_b}(X_L^{ab},X_{ba}^L)+2z^{(1)}+y^{(1)}\;. \qquad \text{(C.9)}
\end{aligned}$$

C.2 Gauge Invariant Function $\bar{F}$

The gauge-invariant extension $\bar{F}$ is a power series expansion in the Φ_a variables, *e.g.* ,

$$\bar{F} \;=\; F+\left\{\begin{array}{c}\Phi_aF_R^a\\ F_L^a\Phi_a\end{array}\right\}+\frac{1}{2}\Phi_a\Phi_bF_R^{ba}+\mathcal{O}(\Phi^3)\;. \qquad \text{(C.10)}$$

The coefficient functions for $\bar{F}$ are uniquely determined by gauge invariance condition (4.10). The first-order coefficient functions read

$$\begin{aligned}
F_R^a &= -\omega^{ab}X_{bc}^L(\Theta^c,F) &=& \; X_R^{ab}E_{bc}(\Theta^c,F)\;, \\
F_L^a &= -(F,\Theta^c)X_{cb}^R\omega^{ba} &=& \; (F,\Theta^c)E_{cb}X_L^{ba}\;,
\end{aligned} \qquad \text{(C.11)}$$

The contracted second-order coefficient function $(-1)^{\varepsilon_a+1}\omega_{ab}F_R^{ba}$ is determined by the following calculation

$$\begin{aligned}
0 &\stackrel{(4.10)}{=} (-1)^{\varepsilon_a}X_{ab}^R\frac{\overrightarrow{\partial^l}}{\partial\Phi_b}\,(T^a,\bar{F})_{\rm ext}\big|_{\Phi=0} \\
&= X_{ba}^L(\Theta^a,F_R^b)(-1)^{\varepsilon_b+1}+(n,F)+n_L^a\omega_{ab}F_R^b+(-1)^{\varepsilon_a+1}\omega_{ab}F_R^{ba}\;. \qquad \text{(C.12)}
\end{aligned}$$

C.3 Gauge Invariant Density $\bar{\rho}$

The (logarithm of the) gauge-invariant density $\bar{\rho}$ is a power series expansion in the Φ_a variables, *e.g.* ,

$$\ln\sqrt{\bar{\rho}} \;\equiv\; \bar{\ell} \;=\; \ell+\left\{\begin{array}{c}\ell_L^a\Phi_a\\ \Phi_a\ell_R^a\end{array}\right\}+\frac{1}{2}\Phi_a\Phi_b\ell_R^{ba}+\mathcal{O}(\Phi^3)\;, \qquad \ell \;\equiv\; \ln\sqrt{\rho j}\;. \qquad \text{(C.13)}$$

The coefficient functions for $\bar{\rho}$ are uniquely determined by the gauge invariance condition (4.11). The first-order coefficient functions ℓ^a can be found from the following Lemma C.1.

Lemma C.1

$$\begin{aligned}\frac{1}{2}n_L^c - \ell_L^c &= (\Delta_\rho \Theta^a) X^R_{ab}\omega^{bc} + \frac{1}{2}(-1)^{\varepsilon_a}(\Theta^a, X^R_{ab})\omega^{bc} \\ &= \frac{(-1)^{\varepsilon_A}}{2\rho}\overrightarrow{\partial^l_A}\rho(\Gamma^A, \Theta^a) X^R_{ab}\omega^{bc} \ , \end{aligned} \tag{C.14}$$

$$\begin{aligned}\frac{1}{2}n_R^c - \ell_R^c &= \omega^{cb} X^L_{ba}(\Delta_\rho \Theta^a)(-1)^{\varepsilon_a} + \frac{1}{2}\omega^{cb}(X^L_{ba}, \Theta^a)(-1)^{\varepsilon_a} \\ &= \omega^{cb} X^L_{ba}(\Theta^a, \Gamma^A)\rho\overleftarrow{\partial^r_A}\frac{(-1)^{\varepsilon_A}}{2\rho} \ . \end{aligned} \tag{C.15}$$

Proof of Lemma C.1: Combine

$$0 \stackrel{(4.11)}{=} (\Delta_{\bar{\rho}} T^a)\Big|_{\Phi=0} = (\Delta_{\rho j}\Theta^a) + \frac{1}{2}(-1)^{\varepsilon_b+1}\omega_{bc}Y_R^{cba} + \ell_L^c\omega_{cb}X_R^{ba} \tag{C.16}$$

and

$$\begin{aligned} 0 &\stackrel{(4.4)}{=} (-1)^{\varepsilon_b} X^R_{bc}\frac{\overrightarrow{\partial^l}}{\partial\Phi_c}(T^b, T^a)_{\rm ext}\Big|_{\Phi=0} \\ &= (-1)^{\varepsilon_c+1}X^L_{cb}(\Theta^b, X_R^{ca}) + (n, \Theta^a) + n_L^c\omega_{cb}X_R^{ba} + (-1)^{\varepsilon_b+1}\omega_{bc}Y_R^{cba} \ . \end{aligned} \tag{C.17}$$

□

The contracted second-order coefficient function $(-1)^{\varepsilon_a+1}\omega_{ab}\ell_R^{ba}$ is determined by the following calculation

$$\begin{aligned} 0 &\stackrel{(4.11)}{=} X^R_{ab}\frac{\overrightarrow{\partial^l}}{\partial\Phi_b}(\Delta_{\bar{\rho}} T^a)\Big|_{\Phi=0} \\ &= (\Delta_{\rho j}X_L^{ab})X^L_{ba}(-1)^{\varepsilon_a} + \frac{1}{2}(-1)^{\varepsilon_b+\varepsilon_c}X^L_{ba}Z_L^{abcd}\omega_{dc} \\ &\quad + X^L_{ba}(\Theta^a, \ell_L^b) + n_L^a\omega_{ab}\ell_R^b + (-1)^{\varepsilon_a+1}\omega_{ab}\ell_R^{ba} \ . \end{aligned} \tag{C.18}$$

C.4 Assembling the Proof

Proof of eq. (4.25):

$$(\bar{F}, \bar{G})_{\rm ext}\big|_{\Phi=0} = (F, G) + F_L^a\omega_{ab}G_R^b \stackrel{(C.11)}{=} (F, G)_D \ . \tag{C.19}$$

□

Proof of eq. (4.26):

$$\begin{aligned} (\Delta_{\bar{\rho}, E_{\rm ext}}\bar{F})\Big|_{\Phi=0} &= (\Delta_{\rho j}F) + \frac{1}{2}(-1)^{\varepsilon_a+1}\omega_{ab}F_R^{ba} + \ell_L^a\omega_{ab}F_R^b \\ &\stackrel{(C.12)}{=} (\Delta_{\rho j}F) - \frac{1}{2}(n, F) + (\ell_L^a - \frac{1}{2}n_L^a)\omega_{ab}F_R^b - \frac{1}{2}X^L_{ba}(\Theta^a, F_R^b)(-1)^{\varepsilon_b+1} \\ &\stackrel{(C.14)}{=} (\Delta_\rho F) - (\Delta_\rho\Theta^a)X^R_{ab}F_R^b - \frac{1}{2}(-1)^{\varepsilon_a}(\Theta^a, X^R_{ab}F_R^b) \\ &= (\Delta_\rho F) - (\Delta_\rho\Theta^a)E_{ab}(\Theta^b, F) - \frac{1}{2}(-1)^{\varepsilon_a}(\Theta^a, E_{ab}(\Theta^b, F)) \\ &= (\Delta_{\rho, E_D}F) \ . \end{aligned} \tag{C.20}$$

□

Proof of eq. (4.28): Using eq. (2.45) it follows that

$$\begin{aligned}
\nu_{\rho j} - \nu_{\rho} &\stackrel{(2.45)}{=} \frac{1}{\sqrt{j}}(\Delta_{\rho}\sqrt{j}) \;=\; \frac{1}{2}(\Delta_{\rho}n) + \frac{1}{8}(n,n) \;=\; \frac{1}{2}(\Delta_{\rho j}n) - \frac{1}{8}(n,n) \\
&= \frac{1}{2}(\Delta_{\rho j}X_L^{ab})X_{ba}^L(-1)^{\varepsilon_a} - \frac{1}{4}(-1)^{\varepsilon_b}(X_L^{ab}, X_{ba}^L) - \frac{1}{8}(n,n) \;,
\end{aligned} \tag{C.21}$$

so that

$$\begin{aligned}
\nu_{\bar{\rho},E_{\rm ext}}\Big|_{\Phi=0} &= \nu_{\rho j} + \frac{1}{2}(-1)^{\varepsilon_a+1}\omega_{ab}\ell_R^{ba} + \frac{1}{2}\ell_L^a\omega_{ab}\ell_R^b \\
&\stackrel{(C.21)}{=} \nu_{\rho} - \frac{1}{8}(n,n) + \frac{1}{2}(\Delta_{\rho j}X_L^{ab})X_{ba}^L(-1)^{\varepsilon_a} - \frac{1}{4}(-1)^{\varepsilon_b}(X_L^{ab}, X_{ba}^L) \\
&\quad + \frac{1}{2}(-1)^{\varepsilon_a+1}\omega_{ab}\ell_R^{ba} + \frac{1}{2}\ell_L^a\omega_{ab}\ell_R^b \\
&\stackrel{(C.18)}{=} \nu_{\rho} - \frac{1}{8}(n,n) - \frac{1}{2}n_L^a\omega_{ab}\ell_R^b + \frac{1}{2}\ell_L^a\omega_{ab}\ell_R^b \\
&\quad - \frac{1}{4}(-1)^{\varepsilon_b}(X_L^{ab}, X_{ba}^L) - \frac{z^{(1)}}{4} - \frac{1}{2}X_{ba}^L(\Theta^a, \ell_L^b) \\
&\stackrel{(C.9)}{=} \nu_{\rho} + \frac{1}{2}(\frac{1}{2}n_L^a - \ell_L^a)\omega_{ab}(\frac{1}{2}n_R^b - \ell_R^b) + \frac{1}{2}(\Theta^a, \frac{1}{2}n_L^b - \ell_L^b)X_{ba}^L \\
&\quad - \frac{1}{8}(-1)^{\varepsilon_b}(X_L^{ab}, X_{ba}^L) + \frac{y^{(1)}}{8} \\
&\stackrel{(C.14)}{=} \nu_{\rho} - \frac{\nu_{\rho,D}^{(6)}}{2} + \frac{1}{2}(\Theta^a, X_{ab}^R)\omega^{bc}(\Delta_{\rho}\Theta^c)(-1)^{\varepsilon_c} + \frac{1}{8}(-1)^{\varepsilon_a+\varepsilon_d}(\Theta^a, X_{ab}^R)\omega^{bc}(X_{cd}^L, \Theta^d) \\
&\quad + \frac{1}{2}(\Theta^a, \frac{1}{2}n_L^b - \ell_L^b)X_{ba}^L - \frac{1}{8}(-1)^{\varepsilon_b}(X_L^{ab}, X_{ba}^L) + \frac{y^{(1)}}{8} \\
&= \nu_{\rho} - \frac{\nu_{\rho,D}^{(6)}}{2} - \frac{\nu_{\rho,D}^{(7)}}{2} + \frac{1}{8}(-1)^{\varepsilon_a+\varepsilon_d}(\Theta^a, X_{ab}^R)\omega^{bc}(X_{cd}^L, \Theta^d) \\
&\quad + \frac{1}{4}(-1)^{\varepsilon_b}(\Theta^a, (\Theta^b, X_{bc}^R))\omega^{cd}X_{da}^L - \frac{1}{8}(-1)^{\varepsilon_b}(X_L^{ab}, X_{ba}^L) + \frac{y^{(1)}}{8} \\
&= \nu_{\rho,E_D} - \frac{\nu_D^{(9)}}{24} - \frac{x^{(1)}}{8} + \frac{y^{(1)}}{8} \;\stackrel{(C.23)}{=}\; \nu_{\rho,E_D} \;,
\end{aligned} \tag{C.22}$$

where the last equality in eq. (C.22) follows from Lemma C.2 below, and the odd quantity $x^{(1)}$ is defined in eq. (C.25).

□

C.5 Lemma C.2

It turns out that the most difficult part in the proof of eq. (4.28) is to eliminate the Y^{abc} dependence from the odd $y^{(1)}$ quantity (C.8). Lemma C.2 gives a formula for $y^{(1)}$ that are manifestly independent of Y^{abc}.

Lemma C.2

$$y^{(1)} \;=\; \frac{\nu_D^{(9)}}{3} + x^{(1)} \;. \tag{C.23}$$

PROOF OF LEMMA C.2: We first decompose the odd $\nu_D^{(9)}$ quantity (3.14) as

$$\nu_D^{(9)} \equiv (-1)^{(\varepsilon_a+1)(\varepsilon_d+1)}(\Theta^d, E_{ab})E^{bc}(E_{cd}, \Theta^a) = -x^{(1)} - 2x^{(2)} - x^{(3)} , \qquad (C.24)$$

where

$$\begin{aligned} x^{(1)} &\equiv (-1)^{(\varepsilon_a+1)(\varepsilon_d+1)}(\Theta^d, X^R_{ab})\omega^{bc}(X^L_{cd}, \Theta^a) , & (C.25)\\ x^{(2)} &\equiv (-1)^{\varepsilon_b(\varepsilon_d+1)}(\Theta^d, X_L^{bc})(X^L_{cd}, \Theta^a)E_{ab} = (-1)^{\varepsilon_b(\varepsilon_d+1)}E_{ba}(\Theta^a, X^R_{dc})(X_R^{cb}, \Theta^d) , & (C.26)\\ x^{(3)} &\equiv (-1)^{\varepsilon_b\varepsilon_e}E_{ef}(\Theta^f, X_L^{bc})\omega_{cd}(X_R^{de}, \Theta^a)E_{ab} . & (C.27)\end{aligned}$$

Secondly, we define

$$\begin{aligned} x^{(4)} &\equiv (-1)^{\varepsilon_c}E_{ab}(\Theta^b, X_L^{cd})(X^L_{dc}, \Theta^a)\\ &= -(-1)^{\varepsilon_c}E_{ab}(\Theta^b, X_R^{cd})(X^R_{dc}, \Theta^a) = x^{(3)} - x^{(1)} . \qquad (C.28)\end{aligned}$$

The third (=last) equality in eq. (C.28) is a non-trivial assertion. To prove it, we define the following quantities:

$$\begin{aligned} x^{(5)} &\equiv (-1)^{\varepsilon_d}(\Theta^f, X^R_{dc})X_R^{cb}E_{ba}(\Theta^a, E^{de})E_{ef}\\ &= -(-1)^{\varepsilon_b\varepsilon_e}E_{ef}(\Theta^f, X_L^{bc})X^L_{cd}(E^{de}, \Theta^a)E_{ab} = x^{(2)} + x^{(3)} , & (C.29)\\ x^{(6)} &\equiv (-1)^{\varepsilon_b\varepsilon_e}E_{ef}(\Theta^f, X_L^{bc})X^L_{cd}(\Theta^d, E^{ea})E_{ab} = -x^{(1)} - x^{(2)} , & (C.30)\\ x^{(7)} &\equiv (-1)^{\varepsilon_d}(\Theta^f, X^R_{dc})X_R^{cb}E_{ba}(E^{ad}, \Theta^e)E_{ef} = -x^{(4)} + x^{(8)} , & (C.31)\\ x^{(8)} &\equiv (-1)^{(\varepsilon_a+1)(\varepsilon_d+1)}E^{ab}(X^R_{bd}, \Theta^e)E_{ef}(\Theta^f, X^R_{ac})\omega^{cd} = 0 , & (C.32)\end{aligned}$$

where eq. (4.8) is used in the second equality of eqs. (C.29), (C.30) and (C.31). Remarkably the quantity $x^{(8)}$ vanishes due to an antisymmetry under the index permutation $ace \leftrightarrow bdf$. One may now check that the Jacobi identity

$$\sum_{\text{cycl. } a,b,c} (-1)^{(\varepsilon_a+1)(\varepsilon_c+1)}(E^{ab}, \Theta^c) = 0 \qquad (C.33)$$

yields eq. (C.28):

$$0 = x^{(5)} + x^{(6)} + x^{(7)} = -x^{(1)} + x^{(3)} - x^{(4)} . \qquad (C.34)$$

Thirdly, we define

$$\begin{aligned} y^{(3)} &\equiv (-1)^{\varepsilon_c}E_{ab}Y_L^{bcd}\omega_{de}X_R^{ef}(X^R_{fc}, \Theta^a) = y^{(1)} + x^{(4)} + x^{(2)} , & (C.35)\\ y^{(4)} &\equiv (-1)^{(\varepsilon_a+1)\varepsilon_d}\omega^{ab}X^L_{bc}Y_L^{cde}\omega_{ef}X_R^{fg}(X^R_{ga}, \Theta^h)X^R_{hd} = y^{(1)} - x^{(1)} + x^{(2)} , & (C.36)\end{aligned}$$

where eq. (4.9) is used in the second equality of eqs. (C.35) and (C.36). Note that $x^{(1)}$ to $x^{(8)}$ are manifestly independent of the Y^{abc} structure functions. We shall soon see that this is also the case for the variables $y^{(1)}$ to $y^{(4)}$. It turns out to be possible to rewrite $y^{(3)}$ as

$$\begin{aligned} y^{(3)} &= \frac{1}{2}(-1)^{(\varepsilon_a+1)(\varepsilon_c+\varepsilon_f+1)+\varepsilon_c}E_{ab}Y_L^{bcd}X^L_{de}(E^{ef}, \Theta^a)X^R_{fc}\\ &= (-1)^{\varepsilon_a\varepsilon_c}E_{ab}Y_L^{bcd}X^L_{de}(E^{ea}, \Theta^f)X^R_{fc} = -y^{(1)} - y^{(4)} . \qquad (C.37)\end{aligned}$$

Here the Jacobi identity (C.33) is used in the second equality of eq. (C.37). Altogether, eqs. (C.35), (C.36) and (C.37) yields

$$3y^{(1)} = x^{(1)} - 2x^{(2)} - x^{(4)} . \qquad (C.38)$$

Now Lemma C.2 follows by combining eqs. (C.24), (C.28) and (C.38).

□

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Paper III

Odd Scalar Curvature in Field-Antifield Formalism

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J. Math. Phys. **49** (2008) 033515

arXiv:0708.0400

Odd Scalar Curvature in Field-Antifield Formalism

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January 5, 2014

Abstract

We consider the possibility of adding a Grassmann-odd function ν to the odd Laplacian. Requiring the total Δ operator to be nilpotent leads to a differential condition for ν, which is integrable. It turns out that the odd function ν is not an independent geometric object, but is instead completely specified by the antisymplectic structure E and the density ρ. The main impact of introducing the ν term is that it makes compatibility relations between E and ρ obsolete. We give a geometric interpretation of ν as (minus 1/8 times) the odd scalar curvature of an arbitrary antisymplectic, torsion-free and ρ-compatible connection. We show that the total Δ operator is a ρ-dressed version of Khudaverdian's Δ_E operator, which takes semidensities to semidensities. We also show that the construction generalizes to the situation where ρ is replaced by a non-flat line bundle connection F. This generalization is implemented by breaking the nilpotency of Δ with an arbitrary Grassmann-even second-order operator source.

PACS number(s): 02.40.-k; 03.65.Ca; 04.60.Gw; 11.10.-z; 11.10.Ef; 11.15.Bt.
Keywords: BV Field-Antifield Formalism; Odd Laplacian; Antisymplectic Geometry; Semidensity; Antisymplectic Connection; Odd Scalar Curvature.

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1 Introduction

Conventionally [1, 2, 3, 4] the geometric arena for quantization of Lagrangian theories in the field-antifield formalism [5, 6, 7] is taken to be an antisymplectic manifold $(M;E)$ with a measure density ρ. Each point in the manifold M with local coordinates Γ^A and Grassmann parity $\varepsilon_A \equiv \varepsilon(\Gamma^A)$ represents a field-antifield configuration $\Gamma^A = \{\phi^\alpha; \phi^*_\alpha\}$, the antisymplectic structure E provides the antibracket $(\cdot,\cdot)$, and the density ρ yields the path integral measure. However, up until recently, it has been necessary to impose a compatibility condition [2, 8] between the two geometric structures E and ρ to ensure nilpotency of the odd Laplacian

$$\Delta_\rho \equiv \frac{(-1)^{\varepsilon_A}}{2\rho} \overrightarrow{\partial^l_A} \rho E^{AB} \overrightarrow{\partial^l_B} , \qquad \overrightarrow{\partial^l_A} \equiv \frac{\overrightarrow{\partial^l}}{\partial\Gamma^A} . \tag{1.1}$$

In this paper, we show that the compatibility condition between E and ρ can be omitted if one adds an odd scalar function ν to the odd Laplacian Δ_ρ,

$$\Delta = \Delta_\rho + \nu \tag{1.2}$$

such that the total Δ operator is nilpotent

$$\Delta^2 = 0 . \tag{1.3}$$

Nilpotency is important for the field-antifield formalism in many ways, for instance in securing that the physical partition function $\mathcal{Z}$ is independent of gauge-choice, see Appendix A. (More precisely, what is really vital is the nilpotency of the underlying Δ_E operator, cf. Sections 8-9.) In physics terms, the addition of the ν function to the odd Laplacian Δ_ρ implies that the quantum master equation

$$\Delta e^{\frac{i}{\hbar}W} = 0 \tag{1.4}$$

is modified with a ν term at the two-loop order $\mathcal{O}(\hbar^2)$:

$$\frac{1}{2}(W,W) = i\hbar\Delta_\rho W + \hbar^2\nu , \tag{1.5}$$

and Δ_ρ is in general no longer a nilpotent operator. It turns out that the zeroth-order ν term is uniquely determined from the nilpotency requirement (1.3) apart from an odd constant. One particular solution to the zeroth-order term, which we call ν_ρ, takes a special form [9]

$$\nu_\rho \equiv \nu^{(0)}_\rho + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24} , \tag{1.6}$$

where $\nu^{(0)}_\rho$, $\nu^{(1)}$ and $\nu^{(2)}$ are defined as

$$\nu^{(0)}_\rho \equiv \frac{1}{\sqrt{\rho}}(\Delta_1\sqrt{\rho}) , \tag{1.7}$$

$$\nu^{(1)} \equiv (-1)^{\varepsilon_A}(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A} E^{AB}) , \tag{1.8}$$

$$\nu^{(2)} \equiv (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_D} E^{AB})E_{BC}(\overrightarrow{\partial^l_A} E^{CD}) \tag{1.9}$$

$$= -(-1)^{\varepsilon_B}(\overrightarrow{\partial^l_A} E_{BC})E^{CD}(\overrightarrow{\partial^l_D} E^{BA}) . \tag{1.10}$$

Here, Δ_1 in eq. (1.7) denotes the expression (1.1) for the odd Laplacian $\Delta_{\rho=1}$ with ρ replaced by 1. In particular, the odd scalar ν_ρ is a function of E and ρ, so there is no call for new independent geometric structures on the manifold M. In Sections 2–6 we show that $\Delta_\rho + \nu$ is the only possible Δ operator within the set of all second-order differential operators. The now obsolete compatibility condition [2, 8] between E and ρ can be recast as $\nu_\rho =$ odd constant, thereby making contact to the previous approach [2], which uses the odd Laplacian Δ_ρ only. The explicit formula (1.6) for ν_ρ is proven in Section 7 and Appendix B. The formula (1.6) first appeared in Ref. [9]. That paper was devoted to Khudaverdian's Δ_E operator [10, 11, 12, 13], which takes semidensities to semidensities. This is no coincidence: At the bare level of mathematical formulas the construction is intimately related to the Δ_E operator, as shown in Sections 8-9. However the starting point is different. On one hand, Ref. [9] studied the Δ_E operator in its minimal and purest setting, which is a manifold with an antisymplectic structure E but without a density ρ. On the other hand, the starting point of the current paper is a Δ operator that takes scalar functions to scalar functions, and this implies that a choice of ρ (or F, cf. below) should be made. Later in Sections 10 and 11 we interpret the odd ν_ρ function as (minus 1/8 times) the odd scalar curvature R of an arbitrary antisymplectic, torsion-free and ρ-compatible connection,

$$\nu_\rho \;=\; -\frac{R}{8}\;. \tag{1.11}$$

One of the main priorities for the current article is to ensure that all arguments are handled in completely general coordinates without resorting to Darboux coordinates at any stage. This is important to give a physical theory a natural, coordinate-independent, geometric status in the antisymplectic phase space. We shall also throughout the paper often address the question of generalizing the density ρ to a non-flat line bundle connection F. It is well-known [2] that a density ρ gives rise to a flat line bundle connection

$$F_A \;=\; (\overrightarrow{\partial^l_A} \ln \rho)\;. \tag{1.12}$$

In fact, several mathematical objects, for instance the odd Laplacian Δ_ρ and the odd scalar ν_ρ, can be formulated entirely using F instead of ρ. Surprisingly, many of these objects continue to be well-defined for non-flat F's as well, where the nilpotency (and the ordinary physical description) is broken down. In Section 5 we shall therefore temporarily digress to contemplate a modification of the nilpotency condition that addresses these mathematical observations. Finally, Section 12 contains our conclusions.

General remark about notation. We have two types of grading: A Grassmann grading ε and an exterior form degree p. The sign conventions are such that two exterior forms ξ and η, of Grassmann parity ε_ξ, ε_η and exterior form degree p_ξ, p_η, respectively, commute in the following graded sense:

$$\eta \wedge \xi \;=\; (-1)^{\varepsilon_\xi \varepsilon_\eta + p_\xi p_\eta} \xi \wedge \eta \tag{1.13}$$

inside the exterior algebra. We will often not write the exterior wedges "$\wedge$" explicitly.

2 General Second-Order Δ operator

We here introduce the setting and notation more carefully, and argue that the Δ operator must be equal to $\Delta_\rho + \nu_\rho$ up to an odd constant. (The undetermined odd constant comes from the fact that the square $\Delta^2 = \frac{1}{2}[\Delta, \Delta]$ does not change if Δ is shifted by an odd constant.) Consider now an arbitrary Grassmann-odd, second-order, differential operator Δ that takes scalar functions to scalar functions. In this paper, we shall only discuss the non-degenerate case, where the second-order term in Δ is of maximal rank, and hence provides for a non-degenerated antibracket $(\cdot, \cdot)$, cf. the Definition (2.6) below. (The non-degeneracy assumption is motivated by the fact that it is satisfied for currently known applications. The degenerate case may be dealt with via for instance the antisymplectic conversion mechanism [14, 15].) Due to the non-degeneracy assumption, it is always possible to organize Δ as

$$\Delta \;=\; \Delta_F + \nu \;, \tag{2.1}$$

where ν is a zeroth-order term and Δ_F is an operator with terms of second and first order [2]

$$\Delta_F \;\equiv\; \frac{(-1)^{\varepsilon_A}}{2}(\overrightarrow{\partial^l_A}+F_A)E^{AB}\overrightarrow{\partial^l_B} \;. \tag{2.2}$$

Here, $E^{AB}=E^{AB}(\Gamma)$, $F_A=F_A(\Gamma)$ and $\nu=\nu(\Gamma)$ is a $(2,0)$-tensor, a line bundle connection, and a scalar, respectively. We shall sometimes use the slightly longer notation $\Delta_F \equiv \Delta_{F,E}$ to acknowledge that it depends on two inputs: F and E. The line bundle connection F_A transforms under general coordinate transformations $\Gamma^A \to \Gamma'^B$ as

$$F_A \;=\; (\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}\Gamma'^B)F'_B + (\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}\ln J) \;, \qquad J \;\equiv\; \mathrm{sdet}\frac{\partial\Gamma'^B}{\partial\Gamma^A} \;. \tag{2.3}$$

These transformation properties guarantee that the expressions (2.1) and (2.2) remain invariant under general coordinate transformations. The Grassmann-parities are

$$\varepsilon(E^{AB}) \;=\; \varepsilon_A+\varepsilon_B+1 \;, \quad \varepsilon(F_A) \;=\; \varepsilon_A \;, \quad \varepsilon(\nu) \;=\; 1 \;. \tag{2.4}$$

One may, without loss of generality, assume that the $(2,0)$-tensor E^{AB} has a Grassmann-graded skewsymmetry

$$E^{AB} \;=\; -\,(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}E^{BA} \;. \tag{2.5}$$

The antibracket (f,g) of two functions $f=f(\Gamma)$ and $g=g(\Gamma)$ is defined via a double commutator* [16] with the Δ-operator, acting on the constant unit function 1,

$$\begin{aligned}(f,g) \;&\equiv\; (-1)^{\varepsilon_f}[[\overrightarrow{\Delta},f],g]1 \;\equiv\; (-1)^{\varepsilon_f}\Delta(fg)-(-1)^{\varepsilon_f}(\Delta f)g - f(\Delta g)+(-1)^{\varepsilon_g}fg(\Delta 1)\\ &=\; (f\overleftarrow{\partial^r_A})E^{AB}(\overrightarrow{\partial^l_B}g) \;=\; -\,(-1)^{(\varepsilon_f+1)(\varepsilon_g+1)}(g,f) \;,\end{aligned} \tag{2.6}$$

where use is made of the skewsymmetry (2.5) in the third equality. By the non-degeneracy assumption, there exists an inverse matrix E_{AB} such that

$$E^{AB}E_{BC} \;=\; \delta^A_C \;=\; E_{CB}E^{BA} \;. \tag{2.7}$$

Since the tensor E^{AB} possesses a graded $A\leftrightarrow B$ skewsymmetry (2.5), the inverse tensor E_{AB} must be skewsymmetric,

$$E_{AB} = -(-1)^{\varepsilon_A\varepsilon_B}E_{BA} \;. \tag{2.8}$$

In other words, E_{AB} is a two-form

$$E \;=\; \frac{1}{2}d\Gamma^A E_{AB}\wedge d\Gamma^B \;. \tag{2.9}$$

The Grassmann parity is

$$\varepsilon(E_{AB}) \;=\; \varepsilon_A+\varepsilon_B+1 \;. \tag{2.10}$$

*Here, and throughout the paper, $[A,B]$ and $\{A,B\}$ denote the graded commutator $[A,B]\equiv AB-(-1)^{\varepsilon_A\varepsilon_B}BA$ and the graded anticommutator $\{A,B\}\equiv AB+(-1)^{\varepsilon_A\varepsilon_B}BA$, respectively.

3 Nilpotency Conditions: Part I

The square $\Delta^2 = \frac{1}{2}[\Delta, \Delta]$ of an odd second-order operator (2.1) is generally a third-order differential operator, which we, for simplicity, imagine has been normal ordered, *i.e.* with all derivatives standing to the right. Nilpotency (1.3) of the Δ operator leads to conditions on E^{AB}, F_A and ν. Let us therefore systematically, over the next four Sections 3–6, discuss order by order the consequences of the nilpotency condition $\Delta^2 = 0$, starting with the highest (third) order terms, and going down until we reach the zeroth order.

The third-order terms of Δ^2 vanish if and only if the Jacobi identity

$$\sum_{\text{cycl. } f,g,h} (-1)^{(\varepsilon_f+1)(\varepsilon_h+1)}(f,(g,h)) \;=\; 0 \tag{3.1}$$

for the antibracket $(\cdot,\cdot)$ holds. We shall always assume this from now on. Equivalently, the two-form E_{AB} is closed,

$$dE \;=\; 0 \;. \tag{3.2}$$

In terms of the matrices E^{AB} and E_{AB}, the Jacobi identity (3.1) and the closeness condition (3.2) read

$$\sum_{\text{cycl. } A,B,C} (-1)^{(\varepsilon_A+1)(\varepsilon_C+1)} E^{AD}(\overrightarrow{\partial^l_D} E^{BC}) \;=\; 0 \;, \tag{3.3}$$

$$\sum_{\text{cycl. } A,B,C} (-1)^{\varepsilon_A\varepsilon_C}(\overrightarrow{\partial^l_A} E_{BC}) \;=\; 0 \;, \tag{3.4}$$

respectively. By definition, a non-degenerate tensor E_{AB} with Grassmann-parity (2.10), skewsymmetry (2.8), and closeness relation (3.4) is called an *antisymplectic* structure.

Granted the Jacobi identity (3.1), the second-order terms of Δ^2 can be written on the form

$$\frac{1}{4}\mathcal{R}^{AB}\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A} \;, \tag{3.5}$$

where $\mathcal{R}^{AB}$ with upper indices is a shorthand for

$$\mathcal{R}^{AD} \;\equiv\; E^{AB}\mathcal{R}_{BC}E^{CD}(-1)^{\varepsilon_C} \;, \tag{3.6}$$

and $\mathcal{R}_{AB}$ with lower indices is the curvature tensor for the line bundle connection F_A:

$$\mathcal{R}_{AB} \;\equiv\; [\overrightarrow{\partial^l_A}+F_A, \overrightarrow{\partial^l_B}+F_B] \;=\; (\overrightarrow{\partial^l_A} F_B) - (-1)^{\varepsilon_A\varepsilon_B}(A \leftrightarrow B) \;. \tag{3.7}$$

Remarkably, the two tensors $\mathcal{R}_{AB}$ and $\mathcal{R}^{AB}$ carry opposite symmetry:

$$\mathcal{R}_{AB} \;=\; -(-1)^{\varepsilon_A\varepsilon_B}\mathcal{R}_{BA} \;, \tag{3.8}$$

$$\mathcal{R}^{AB} \;=\; (-1)^{\varepsilon_A\varepsilon_B}\mathcal{R}^{BA} \;. \tag{3.9}$$

It follows that in the non-degenerate case, the second-order terms of Δ^2 vanish if and only if the line bundle connection F_A has vanishing curvature

$$\mathcal{R}_{AB} \;=\; 0 \;. \tag{3.10}$$

The zero curvature condition (3.10) is an integrability condition for the local existence of a density ρ,

$$F_A \;=\; (\overrightarrow{\partial^l_A} \ln \rho) \;. \qquad (3.11)$$

Under the $F \leftrightarrow \rho$ identification (3.11) the Δ_F operator (2.2) just becomes the ordinary odd Laplacian Δ_ρ from eq. (1.1),

$$\Delta_F \;=\; \Delta_\rho \;. \qquad (3.12)$$

Conventionally the field-antifield formalism requires the $F \leftrightarrow \rho$ identification (3.11) to hold globally. Nevertheless, we shall present many of the constructions below using F rather than ρ, to be as general as possible.

There exists a descriptive characterization: Granted the Jacobi identity (3.1), the second-order terms of Δ^2 vanish if and only if there is a Leibniz rule for the interplay of the so-called "one-bracket" $\Phi^1_\Delta \equiv \Delta - (\Delta 1) = \Delta_F$ and the "two-bracket" $(\cdot,\cdot)$

$$\Delta_F(f,g) \;=\; (\Delta_F f,g) - (-1)^{\varepsilon_f}(f,\Delta_F g) \;. \qquad (3.13)$$

See Ref. [16, 17] for more details.

4 A Non-Zero F-Curvature?

In eq. (3.10) of the previous Section 3 we learned that the nilpotency condition (1.3) completely kills the line bundle curvature $\mathcal{R}$. Nevertheless, several constructions continue to be well-defined for non-zero $\mathcal{R}$. For instance, both the important scalars ν_F and R fall into this category, cf. eqs. (7.1) and (11.7) below. Another example, which turns out to be related to our discussion, is the Grassmann-odd 2-cocycle of Khudaverdian and Voronov [8, 11, 18]. It is defined using two (possibly non-flat) line bundle connections $F^{(1)}$ and $F^{(2)}$ as follows:

$$\nu(F^{(1)};F^{(2)},E) \;\equiv\; \frac{1}{4}{\rm div}_{F^{(12)}} X_{(12)} \;\equiv\; \frac{(-1)^{\varepsilon_A}}{4}(\overrightarrow{\partial^l_A} + \frac{F^{(1)}+F^{(2)}}{2})(E^{AB}(F^{(1)}_B - F^{(2)}_B)) \;, \qquad (4.1)$$

where the divergence "div" is defined in eq. (10.13),

$$F^{(12)} \;\equiv\; \frac{F^{(1)}+F^{(2)}}{2} \;, \qquad (4.2)$$

and

$$X^A_{(12)} \;\equiv\; E^{AB}(F^{(1)}_B - F^{(2)}_B) \;. \qquad (4.3)$$

It is clear from Definition (4.1) that $\nu(F^{(1)};F^{(2)},E)$ behaves as a scalar under general coordinate transformations. This is because the average $F^{(12)}$ is again a line bundle connection, and $X_{(12)}$ is a vector field since the difference $F^{(1)}_B - F^{(2)}_B$ is a co-vector (=one-form), cf. eq. (2.3). That $\nu(F^{(1)};F^{(2)},E)$ is a 2-cocycle

$$\nu(F^{(1)};F^{(2)},E) + \nu(F^{(2)};F^{(3)},E) + \nu(F^{(3)};F^{(1)},E) \;=\; 0 \qquad (4.4)$$

follows easily by rewriting Definition (4.1) as

$$\nu(F^{(1)};F^{(2)},E) \;=\; \nu^{(0)}_{F^{(1)}} - \nu^{(0)}_{F^{(2)}} \;, \qquad (4.5)$$

where $\nu^{(0)}_F$ generalizes eq. (1.7):

$$\nu^{(0)}_F \;\equiv\; \frac{(-1)^{\varepsilon_A}}{4}(\overrightarrow{\partial^l_A} + \frac{F_A}{2})(E^{AB}F_B) \;. \qquad (4.6)$$

Note that Definitions (4.1) and (4.6) continue to make sense for non-flat F's. We should stress that $\nu_F^{(0)}$ itself is *not* a scalar, but we shall soon see that it can be replaced in eq. (4.5) by a scalar ν_F, cf. eq. (7.1) below. In other words, $\nu(F^{(1)};F^{(2)},E)$ is a 2-coboundary.

The F-curvature $\mathcal{R}_{AB}$ is also an interesting geometric object in its own right. It can be identified with a Ricci two-form of a tangent bundle connection ∇, cf. eq. (11.4) in Section 11 below. The Ricci two-form

$$\mathcal{R} \;=\; \frac{1}{2} d\Gamma^A \mathcal{R}_{AB} \wedge d\Gamma^B (-1)^{\varepsilon_B} \tag{4.7}$$

is closed

$$d\mathcal{R} \;=\; 0 \;, \tag{4.8}$$

due to the Bianchi identity

$$\sum_{\text{cycl. } A,B,C} (-1)^{\varepsilon_A \varepsilon_C} (\overrightarrow{\partial^l_A} \mathcal{R}_{BC}) \;=\; 0 \;, \tag{4.9}$$

so the two-form (4.7) defines a cohomology class.

5 Breaking the Nilpotency

Due to the above mathematical reasons we shall digress in this Section 5 to contemplate how a non-zero F-curvature could arise in antisymplectic geometry, although we should stress that it remains unclear if it is useful in physics. Nevertheless, the strategy that we shall adapt here is to append a general Grassmann-even (possibly degenerate) second-order operator source $\frac{1}{2}\Delta_{\mathcal{R}}$ to the right-hand side of the nilpotency condition (1.3):

$$\Delta^2 \;=\; \frac{1}{2}\Delta_{\mathcal{R}} \;. \tag{5.1}$$

A covariant and general way of realizing the second-order $\Delta_{\mathcal{R}}$ operator is to write

$$\Delta_{\mathcal{R}} \;\equiv\; \Delta_{F,\mathcal{R}} + V_{\mathcal{R}} + n_{\mathcal{R}} \;, \tag{5.2}$$

where

$$\Delta_{F,\mathcal{R}} \;\equiv\; \frac{(-1)^{\varepsilon_A}}{2} (\overrightarrow{\partial^l_A} + F_A) \mathcal{R}^{AB} \overrightarrow{\partial^l_B} \tag{5.3}$$

is an Grassmann-even Laplacian based on F_A and $\mathcal{R}^{AB}$. We have included a Grassmann-even vector field

$$V_{\mathcal{R}} \;\equiv\; V^A_{\mathcal{R}} \overrightarrow{\partial^l_A} \tag{5.4}$$

and a scalar function $n_{\mathcal{R}}$ to give a systematic treatment. Note that the vector field $V_{\mathcal{R}}$ is the difference of the subleading connection terms inside $\Delta_{\mathcal{R}}$ and $\Delta_{F,\mathcal{R}}$. We shall show below that the $n_{\mathcal{R}}$ term is completely determined by consistency, while $V_{\mathcal{R}}$ in principle can be any locally Hamiltonian vector field subjected to the following restriction: Both $V^A_{\mathcal{R}}$ and $n_{\mathcal{R}}$ should be proportional to the $\mathcal{R}$-source (or its derivatives) in order to restore nilpotency (1.3) in the limit $\mathcal{R} \to 0$.

The new condition (5.1) still imposes the Jacobi identity (3.1) for the antibracket $(\cdot,\cdot)$ at the third order, since the modification is just of second order. (We mention, for later, that the Jacobi identity alone guarantees the existence of a nilpotent Δ_E operator and its quantization scheme, cf. Sections 8-9, regardless of how the nilpotency (5.1) of Δ is broken at lower orders.) The second-order terms in eq. (5.1) implies that the F-curvature $\mathcal{R}^{AB}$ defined in eq. (3.7) should be identified with the principal symbol $\mathcal{R}^{AB}$ appearing inside the $\Delta_{F,\mathcal{R}}$ operator (5.3), thereby justifying the notation. Note that the Leibniz rule (3.13) is *no* longer valid. To see this, it is useful to define an even $\mathcal{R}$-bracket [19]

$$\begin{aligned} (f,g)_{\mathcal{R}} &\equiv [[\overrightarrow{\Delta}_{\mathcal{R}},f],g]1 \equiv \Delta_{\mathcal{R}}(fg)-(\Delta_{\mathcal{R}}f)g-f(\Delta_{\mathcal{R}}g)+fg(\Delta_{\mathcal{R}}1) \\ &= (f\overleftarrow{\partial}^{r}_{A})\mathcal{R}^{AB}(\overrightarrow{\partial}^{l}_{B}g) = (-1)^{\varepsilon_f\varepsilon_g}(g,f)_{\mathcal{R}} . \end{aligned} \tag{5.5}$$

It turns out that the $\mathcal{R}$-bracket $(\cdot,\cdot)_{\mathcal{R}}$ measures the failure of the Leibniz rule:

$$\frac{1}{2}(f,g)_{\mathcal{R}} = (-1)^{\varepsilon_f}\Delta_F(f,g)-(-1)^{\varepsilon_f}(\Delta_F f,g)+(f,\Delta_F g) . \tag{5.6}$$

Note that this $\mathcal{R}$-bracket $(\cdot,\cdot)_{\mathcal{R}}$ does *not* satisfy a Jacobi identity. (In fact, we shall see that the closeness relation (4.8) for $\mathcal{R}_{AB}$ will instead lead to a compatibility relation (5.8) below.) Since $\Delta_F^2-\frac{1}{2}\Delta_{F,\mathcal{R}}$ is a first-order operator, cf. eqs. (2.1) and (5.1), the commutator

$$\frac{1}{2}[\Delta_{F,\mathcal{R}},\Delta_F] = [\Delta_F,\Delta_F^2-\frac{1}{2}\Delta_{F,\mathcal{R}}] \tag{5.7}$$

becomes a second-order operator at most. (We shall improve this estimate in Lemma 5.1 below.) This fact already implies that the two brackets $(\cdot,\cdot)$ and $(\cdot,\cdot)_{\mathcal{R}}$ are compatible in the sense that

$$\sum_{\text{cycl. } f,g,h}(-1)^{\varepsilon_f(\varepsilon_h+1)}((f,g),h)_{\mathcal{R}} = \sum_{\text{cycl. } f,g,h}(-1)^{\varepsilon_f(\varepsilon_h+1)+\varepsilon_g}((f,g)_{\mathcal{R}},h) . \tag{5.8}$$

Phrased differently, one may define a one-parameter family of antisymplectic two-forms

$$E(\theta) \equiv E+\theta\mathcal{R} \equiv E+\mathcal{R}\theta = \frac{1}{2}d\Gamma^A E_{AB}(\theta)\wedge d\Gamma^B , \qquad dE(\theta) = 0 , \tag{5.9}$$

which depends on a Grassmann-odd parameter θ. In components it reads

$$E_{AB}(\theta) = E_{AB}+\mathcal{R}_{AB}\theta , \tag{5.10}$$

$$E^{AB}(\theta) = E^{AB}+(-1)^{\varepsilon_A}\theta\mathcal{R}^{AB} = E^{AB}+\mathcal{R}^{AB}\theta(-1)^{\varepsilon_B} . \tag{5.11}$$

There exists locally an antisymplectic one-form potential

$$\begin{aligned} U(\theta) &\equiv U_A(\theta)d\Gamma^A , & U_A(\theta) &\equiv U_A+F_A\theta , \\ dU(\theta) &= E(\theta) , & \overrightarrow{\partial}^{l}_{A}U_B(\theta)-(-1)^{\varepsilon_A\varepsilon_B}(A\leftrightarrow B) &= E_{AB}(\theta) . \end{aligned} \tag{5.12}$$

We will now improve the estimate from eq. (5.7):

Lemma 5.1 *The commutator $[\Delta_F,\Delta_{F,\mathcal{R}}]$ is always a first-order operator at most.*

PROOF OF LEMMA 5.1: Note that the commutator $[\Delta_F,\Delta_{F,\mathcal{R}}]$ appears inside the square

$$(\Delta_F(\theta))^2 = \Delta_F^2+\theta[\Delta_{F,\mathcal{R}},\Delta_F] = \Delta_F^2+[\Delta_F,\Delta_{F,\mathcal{R}}]\theta \tag{5.13}$$

of the Grassmann-odd second-order operator

$$\Delta_F(\theta) \equiv \Delta_F+\theta\Delta_{F,\mathcal{R}} \equiv \Delta_F+\Delta_{F,\mathcal{R}}\theta = \frac{(-1)^{\varepsilon_A}}{2}(\overrightarrow{\partial}^{l}_{A}+F_A)E^{AB}(\theta)\overrightarrow{\partial}^{l}_{B} . \tag{5.14}$$

One knows from the general discussion in the previous Section 3 that the third-order terms in the square (5.13) vanish because $E^{AB}(\theta)$ satisfies the Jacobi identity (3.3). Moreover, the second-order terms in the square (5.13) are of the form

$$\frac{(-1)^{\varepsilon_C}}{4} E^{AB}(\theta)\ \mathcal{R}_{BC}\ E^{CD}(\theta)\ \overrightarrow{\partial^l_D}\overrightarrow{\partial^l_A} \ = \ \frac{1}{4}\mathcal{R}^{AB}\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A}\ , \tag{5.15}$$

cf. eqs. (3.5) and (3.6). It is easy to see that the two θ-dependent terms inside the left-hand side of eq. (5.15) cancel against each other. In fact, each of the two terms vanish separately due to skewsymmetry:

$$(-1)^{\varepsilon_C+\varepsilon_F} E^{AB}\mathcal{R}_{BC}E^{CD}\mathcal{R}_{DF}E^{FG} \ = \ \mathcal{R}^{AC}E_{CD}\mathcal{R}^{DG} \ = \ (-1)^{(\varepsilon_A+1)(\varepsilon_G+1)}(A\leftrightarrow G)\ . \tag{5.16}$$

Therefore, the θ-dependent part of the square (5.13) must be of first order at most.

□

(One may also give a proof of Lemma 5.1 based on Lemma B.1 in Appendix B.) Lemma 5.1 implies (for instance via the technology of Ref. [16]) that

$$\begin{aligned}\Delta_{F,\mathcal{R}}(f,g)-(\Delta_{F,\mathcal{R}}f,g)-(f,\Delta_{F,\mathcal{R}}g) \ &= \ (-1)^{\varepsilon_f}\Delta_F(f,g)_{\mathcal{R}}-(-1)^{\varepsilon_f}(\Delta_F f,g)_{\mathcal{R}} \\ &\quad -(f,\Delta_F g)_{\mathcal{R}}\ , \end{aligned}\tag{5.17}$$

$$(\Delta_F^2-\frac{1}{2}\Delta_{F,\mathcal{R}})(f,g) \ = \ ((\Delta_F^2-\frac{1}{2}\Delta_{F,\mathcal{R}})f,g)+(f,(\Delta_F^2-\frac{1}{2}\Delta_{F,\mathcal{R}})g)\ . \tag{5.18}$$

More generally, there exists a superformulation

$$\Delta(\theta) \ \equiv \ \Delta+\theta\Delta_{\mathcal{R}} \ \equiv \ \Delta+\Delta_{\mathcal{R}}\theta \ = \ \frac{(-1)^{\varepsilon_A}}{2}(\overrightarrow{\partial^l_A}+F_A(\theta))E^{AB}(\theta)\overrightarrow{\partial^l_B}+\nu(\theta)\ \ , \tag{5.19}$$

where

$$\nu(\theta) \ \equiv \ \nu+\theta n_{\mathcal{R}} \ \equiv \ \nu+n_{\mathcal{R}}\theta\ , \tag{5.20}$$

and

$$F_A(\theta) \ \equiv \ F_A+2E_{AB}V^B_{\mathcal{R}}\theta \ \equiv \ F_A-2V^B_{\mathcal{R}}E_{BA}\theta\ . \tag{5.21}$$

The nilpotency condition

$$\left(\Delta(\theta)-\frac{1}{2}\frac{\partial}{\partial\theta}\right)^2 \ = \ 0 \tag{5.22}$$

precisely encodes the deformed condition (5.1) and its consistency relation

$$\begin{aligned}0 \ &= \ [\Delta,[\Delta,\Delta]] \ = \ [\Delta,\Delta_{\mathcal{R}}] \ = \ [\Delta_F+\nu,\Delta_{F,\mathcal{R}}+V_{\mathcal{R}}+n_{\mathcal{R}}] \\ &= \ [\Delta_F,\Delta_{F,\mathcal{R}}]+[\Delta_F,V_{\mathcal{R}}]+[\Delta_F,n_{\mathcal{R}}]-[\Delta_{F,\mathcal{R}}+V_{\mathcal{R}},\nu]\ .\end{aligned}\tag{5.23}$$

Note in the last line of eq. (5.23) that the first term $[\Delta_F,\Delta_{F,\mathcal{R}}]$ and the two last terms $[\Delta_F,n_{\mathcal{R}}]$ and $[\Delta_{F,\mathcal{R}}+V_{\mathcal{R}},\nu]$ are all of first order. Hence, the second term $[\Delta_F,V_{\mathcal{R}}]$ must be of first order as well. This in turn implies that $V_{\mathcal{R}}$ should be a generating vector field for an anticanonical transformation:

$$V_{\mathcal{R}}(f,g) \ = \ (V_{\mathcal{R}}(f),g)+(f,V_{\mathcal{R}}(g))\ . \tag{5.24}$$

Since the antibracket is non-degenerated, it follows that $V_{\mathcal{R}}$ must be a locally Hamiltonian vector field, which we, for simplicity, will assume is a globally Hamiltonian vector field

$$V_{\mathcal{R}} \ = \ -2(\nu_{\mathcal{R}},\cdot)\ , \tag{5.25}$$

with some Fermionic globally defined Hamiltonian $\nu_{\mathcal{R}}$. The factor "-2" in eq. (5.25) is chosen for later convenience. The Hamiltonian $\nu_{\mathcal{R}}$ in eq. (5.25) should be considered as an additional geometric input, which labels the different ways (5.1) of breaking the nilpotency of Δ. It is a priori only defined in eq. (5.25) up to an odd constant. We fix this constant by requiring that

$$\nu_{\mathcal{R}} \;\to\; 0 \qquad \text{for} \qquad \mathcal{R} \;\to\; 0 \;. \tag{5.26}$$

Altogether, the Hamiltonian $\nu_{\mathcal{R}}$ does not contribute to the curvature

$$\overrightarrow{\partial^l_A} F_B(\theta) - (-1)^{\varepsilon_A \varepsilon_B}(A \leftrightarrow B) \;=\; \mathcal{R}_{AB} \tag{5.27}$$

of the line bundle connection

$$F_A(\theta) \;=\; F_A + 4(\overrightarrow{\partial^l_A}\nu_{\mathcal{R}})\theta \;. \tag{5.28}$$

Now let us continue the investigation of the deformed condition (5.1). The first-order terms of eq. (5.1) cancel if and only if

$$\Delta_F^2 - \frac{1}{2}\Delta_{F,\mathcal{R}} \;=\; (\nu{-}\nu_{\mathcal{R}}, \cdot) \;. \tag{5.29}$$

This is a differential equation for the function $\nu{=}\nu(\Gamma)$, or, equivalently, for the difference $\nu{-}\nu_{\mathcal{R}}$. It now becomes clear that the $\nu_{\mathcal{R}}$ function provides an auxiliary curvature background for the ν function. Since we assume that $\nu_{\mathcal{R}}$ is given, we will now focus on the difference $\nu{-}\nu_{\mathcal{R}}$ rather than on ν itself. The Frobenius integrability condition for eq. (5.29) comes from the fact that the operator $\Delta_F^2 - \frac{1}{2}\Delta_{F,\mathcal{R}}$ differentiates the antibracket, cf. eq. (5.18). This implies that the difference $\nu{-}\nu_{\mathcal{R}}$ can be written as a contour integral

$$(\nu{-}\nu_{\mathcal{R}})(\Gamma) \;=\; (\nu{-}\nu_{\mathcal{R}})(\Gamma_0) + \int_{\Gamma_0}^{\Gamma} ((\Delta_F^2{-}\frac{1}{2}\Delta_{F,\mathcal{R}})\Gamma^A)E_{AB}\Big|_{\Gamma\to\Gamma'} d\Gamma'^B \tag{5.30}$$

that is independent of the curve (aside from the two endpoints). It only depends on E, F, and an odd integration constant $(\nu{-}\nu_{\mathcal{R}})(\Gamma_0)$. In particular, we conclude that the difference $\nu{-}\nu_{\mathcal{R}}$ does not introduce any new geometric structures. The first-order commutator from Lemma 5.1 can now be expressed in terms of the difference $\nu{-}\nu_{\mathcal{R}}$ as follows:

$$\begin{aligned}\frac{1}{2}[\Delta_{F,\mathcal{R}}, \Delta_F] \;&=\; [\Delta_F, \Delta_F^2 - \frac{1}{2}\Delta_{F,\mathcal{R}}] \;=\; \Delta_F(\nu{-}\nu_{\mathcal{R}}, \cdot) - (\nu{-}\nu_{\mathcal{R}}, \Delta_F(\cdot)) \\ &=\; (\Delta_F(\nu{-}\nu_{\mathcal{R}}), \cdot) - \frac{1}{2}(\nu{-}\nu_{\mathcal{R}}, \cdot)_{\mathcal{R}} \;.\end{aligned} \tag{5.31}$$

Here, eq. (5.29) is used in the second equality and the deformed Leibniz rule (5.6) is used in the third (=last) equality.

Finally, the zeroth-order terms of eq. (5.1) cancel if and only if

$$n_{\mathcal{R}} \;=\; 2(\Delta_F \nu) \;, \tag{5.32}$$

so this fixes completely the Grassmann-even function $n_{\mathcal{R}}$. One can show that if the Hamiltonian vector field $V_{\mathcal{R}}^A$ vanishes in the flat limit $\mathcal{R} \to 0$, then the $n_{\mathcal{R}}$ function, defined via eq. (5.32), automatically does the same, cf. eq. (6.2) below. The nilpotency-breaking operator $\Delta_{\mathcal{R}}$ will therefore vanish for $\mathcal{R} \to 0$, as it should.

6 Nilpotency Conditions: Part II

After this digression into non-zero $\mathcal{R}$ curvature, let us now return to the nilpotent (and ordinary physical) situation $\Delta^2 = 0$, where $\mathcal{R}$, $V_{\mathcal{R}}^A$ and $n_{\mathcal{R}}$ are all zero. Not much changes for the condition

(5.29) for the first-order terms other that one should remove the $\nu_{\mathcal{R}}$ function and the $\Delta_{F,\mathcal{R}}$ operator from the Frobenius integrability condition (5.18), the differential eq. (5.29), and the contour integral (5.30). (Of course, now the Frobenius integrability condition is just an easy consequence of the Leibniz rule (3.13) applied twice.) The condition (5.32) for the zeroth-order terms becomes

$$(\Delta_F \nu) \;=\; 0 \; . \tag{6.1}$$

Equation (6.1) is not an independent condition but it follows instead automatically from the previous requirements. PROOF:

$$\begin{aligned}
-(\Delta_F \nu) &= \frac{(-1)^{\varepsilon_A}}{2}(\overrightarrow{\partial^l_A}+F_A)(\nu,\Gamma^A) \;=\; \frac{(-1)^{\varepsilon_A}}{2}(\overrightarrow{\partial^l_A}+F_A)\Delta_F^2\Gamma^A \\
&= \frac{(-1)^{\varepsilon_A+\varepsilon_B}}{4}(\overrightarrow{\partial^l_A}+F_A)(\overrightarrow{\partial^l_B}+F_B)(\Gamma^B,\Delta_F\Gamma^A) \\
&= -\frac{(-1)^{\varepsilon_A}}{8}(\overrightarrow{\partial^l_A}+F_A)(\overrightarrow{\partial^l_B}+F_B)\Delta_F(\Gamma^B,\Gamma^A) \\
&= \frac{(-1)^{\varepsilon_A\varepsilon_C}}{16}(\overrightarrow{\partial^l_A}+F_A)(\overrightarrow{\partial^l_B}+F_B)(\overrightarrow{\partial^l_C}+F_C)(\Gamma^C,(\Gamma^B,\Gamma^A))(-1)^{(\varepsilon_A+1)(\varepsilon_C+1)} \;=\; 0 \; .
\end{aligned} \tag{6.2}$$

Here, the ν eq. (5.29) is used in the second equality, the Leibniz rule (3.13) in the fourth equality, the Jacobi identity (3.1) in the sixth (=last) equality, and the zero curvature condition (3.10) in the second, fourth and sixth equality.

□

7 An Explicit Solution ν_F

Remarkably, the integral (5.30) can be performed.

Proposition 7.1 *The odd quantity*

$$\nu_F \;\equiv\; \nu_F^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24} \tag{7.1}$$

is a solution to the differential eq. (5.29) for the difference $\nu-\nu_{\mathcal{R}}$, even if the line bundle connection F is not flat.

Here, $\nu_F^{(0)}$, $\nu^{(1)}$ and $\nu^{(2)}$ are given by eqs. (4.6), (1.8) and (1.9), respectively. Proposition 7.1 is proven in Appendix B by repeated use of the Jacobi identity (3.3) and the closeness relation (3.4). Notice that under the $F \leftrightarrow \rho$ identification (3.11), the F-dependent Definitions (4.6) and (7.1) reduce to their ρ counterparts (1.6) and (1.7),

$$\nu_F \;=\; \nu_\rho \; , \qquad \nu_F^{(0)} \;=\; \nu_\rho^{(0)} \; . \tag{7.2}$$

Notation: ν_F or ν_ρ with subscript "F" or "ρ" denotes one particular solution (7.1) or (1.6) to the difference $\nu-\nu_{\mathcal{R}}$ in eq. (5.29), respectively.

Proposition 7.2 *The ν_F quantity (7.1) is invariant under general coordinate transformations, i.e. it is a scalar, even if the line bundle connection F is not flat.*

PROOF OF PROPOSITION 7.2: Under an arbitrary infinitesimal coordinate transformation $\delta\Gamma^A = X^A$, one calculates [9]

$$\delta\nu_F^{(0)} \;=\; -\frac{1}{2}\Delta_1 \mathrm{div}_1 X \;, \tag{7.3}$$

$$\delta\nu^{(1)} \;=\; 4\Delta_1 \mathrm{div}_1 X + (-1)^{\varepsilon_A}(\overrightarrow{\partial^l_C} E^{AB})(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A} X^C) \;, \tag{7.4}$$

$$\delta\nu^{(2)} \;=\; 3(-1)^{\varepsilon_A}(\overrightarrow{\partial^l_C} E^{AB})(\overrightarrow{\partial^l_B}\overrightarrow{\partial^l_A} X^C) \;, \tag{7.5}$$

where Δ_1 and div_1 denote the expressions (1.1) and (10.14) for the odd Laplacian $\Delta_{\rho=1}$ and the divergence $\mathrm{div}_{\rho=1}$ with ρ replaced by 1. One easily sees that while the three constituents $\nu_F^{(0)}$, $\nu^{(1)}$ and $\nu^{(2)}$ separately have non-trivial transformation properties, the linear combination ν_F in eq. (7.1) is indeed a scalar. Proposition 7.2 also follows from the identification of ν_F as an odd scalar curvature, cf. eq. (11.8) below.

□

The difference $\nu-\nu_{\mathcal{R}}$ is only determined up to an odd integration constant because the defining relation (5.29) is a differential relation. The explicit solution ν_F in (7.1) provides us with an opportunity to fix this odd integration constant once and for all. Out of all the solutions to the difference $\nu-\nu_{\mathcal{R}}$, we choose the ν_F solution (7.1), *i.e.* we identify from now on

$$\nu \;\equiv\; \nu_F + \nu_{\mathcal{R}} \;. \tag{7.6}$$

We do this for two reasons. Firstly, any odd constants inside the ν_F expression (7.1) can only arise implicitly through E and F, which means that if E and F do not carry any odd constants, then the ν_F solution (7.1) will be free of odd constants as well. Similarly, the $\nu_{\mathcal{R}}$ part does not contain odd constants because of the boundary condition (5.26). Secondly, the expression ν_F is the only solution that has an interpretation as an odd scalar curvature, cf. eq. (11.8) below. This completes the reduction of a general second-order Δ operator to

$$\Delta \;=\; \Delta_F + \nu \;=\; \Delta_F + \nu_F + \nu_{\mathcal{R}} \quad\longrightarrow\quad \Delta_\rho + \nu_\rho \qquad \text{for} \qquad \mathcal{R} \;\to\; 0 \;. \tag{7.7}$$

8 The Δ_E operator

Let us briefly outline the connection to Khudaverdian's Δ_E operator [10, 11, 12, 13], which takes semidensities to semidensities. The Δ_E operator was defined in Ref. [9] as

$$\Delta_E \;\equiv\; \Delta_1 + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24} \;, \tag{8.1}$$

where Δ_1 denotes the expression (1.1) for the odd Laplacian $\Delta_{\rho=1}$ with ρ replaced by 1. Some of the strengths of Definition (8.1) are that it works in any coordinate system and that it is manifestly independent of ρ or F. However, it is a rather lengthy calculation to demonstrate in a ρ-less or F-less environment that Δ_E has the pertinent transformation property under general coordinate transformations, and that it is nilpotent

$$\Delta_E^2 \;=\; 0 \;, \tag{8.2}$$

cf. Ref. [9]. Once we are given a density ρ, the situation simplifies considerably. Then, the Δ_E operator becomes just the operator $\Delta \equiv \Delta_\rho + \nu_\rho$ conjugated with the square root of ρ:

$$\Delta_E \;=\; \sqrt{\rho}\,\Delta\frac{1}{\sqrt{\rho}} \;. \tag{8.3}$$

PROOF OF EQ. (8.3): Let σ denote an arbitrary semidensity. Then, it follows from the explicit ν_ρ formula (1.6) that

$$\begin{aligned}(\Delta_E\sigma) &= (\Delta_1\sigma)+(\frac{\nu^{(1)}}{8}-\frac{\nu^{(2)}}{24})\sigma \;=\; (\Delta_1\sigma)-(\Delta_1\sqrt{\rho})\frac{\sigma}{\sqrt{\rho}}+\nu_\rho\sigma \\ &= \sqrt{\rho}(\Delta_1\frac{\sigma}{\sqrt{\rho}})+(\sqrt{\rho},\frac{\sigma}{\sqrt{\rho}})+\nu_\rho\sigma \;=\; \sqrt{\rho}(\Delta_\rho\frac{\sigma}{\sqrt{\rho}})+\nu_\rho\sigma \;=\; \sqrt{\rho}(\Delta\frac{\sigma}{\sqrt{\rho}}) \;. \end{aligned} \tag{8.4}$$

It is remarkable that the $\sqrt{\rho}$-conjugated Δ operator $\sqrt{\rho}\Delta\frac{1}{\sqrt{\rho}}$ does not depend on ρ at all! On the other hand, it is obvious that the operator $\sqrt{\rho}\Delta\frac{1}{\sqrt{\rho}}$ is nilpotent and that it satisfies the required transformation law under general coordinate transformations, *i.e.* that it takes semidensities to semidensities. This is because the Δ operator itself is a nilpotent operator and Δ takes scalar functions to scalar functions. Let us also mention that

$$\nu_\rho \;=\; (\Delta 1) \;=\; \frac{1}{\sqrt{\rho}}(\Delta_E\sqrt{\rho}) \;. \tag{8.5}$$

The right-hand side of eq. (8.5) served as a definition of the odd scalar ν_ρ in Ref. [9].

More generally, the operators Δ_E and $\Delta\equiv\Delta_F+\nu_F+\nu_{\mathcal{R}}$ are linked via

$$\Delta_E \;=\; \Delta-\frac{(-1)^{\varepsilon_A}}{2}F_A(\Gamma^A,\cdot)-\nu_F^{(0)}-\nu_{\mathcal{R}} \;. \tag{8.6}$$

Equation (8.6) may be viewed as a generalization of eq. (8.3) to non-flat F's, or, equivalently, to non-nilpotent Δ's, cf. eq. (3.10). It might be worth emphasizing that Δ_E is nilpotent even in this situation, since Δ_E only depends on E.

9 F-Independent Formalism

There exists [9, 15] a manifestly F-independent quantization scheme based on the Δ_E operator. Since we will demand that the quantization is covariant with respect to the antisymplectic phase space, it will be necessary to use first-level formalism or one of its higher-level generalizations [2, 20]. See Ref. [8] for a review of the multi-level formalism. It turns out to be most efficient to use the second-level formalism in order not to deal directly with weak quantum master equations [21]. Let Γ^A denote all the zeroth- and first-level fields and antifields, and let λ^α denote the second-level Lagrange multipliers for the first-level gauge-fixing constraints. Assume also that there is no dependence on the corresponding second-level antifields λ^*_α. The second-level partition function

$$\mathcal{Z} \;=\; \int[d\Gamma][d\lambda]\; e^{\frac{i}{\hbar}(W_E+X_E)} \tag{9.1}$$

contains two Boltzmann semidensities: a gauge-generating semidensity $e^{\frac{i}{\hbar}W_E}$ and a gauge-fixing semidensity $e^{\frac{i}{\hbar}X_E}$, where W_E and X_E denote the corresponding quantum actions. The two Boltzmann semidensities are both required to satisfy strong quantum master equations

$$\Delta_E e^{\frac{i}{\hbar}W_E} \;=\; 0 \;, \qquad\qquad \Delta_E e^{\frac{i}{\hbar}X_E} \;=\; 0 \;, \tag{9.2}$$

or equivalently,

$$\frac{1}{2}(W_E,W_E) \;=\; i\hbar\Delta_1 W_E+\hbar^2\Delta_E 1 \;, \qquad\qquad \frac{1}{2}(X_E,X_E) \;=\; i\hbar\Delta_1 X_E+\hbar^2\Delta_E 1 \;, \tag{9.3}$$

where

$$\Delta_E 1 \;=\; \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24} \; . \tag{9.4}$$

The caveat is that the quantum actions W_E and X_E are *not* scalars. They obey non-trivial transformation laws under general coordinate transformations, since they are logarithms of semidensities. It is shown in Appendix A that the partition function (9.1) is independent of the gauge choice X_E.

If we are given a density ρ, we may introduce a nilpotent Δ operator (8.3) and Boltzmann scalars $e^{\frac{i}{\hbar}W}$ and $e^{\frac{i}{\hbar}X}$ by dressing appropriately with square roots of ρ:

$$\sqrt{\rho}\Delta \;=\; \Delta_E\sqrt{\rho} \; , \qquad e^{\frac{i}{\hbar}W_E} \;=\; \sqrt{\rho}e^{\frac{i}{\hbar}W} \; , \qquad e^{\frac{i}{\hbar}X_E} \;=\; \sqrt{\rho}e^{\frac{i}{\hbar}X} \; . \tag{9.5}$$

Then $\Delta = \Delta_\rho + \nu_\rho$ and the two scalar actions W and X will satisfy the strong quantum master eq. (1.4) from the Introduction, which in non-exponential form reads

$$\frac{1}{2}(W,W) \;=\; i\hbar\Delta_\rho W + \hbar^2\nu_\rho \; , \qquad \frac{1}{2}(X,X) \;=\; i\hbar\Delta_\rho X + \hbar^2\nu_\rho \; . \tag{9.6}$$

The partition function (9.1) then reduces to the familiar W-X form:

$$\mathcal{Z} \;=\; \int \rho[d\Gamma][d\lambda] \; e^{\frac{i}{\hbar}(W+X)} \; . \tag{9.7}$$

Conversely, since the partition function (9.7) via the above identifications (9.5) can be written in the manifestly ρ-independent form (9.1), one may state that in this sense the partition function (9.7) does not depend on ρ. The point is that the well-known ambiguity in the choice of measure that exists in the field-antifield formalism has been fully transcribed into an ambiguity in the choice of the Boltzmann semidensity $e^{\frac{i}{\hbar}W_E}$. Put differently, if one splits the Boltzmann semidensity $e^{\frac{i}{\hbar}W_E}$ into a Boltzmann scalar $e^{\frac{i}{\hbar}W}$ and a density ρ as done in eq. (9.5), the measure ambiguity sits inside the scalar $e^{\frac{i}{\hbar}W}$, not in ρ, as ρ actually drops out of $\mathcal{Z}$.

More generally, imagine that we are given a non-nilpotent operator $\Delta \equiv \Delta_F + \nu_F + \nu_\mathcal{R}$ with a non-flat line bundle connection F that satisfies the deformed nilpotency condition (5.1). We can still define the partition function in this situation via the above quantization scheme (9.1) based on the nilpotent Δ_E operator. Such an approach will of course be manifestly F-independent by construction.

10 Connection

We now introduce a connection $\nabla : TM \times TM \to TM$. See Ref. [19, 22] for related discussions. The left covariant derivative $(\nabla_A X)^B$ of a left vector field X^A is defined as [19]

$$(\nabla_A X)^B \;\equiv\; (\overrightarrow{\partial^l_A} X^B) + (-1)^{\varepsilon_X(\varepsilon_B+\varepsilon_C)}\Gamma_A{}^B{}_C X^C \; , \qquad \varepsilon(X^A) \;=\; \varepsilon_X + \varepsilon_A \; , \tag{10.1}$$

The word "left" implies that X^A and $(\nabla_A X)^B$ transform with left derivatives

$$X'^B \;=\; X^A(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}\Gamma'^B) \; , \qquad (\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}\Gamma'^B)(\nabla'_B X)'^C \;=\; (\nabla_A X)^B(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^B}\Gamma'^C) \; , \tag{10.2}$$

under general coordinate transformations $\Gamma^A \to \Gamma'^B$. It is convenient to introduce a reordered Christoffel symbol

$$\Gamma^A{}_{BC} \;\equiv\; (-1)^{\varepsilon_A\varepsilon_B}\Gamma_B{}^A{}_C \tag{10.3}$$

to minimize the appearances of sign factors. On an antisymplectic manifold $(M;E)$, it is furthermore possible to define a Christoffel symbol with three lower indices

$$\Gamma_{ABC} \;\equiv\; E_{AD}\Gamma^{D}{}_{BC}(-1)^{\varepsilon_B} \; . \tag{10.4}$$

Let us also define

$$\gamma_{ABC} \;\equiv\; \Gamma_{ABC} - \frac{1}{3}(E_{A\{B}\overleftarrow{\partial^r_{C\}}}) \;\equiv\; \Gamma_{ABC} - \frac{1}{3}(E_{AB}\overleftarrow{\partial^r_C} + E_{AC}\overleftarrow{\partial^r_B}(-1)^{\varepsilon_B\varepsilon_C}) \; . \tag{10.5}$$

γ_{ABC} is *not* a tensor but it still has some useful properties, see eqs. (10.8) and (10.11) below. One can think of γ_{ABC} as parametrizing all the possible connections ∇ on $(M;E)$.

An *antisymplectic connection* $\Gamma_A{}^B{}_C$ satisfies by definition [19]

$$0 \;=\; (\nabla_A E)^{BC} \;\equiv\; (\overrightarrow{\partial^l_A}E^{BC}) + \Big(\Gamma_A{}^B{}_D E^{DC} - (-1)^{(\varepsilon_B+1)(\varepsilon_C+1)}(B \leftrightarrow C)\Big) \; , \tag{10.6}$$

so that the antisymplectic metric E^{AB} is covariantly preserved. In terms of the two-form E_{AB}, the antisymplectic condition reads

$$0 \;=\; (\nabla_A E)_{BC} \;\equiv\; (\overrightarrow{\partial^l_A}E_{BC}) - ((-1)^{\varepsilon_A\varepsilon_B}\Gamma_{BAC} - (-1)^{\varepsilon_B\varepsilon_C}(B \leftrightarrow C)) \; . \tag{10.7}$$

Written in terms of the γ_{ABC} symbol, the antisymplectic condition (10.7) becomes a purely algebraic equation, due to the closeness relation (3.4):

$$\gamma_{ABC} \;=\; (-1)^{\varepsilon_A\varepsilon_B+\varepsilon_B\varepsilon_C+\varepsilon_C\varepsilon_A}\gamma_{CBA} \; . \tag{10.8}$$

A *torsion-free* connection has the following symmetry in the lower indices:

$$\Gamma^A{}_{BC} \;=\; -(-1)^{(\varepsilon_B+1)(\varepsilon_C+1)}\Gamma^A{}_{CB} \; , \tag{10.9}$$

$$\Gamma_{ABC} \;=\; (-1)^{\varepsilon_B\varepsilon_C}\Gamma_{ACB} \; , \tag{10.10}$$

$$\gamma_{ABC} \;=\; (-1)^{\varepsilon_B\varepsilon_C}\gamma_{ACB} \; . \tag{10.11}$$

Note that $(-1)^{\varepsilon_A\varepsilon_B}\gamma_{BAC} = \gamma_{ABC} = (-1)^{\varepsilon_B\varepsilon_C}\gamma_{ACB}$ is totally symmetric for an antisymplectic torsion-free connection. (Similar results hold for even symplectic structures.)

A connection ∇ can be used to define a divergence of a Bosonic vector field X^A as

$$\mathrm{str}(\nabla X) \;\equiv\; (-1)^{\varepsilon_A}(\nabla_A X)^A \;=\; ((-1)^{\varepsilon_A}\overrightarrow{\partial^l_A} + \Gamma^B{}_{BA})X^A \; , \qquad \varepsilon_X \;=\; 0 \; . \tag{10.12}$$

On the other hand, the divergence is defined in terms of F or ρ as

$$\mathrm{div}_F X \;\equiv\; (-1)^{\varepsilon_A}(\overrightarrow{\partial^l_A} + F_A)X^A \; , \tag{10.13}$$

$$\mathrm{div}_\rho X \;\equiv\; \frac{(-1)^{\varepsilon_A}}{\rho}\overrightarrow{\partial^l_A}(\rho X^A) \; . \tag{10.14}$$

See Ref. [23] for a mathematical exposition of divergence operators on supermanifolds. Under the $F \leftrightarrow \rho$ identification (3.11), the last two Definitions (10.13) and (10.14) agree:

$$\mathrm{div}_F X \;=\; \mathrm{div}_\rho X \; . \tag{10.15}$$

In order to have a unique divergence operator (and hence a unique notion of volume), it is necessary to impose the following compatibility condition between F_A and the Christoffel symbols $\Gamma^A{}_{BC}$:

$$\Gamma^B{}_{BA} \;=\; (-1)^{\varepsilon_A} F_A \;. \tag{10.16}$$

We shall only consider antisymplectic, torsion-free, and F-compatible connections ∇, *i.e.* connections that satisfy the three conditions (10.6), (10.9) and (10.16). The first and third condition ensure the compatibility with E and F, respectively. The second (the torsion-free condition) guarantees compatibility with the closeness relation (3.4). It can be demonstrated that connections satisfying these three conditions exist locally for $N > 1$, where $2N$ denotes the number of antisymplectic variables Γ^A, $A = 1, \ldots, 2N$. (There are counterexamples for $N = 1$ where ∇ need not exist.) For connections satisfying the three conditions, the Δ_F operator can be written on a manifestly covariant form

$$\Delta_F \;=\; \frac{(-1)^{\varepsilon_A}}{2} \nabla_A E^{AB} \nabla_B \;=\; \frac{(-1)^{\varepsilon_B}}{2} E^{BA} \nabla_A \nabla_B \;. \tag{10.17}$$

11 Curvature

The Riemann curvature tensor $R_{AB}{}^C{}_D$ is defined as the commutator of the ∇ connection

$$([\nabla_A, \nabla_B]X)^C \;=\; R_{AB}{}^C{}_D X^D (-1)^{\varepsilon_X(\varepsilon_C+\varepsilon_D)} \;, \tag{11.1}$$

so that

$$R_{AB}{}^C{}_D \;=\; (\stackrel{\rightarrow}{\partial^l_A} \Gamma_B{}^C{}_D) + (-1)^{\varepsilon_B\varepsilon_C} \Gamma_A{}^C{}_E \Gamma^E{}_{BD} - (-1)^{\varepsilon_A\varepsilon_B}(A \leftrightarrow B) \;. \tag{11.2}$$

It is useful to define a reordered Riemann curvature tensor $R^A{}_{BCD}$ as

$$R^A{}_{BCD} \;\equiv\; (-1)^{\varepsilon_A(\varepsilon_B+\varepsilon_C)} R_{BC}{}^A{}_D \;=\; (-1)^{\varepsilon_A\varepsilon_B} (\stackrel{\rightarrow}{\partial^l_B} \Gamma^A{}_{CD}) + \Gamma^A{}_{BE}\Gamma^E{}_{CD} - (-1)^{\varepsilon_B\varepsilon_C}(B \leftrightarrow C) \;. \tag{11.3}$$

It is interesting to consider the various contractions of the Riemann curvature tensor. There are two possibilities. Firstly, there is the Ricci two-form

$$\mathcal{R}_{AB} \;\equiv\; R_{AB}{}^C{}_C (-1)^{\varepsilon_C} \;=\; (\stackrel{\rightarrow}{\partial^l_A} F_B) - (-1)^{\varepsilon_A\varepsilon_B}(A \leftrightarrow B) \;. \tag{11.4}$$

However, the Ricci two-form $\mathcal{R}_{AB}$ typically vanishes, cf. eq. (3.10), and even if it does not vanish, its antisymmetry (3.8) means that $\mathcal{R}_{AB}$ cannot successfully be contracted with the antisymplectic metric E^{AB} to yield a non-zero scalar curvature, cf. eq. (2.5). Secondly, there is the Ricci tensor

$$R_{AB} \;\equiv\; R^C{}_{CAB} \;=\; (-1)^{\varepsilon_C} (\stackrel{\rightarrow}{\partial^l_C} + F_C) \Gamma^C{}_{AB} - (\stackrel{\rightarrow}{\partial^l_A} F_B)(-1)^{\varepsilon_B} - \Gamma_A{}^C{}_D \Gamma^D{}_{CB} \;. \tag{11.5}$$

Note that when the torsion tensor and Ricci two-form vanish, the Ricci tensor R_{AB} possesses exactly the same $A \leftrightarrow B$ symmetry (2.5) as the antisymplectic metric E^{AB} with upper indices

$$R_{AB} \;=\; -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} R_{BA} \;. \tag{11.6}$$

The *odd scalar curvature* R is therefore defined in antisymplectic geometry as the contraction of the Ricci tensor R_{AB} and the antisymplectic metric E^{BA},

$$R \;\equiv\; R_{AB} E^{BA} \;=\; E^{AB} R_{BA} \;. \tag{11.7}$$

Proposition 11.1 *For an arbitrary, antisymplectic, torsion-free, and F-compatible connections ∇, the scalar curvature R does only depend on E and F through the odd scalar ν_F,*

$$R \;=\; -8\nu_F \;, \tag{11.8}$$

even if the line bundle connection F is not flat.

Proposition 11.1 is shown in Appendix C. In particular, one concludes that the scalar curvature R does not depend on the connection $\Gamma^{A}{}_{BC}$ used.

One can perform various consistency checks on the formalism. Here, let us just mention one. For an antisymplectic connection ∇, one has

$$0 \;=\; [\nabla_{A},\nabla_{B}]E^{CD} \;=\; R_{AB}{}^{C}{}_{F}E^{FD} - (-1)^{(\varepsilon_{C}+1)(\varepsilon_{D}+1)}(C\leftrightarrow D) \;, \tag{11.9}$$

or, equivalently,

$$R^{C}{}_{ABF}E^{FD} \;=\; -\,(-1)^{\varepsilon_{A}\varepsilon_{B}+(\varepsilon_{C}+1)(\varepsilon_{D}+1)+(\varepsilon_{A}+\varepsilon_{B})(\varepsilon_{C}+\varepsilon_{D})}R^{D}{}_{BAF}E^{FC} \;. \tag{11.10}$$

Contracting the $A\leftrightarrow C$ and $B\leftrightarrow D$ indices in eq. (11.10) indeed produces the identity $R=R$. Had the signs turn out differently, the odd scalar curvature (11.7) would have been stillborn, *i.e.* always zero.

12 Conclusions

In this paper, we have first of all analyzed a general non-degenerate, second-order Δ operator, and found that nilpotency determines the Δ operator uniquely (after dismissing an odd constant). The result is that Δ has to be $\Delta_{\rho}+\nu_{\rho}$, where Δ_{ρ} is the odd Laplacian, and ν_{ρ} is an odd scalar function (=zeroth-order operator) that only depends on the density ρ and the antisymplectic structure E. Secondly, we have shown that several constructions in antisymplectic geometry can be extended to a non-flat line bundle connection F, which replaces ρ. We did this by breaking the nilpotency $\Delta^{2}=\frac{1}{2}\Delta_{\mathcal{R}}$ by a general second-order operator $\Delta_{\mathcal{R}}$, which acts as a source for the F-curvature $\mathcal{R}$. In this more general case, the Δ operator takes the form $\Delta_{F}+\nu_{F}+\nu_{\mathcal{R}}$, where Δ_{F} and ν_{F} are generalizations of the odd Laplacian Δ_{ρ} and the odd scalar ν_{ρ}, respectively. The $\nu_{\mathcal{R}}$ term is an auxiliary curvature background encoded in the $\Delta_{\mathcal{R}}$ operator. Thirdly, we have identified the ν_{F} function with (minus 1/8 times) the odd scalar curvature R of an arbitrary antisymplectic, torsion-free, and F-compatible connection.

One may summarize by saying that two notions of curvature play an important rôle in this paper: 1) a line bundle curvature $\mathcal{R}_{AB}$ defined in eq. (3.7) and 2) an odd scalar curvature R defined in eq. (11.7). The former provides a natural framework for several mathematical constructions, but it remains currently unclear if it would be useful in physics. On the other hand, the field-antifield formalism naturally embraces the latter type of curvature both physically and mathematically. Concretely, we saw that the odd scalar curvature R manifests itself via a zeroth-order term ν_{F} in the Δ operator, which could potentially be used in a physical application some day. Altogether, the odd scalar curvature R and ν_{F} represent an important milestone in our understanding of the symmetries and the supergeometric structures behind the powerful field-antifield formalism.

ACKNOWLEDGEMENTS: We would like to thank P.H. Damgaard for discussions and the Niels Bohr institute for warm hospitality. The work of I.A.B. is supported by grants RFBR 05-01-00996, RFBR 05-02-17217 and LSS-4401.2006.2. The work of K.B. is supported by the Ministry of Education of the Czech Republic under the project MSM 0021622409.

A Independence of Gauge-Fixing in the F-Independent Formalism

In this Appendix A, we prove in two different ways that the partition function (9.1) is independent of gauge-fixing. Let us introduce the following shorthand notation

$$w \equiv e^{\frac{i}{\hbar}W_E} , \qquad x \equiv e^{\frac{i}{\hbar}X_E} , \tag{A.1}$$

for the two Boltzmann semidensities, so that the partition function (9.1) simply becomes

$$\mathcal{Z} = \int [d\Gamma][d\lambda] \ w \ x . \tag{A.2}$$

The Boltzmann semidensities w and x are Δ_E-closed because of the two quantum master eqs. (9.2). Since the Δ_E operator is nilpotent, one may argue on general grounds that an arbitrary infinitesimal variation of x should be Δ_E-exact, which may be written as

$$\delta x = [\overrightarrow{\Delta}_E, \delta\Psi]x \equiv \Delta_E(\delta\Psi \ x) + \delta\Psi(\Delta_E x) , \tag{A.3}$$

if one assumes that x is invertible and satisfies the quantum master eq. (9.2). Phrased equivalently, the variation δX_E of the quantum action is BRST-exact,

$$\delta X_E = (X_E, \delta\Psi) + \frac{\hbar}{i}\Delta_1(\delta\Psi) = \sigma_{X_E}(\delta\Psi) , \tag{A.4}$$

where $\sigma_{X_E} = (X_E, \cdot) + \frac{\hbar}{i}\Delta_1$ is a quantum BRST-operator. One may now proceed in at least two ways. One axiomatic way [24] uses that the Δ_E operator (8.1) is symmetric,

$$\Delta_E^T = \Delta_E , \tag{A.5}$$

i.e. stabile under integration by part. Then, an infinitesimal variation (A.3) of the gauge-fixing Boltzmann semidensity x changes the partition function as

$$\begin{aligned} \delta\mathcal{Z} &= \int [d\Gamma][d\lambda] \ w \ \delta x = \int [d\Gamma][d\lambda] \ w \ [\overrightarrow{\Delta}_E, \delta\Psi]x \\ &= \int [d\Gamma][d\lambda] \left[(\Delta_E w) \ \delta\Psi \ x + w \ \delta\Psi \ (\Delta_E x)\right] = 0 , \end{aligned} \tag{A.6}$$

where the symmetry property (A.5) is used in the third equality and the two quantum master equations (9.2) in the fourth (= last) equality. Notice how this proof requires very little knowledge of the detailed form of Δ_E. Another proof [2, 5, 21] uses an intrinsic infinitesimal redefinition of the integration variables,

$$\delta\Gamma^A = \frac{i}{2\hbar}(\Gamma^A, X_E - W_E)\delta\Psi + \frac{1}{2}(\Gamma^A, \delta\Psi) = \frac{w}{2x}(\Gamma^A, \frac{x \ \delta\Psi}{w}) , \qquad \delta\lambda^\alpha = 0 , \tag{A.7}$$

to induce the allowed variation (A.3) of x. Now it is instructive to write the path integral integrand as a volume form $\Omega \equiv wx[d\Gamma][d\lambda]$ with measure density wx. The Lie-derivative is

$$\delta\Omega = (\mathrm{div}_{wx}\delta\Gamma)\Omega . \tag{A.8}$$

In detail, the field-antifield redefinition (A.7) yields the following logarithmic variation of Ω:

$$\begin{aligned} \mathrm{div}_{wx}\delta\Gamma &\equiv \frac{(-1)^{\varepsilon_A}}{wx}\overrightarrow{\partial^l_A}(wx \ \delta\Gamma^A) = \frac{(-1)^{\varepsilon_A}}{2wx}\overrightarrow{\partial^l_A} w^2(\Gamma^A, \frac{x \ \delta\Psi}{w}) = \frac{w}{x}\Delta_{w^2}\frac{x \ \delta\Psi}{w} \\ &= \frac{1}{x}\Delta_1(x \ \delta\Psi) - (\Delta_1 w)\frac{\delta\Psi}{w} = \frac{1}{x}\Delta_1(x \ \delta\Psi) + (\frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24})\delta\Psi = \frac{1}{x}\Delta_E(x \ \delta\Psi) \end{aligned}$$

$$= \frac{1}{x}[\overrightarrow{\Delta}_E, \delta\Psi]x \;=\; \delta\ln x \;. \tag{A.9}$$

Here, a non-trivial property of the odd Laplacian (1.1) is used in the fourth equality, the two quantum master equations (9.2) are used in the fifth and seventh equality, and the formula (A.3) for the allowed variation of x is used in the eighth (=last) equality. If one reads the above eq. (A.9) in the opposite direction, one sees that all allowed variations (A.3) of the gauge-fixing Boltzmann semidensity x can be reproduced by an intrinsic field-antifield redefinition (A.7),

$$\delta\mathcal{Z} \;=\; \int [d\Gamma][d\lambda]\; w\;\delta x \;=\; \int \Omega\;\delta\ln x \;=\; \int \Omega\;\mathrm{div}_{wx}\delta\Gamma \;=\; \int \delta\Omega \;=\; 0\;. \tag{A.10}$$

One concludes that the partition function $\mathcal{Z} = \int\Omega$ must be independent of the gauge-fixing x part since an intrinsic redefinition of dummy integration variables cannot change the value of the path integral.

B Proof of Proposition 7.1

In this Appendix B, we show that the ν_F expression (7.1) satisfies the differential eq. (5.29) for the difference $\nu - \nu_{\mathcal{R}}$. We start by recalling that the Δ_F operator (2.2) is

$$\Delta_F \;\equiv\; \Delta_1 + V\;, \tag{B.1}$$

where Δ_1 denotes the expression (1.1) for the odd Laplacian $\Delta_{\rho=1}$ with ρ replaced by 1, and where we, for convenience, have defined

$$V \;\equiv\; \frac{(-1)^{\varepsilon_A}}{2} F_A(\Gamma^A, \cdot)\;. \tag{B.2}$$

Lemma B.1 *The square of the Δ_F operator is*

$$\Delta_F^2 \;\equiv\; \Delta_1^2 + [\Delta_1, V] + V^2 \;=\; \Delta_1^2 + \frac{1}{2}\Delta_{F,\mathcal{R}} + (\nu_F^{(0)}, \cdot)\;. \tag{B.3}$$

Proof of Lemma B.1: One finds by straightforward calculations that

$$\begin{aligned}
4V^2 \;&=\; (-1)^{\varepsilon_A+\varepsilon_B} F_A(\Gamma^A, F_B(\Gamma^B, \cdot)) \\
&=\; (-1)^{\varepsilon_A} F_B F_A(\Gamma^A, (\Gamma^B, \cdot)) + (-1)^{\varepsilon_A+\varepsilon_B} F_A E^{AC}(\overrightarrow{\partial^l_C} F_B)(\Gamma^B, \cdot) \\
&=\; \frac{(-1)^{\varepsilon_A}}{2} F_B F_A((\Gamma^A, \Gamma^B), \cdot) + (-1)^{\varepsilon_A} F_A E^{AC}[F_C \overleftarrow{\partial^r_B} + \mathcal{R}_{CB}(-1)^{\varepsilon_B}](\Gamma^B, \cdot) \\
&=\; \frac{(-1)^{\varepsilon_A}}{2}(F_A E^{AB} F_B, \cdot) + (-1)^{\varepsilon_A+\varepsilon_C} F_A E^{AB}\mathcal{R}_{BC}(\Gamma^C, \cdot)\;,
\end{aligned} \tag{B.4}$$

and

$$\begin{aligned}
2[\Delta_1, V] \;&=\; (-1)^{\varepsilon_A}\Delta_1 F_A(\Gamma^A, \cdot) + (-1)^{\varepsilon_A} F_A(\Gamma^A, \Delta_1(\cdot)) \\
&=\; (-1)^{\varepsilon_A}(\Delta_1 F_A)(\Gamma^A, \cdot) + (F_A, (\Gamma^A, \cdot)) + F_A\Delta_1(\Gamma^A, \cdot) + (-1)^{\varepsilon_A} F_A(\Gamma^A, \Delta_1(\cdot)) \\
&=\; \frac{(-1)^{\varepsilon_A+\varepsilon_B}}{2}(\overrightarrow{\partial^l_B} E^{BC}\overrightarrow{\partial^l_C} F_A)(\Gamma^A, \cdot) + (F_A\overleftarrow{\partial^r_B})(\Gamma^B, (\Gamma^A, \cdot)) + F_A(\Delta_1\Gamma^A, \cdot) \\
&=\; \frac{(-1)^{\varepsilon_B}}{2}(\overrightarrow{\partial^l_B} E^{BC}[F_C\overleftarrow{\partial^r_A} + \mathcal{R}_{CA}(-1)^{\varepsilon_A}])(\Gamma^A, \cdot) \\
&\quad + \frac{1}{2}[F_A\overleftarrow{\partial^r_B} + (-1)^{\varepsilon_B}\overrightarrow{\partial^l_A} F_B + (-1)^{(\varepsilon_A+1)\varepsilon_B}\mathcal{R}_{BA}](\Gamma^B, (\Gamma^A, \cdot)) + F_A(\Delta_1\Gamma^A, \cdot)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^{\varepsilon_C}}{2} E^{CB}(\overrightarrow{\partial^{l}_{B}} F_C, \cdot) + (\Delta_1 \Gamma^C)(F_C, \cdot) + \frac{(-1)^{\varepsilon_A+\varepsilon_B}}{2}(\overrightarrow{\partial^{l}_{B}} E^{BC} \mathcal{R}_{CA})(\Gamma^A, \cdot) \\
&\quad + \frac{(-1)^{\varepsilon_B}}{2}(\overrightarrow{\partial^{l}_{A}} F_B)((\Gamma^B, \Gamma^A), \cdot) - \frac{(-1)^{(\varepsilon_A+1)(\varepsilon_C+1)}}{2} E^{CB} \mathcal{R}_{BA} \overrightarrow{\partial^{l}_{C}}(\Gamma^A, \cdot) + F_A(\Delta_1 \Gamma^A, \cdot) \\
&= \frac{(-1)^{\varepsilon_B}}{2}(E^{BA} \overrightarrow{\partial^{l}_{A}} F_B, \cdot) + (F_A \Delta_1 \Gamma^A, \cdot) + \frac{(-1)^{\varepsilon_A+\varepsilon_C}}{2} \overrightarrow{\partial^{l}_{A}} E^{AB} \mathcal{R}_{BC}(\Gamma^C, \cdot) \\
&= \frac{(-1)^{\varepsilon_A}}{2}(\overrightarrow{\partial^{l}_{A}}(E^{AB} F_B), \cdot) + \frac{(-1)^{\varepsilon_A+\varepsilon_C}}{2} \overrightarrow{\partial^{l}_{A}} E^{AB} \mathcal{R}_{BC}(\Gamma^C, \cdot) \ , \qquad \text{(B.5)}
\end{aligned}
$$

where the Jacobi identity (3.1) has been applied in the third and fifth equality of eqs. (B.4) and (B.5), respectively.

□

(As an aside, we mention that Lemma B.1 can be used to prove Lemma 5.1 in Section 5.) When one compares Lemma B.1 with the ν differential eq. (5.29), one sees the first clue that the ν_F expression (7.1) is a solution. More precisely, Lemma B.1 has extracted the $\nu_F^{(0)}$ part for us. Next task is to uncover the $\nu^{(1)}$ term (1.8).

Lemma B.2

$$
8(\Delta_1^2 \Gamma^A) \;=\; (\nu^{(1)}, \Gamma^A) - (-1)^{\varepsilon_C}(\overrightarrow{\partial^{l}_{B}} E^{CD})(\overrightarrow{\partial^{l}_{D}} \overrightarrow{\partial^{l}_{C}} E^{BA}) \ . \qquad \text{(B.6)}
$$

Proof of Lemma B.2: Combine

$$
(\overrightarrow{\partial^{l}_{B}} \Delta_1 E^{BA}) - 2(\Delta_1^2 \Gamma^A) \;=\; [\overrightarrow{\partial^{l}_{B}}, \Delta_1] E^{BA} \;=\; \frac{1}{2}(-1)^{\varepsilon_C}(\overrightarrow{\partial^{l}_{B}} E^{CD})(\overrightarrow{\partial^{l}_{D}} \overrightarrow{\partial^{l}_{C}} E^{BA}) + (\overrightarrow{\partial^{l}_{B}} \Delta_1 \Gamma^C) \overrightarrow{\partial^{l}_{C}} E^{BA} \ , \qquad \text{(B.7)}
$$

and

$$
\begin{aligned}
(\overrightarrow{\partial^{l}_{B}} \Delta_1 E^{BA}) &= \overrightarrow{\partial^{l}_{B}} \Delta_1(\Gamma^B, \Gamma^A) \;=\; \overrightarrow{\partial^{l}_{B}}(\Delta_1 \Gamma^B, \Gamma^A) - (-1)^{\varepsilon_B} \overrightarrow{\partial^{l}_{B}}(\Gamma^B, \Delta_1 \Gamma^A) \\
&= \frac{1}{2}(\nu^{(1)}, \Gamma^A) + (\overrightarrow{\partial^{l}_{C}} \Delta_1 \Gamma^B)(\overrightarrow{\partial^{l}_{B}} E^{CA}) - 2(\Delta_1^2 \Gamma^A) \ . \qquad \text{(B.8)}
\end{aligned}
$$

□

So far, we have reproduced the $\nu_F^{(0)}$ and the $\nu^{(1)}$ part of the ν_F solution to the ν differential eq. (5.29). Finally, we should extract the $\nu^{(2)}$ term (1.9). The prefactor 1/24 in the ν_F formula (7.1) hints that such a calculation is going be lengthy. Rewrite first Lemma B.2 as

$$
8(\Delta_1^2 \Gamma^B) E_{BA} \;=\; (\overrightarrow{\partial^{l}_{A}} \nu^{(1)}) - \nu_A^I \ , \qquad \text{(B.9)}
$$

where

$$
\begin{aligned}
\nu_A^I &\equiv (-1)^{\varepsilon_D}(\overrightarrow{\partial^{l}_{C}} E^{DF})(\overrightarrow{\partial^{l}_{F}} \overrightarrow{\partial^{l}_{D}} E^{CB}) E_{BA} \;=\; \nu_A^{II} + \nu_A^{III} \ , \qquad \text{(B.10)} \\
\nu_A^{II} &\equiv (-1)^{\varepsilon_B \varepsilon_D}(\overrightarrow{\partial^{l}_{D}} E^{BC})(\overrightarrow{\partial^{l}_{C}} E^{DF})(\overrightarrow{\partial^{l}_{F}} E_{BA}) \;=\; -\nu_A^{II} - \nu_A^{IV} \ , \qquad \text{(B.11)} \\
\nu_A^{III} &\equiv (-1)^{\varepsilon_D}(\overrightarrow{\partial^{l}_{C}} E^{DF}) \overrightarrow{\partial^{l}_{F}}((\overrightarrow{\partial^{l}_{D}} E^{CB}) E_{BA})
\end{aligned}
$$

$$= \; -(-1)^{(\varepsilon_B+\varepsilon_C)\varepsilon_D}(\overrightarrow{\partial^l_C}E^{DF})\overrightarrow{\partial^l_F}(E^{CB}\overrightarrow{\partial^l_D}E_{BA}) \; = \; \nu_A^{II}+\nu_A^{V} \; , \tag{B.12}$$

$$\nu_A^{IV} \; \equiv \; (-1)^{\varepsilon_C\varepsilon_D}(\overrightarrow{\partial^l_A}E_{BC})(\overrightarrow{\partial^l_D}E^{CF})(\overrightarrow{\partial^l_F}E^{DB}) \; , \tag{B.13}$$

$$\nu_A^{V} \; \equiv \; (-1)^{\varepsilon_C}E^{BF}(\overrightarrow{\partial^l_F}E^{CD})(\overrightarrow{\partial^l_D}\overrightarrow{\partial^l_C}E_{BA}) \; = \; -2\nu_A^{VI} \; , \tag{B.14}$$

$$\nu_A^{VI} \; \equiv \; (-1)^{\varepsilon_B(\varepsilon_C+1)}E^{CF}(\overrightarrow{\partial^l_F}E^{BD})(\overrightarrow{\partial^l_D}\overrightarrow{\partial^l_C}E_{BA}) \; = \; \nu_A^{V}+\nu_A^{VII} \; , \tag{B.15}$$

$$\nu_A^{VII} \; \equiv \; (-1)^{\varepsilon_C}(\overrightarrow{\partial^l_A}\overrightarrow{\partial^l_B}E_{CD})E^{DF}(\overrightarrow{\partial^l_F}E^{CB}) \; . \tag{B.16}$$

Here, the Jacobi identity (3.3) is used in the second equality of eq. (B.14), and the closeness relation (3.4) is used in the second equalities of eqs. (B.11) and (B.15). Altogether eqs. (B.10)–(B.16) yield

$$\nu_A^{I} \; = \; \nu_A^{II}+\nu_A^{III} \; = \; 2\nu_A^{II}+\nu_A^{V} \; = \; -\nu_A^{IV}+\nu_A^{V} \; = \; -\nu_A^{IV}-\frac{2}{3}\nu_A^{VII} \; . \tag{B.17}$$

Ultimately, we would like to show that ν_A^I is equal to $(\overrightarrow{\partial^l_A}\nu^{(2)})/3$. The achievement in eq. (B.17) is more modest: The free "A" index on the ν_A^I expression has been moved to a derivative $\overrightarrow{\partial^l_A}$ in ν_A^{IV} and ν_A^{VII}. On the other hand, differentiation with respect to Γ^A of the two expressions (1.9) and (1.10) for the $\nu^{(2)}$ quantity (1.9) yields two more relations

$$\nu_A^{IV}+2\nu_A^{VIII} \; = \; (\overrightarrow{\partial^l_A}\nu^{(2)}) \; = \; \nu_A^{VIII}-\nu_A^{VII}-\nu_A^{IX} \; , \tag{B.18}$$

where

$$\nu_A^{VIII} \; \equiv \; (-1)^{\varepsilon_C\varepsilon_F}(\overrightarrow{\partial^l_A}\overrightarrow{\partial^l_B}E^{CD})E_{DF}(\overrightarrow{\partial^l_C}E^{FB}) \; , \tag{B.19}$$

$$\begin{aligned}
\nu_A^{IX} \; &\equiv \; (-1)^{\varepsilon_C}(\overrightarrow{\partial^l_A}E^{DF})(\overrightarrow{\partial^l_F}E^{CB})(\overrightarrow{\partial^l_B}E_{CD}) \\
&= \; -(-1)^{\varepsilon_B\varepsilon_G}(\overrightarrow{\partial^l_A}E^{DF})(\overrightarrow{\partial^l_F}E^{BC})E_{CG}(\overrightarrow{\partial^l_B}E^{GH})E_{HD} \\
&= \; (-1)^{\varepsilon_B\varepsilon_G}(\overrightarrow{\partial^l_A}E_{HD})E^{DF}(\overrightarrow{\partial^l_F}E^{BC})E_{CG}(\overrightarrow{\partial^l_B}E^{GH}) \; = \; \nu_A^{IV}-\nu_A^{X} \; ,
\end{aligned} \tag{B.20}$$

$$\begin{aligned}
\nu_A^{X} \; &\equiv \; (-1)^{\varepsilon_B\varepsilon_G+(\varepsilon_B+\varepsilon_C)(\varepsilon_D+1)}(\overrightarrow{\partial^l_A}E_{HD})E^{BF}(\overrightarrow{\partial^l_F}E^{CD})E_{CG}(\overrightarrow{\partial^l_B}E^{GH}) \\
&= \; (-1)^{(\varepsilon_B+1)\varepsilon_D+\varepsilon_C(\varepsilon_B+\varepsilon_H+1)}(\overrightarrow{\partial^l_A}E_{HD})E^{BF}(\overrightarrow{\partial^l_F}E^{DC})(\overrightarrow{\partial^l_B}E^{HG})E_{GC} \; = \; 0 \; .
\end{aligned} \tag{B.21}$$

Here, the Jacobi identity (3.1) is used in the fourth equality of eq. (B.20). Remarkably, the ν_A^X term vanishes due to an antisymmetry under the index permutation $FDC \leftrightarrow BHG$. Altogether, $\nu_A^{IX}=\nu_A^{IV}$ and

$$\nu_A^{I} \; = \; -\nu_A^{IV}-\frac{2}{3}\nu_A^{VII} \; = \; -\nu_A^{IV}-\frac{2}{3}(\nu_A^{VIII}-\nu_A^{IV}-\overrightarrow{\partial^l_A}\nu^{(2)}) \; = \; \frac{1}{3}(\overrightarrow{\partial^l_A}\nu^{(2)}) \; . \tag{B.22}$$

Combining eqs. (B.3), (B.9) and (B.22) shows that the ν_F expression (7.1) satisfies the ν differential eq. (5.29).

C Proof of Proposition 11.1

In this Appendix C, we prove that the odd scalar curvature R is minus eight times the odd scalar ν_F. The odd scalar curvature

$$R \; \equiv \; R_{AB}E^{BA} \; = \; R_I+R_{II}-R_{III}-R_{IV} \tag{C.1}$$

inherits four terms R_I, R_{II}, R_{III} and R_{IV} from the expression (11.5) for the Ricci tensor R_{AB}. They are defined as

$$\begin{aligned}
R_I &\equiv (-1)^{\varepsilon_A}(\overrightarrow{\partial^l_A}\Gamma^A{}_{BC})E^{CB} \;=\; R_V - R_{VI} \;, && \text{(C.2)}\\
R_{II} &\equiv (-1)^{\varepsilon_A}F_A\Gamma^A{}_{BC}E^{CB} \;=\; -(-1)^{\varepsilon_B}F_A(\overrightarrow{\partial^l_B}+F_B)E^{BA} \;, && \text{(C.3)}\\
R_{III} &\equiv (-1)^{\varepsilon_B}E^{BA}(\overrightarrow{\partial^l_A}F_B) \;, && \text{(C.4)}\\
R_{IV} &\equiv \Gamma_A{}^C{}_D\Gamma^D{}_{CB}E^{BA} \;=\; -R_{IV} - R_{VI} \;, && \text{(C.5)}\\
R_V &\equiv (-1)^{\varepsilon_A}\overrightarrow{\partial^l_A}(\Gamma^A{}_{BC}E^{CB}) \;=\; -(-1)^{\varepsilon_B}\overrightarrow{\partial^l_A}(\overrightarrow{\partial^l_B}+F_B)E^{BA} \\
&= -\nu^{(1)} - (-1)^{\varepsilon_A}\overrightarrow{\partial^l_A}(E^{AB}F_B) \;, && \text{(C.6)}\\
R_{VI} &\equiv \Gamma^A{}_{BC}(E^{CB}\overleftarrow{\partial^r_A}) \;. && \text{(C.7)}
\end{aligned}$$

Here, the antisymplectic and the torsion-free conditions (10.6) and (10.9) are used in the second equality of eq. (C.5), and a contracted version of the antisymplectic condition (10.6)

$$(-1)^{\varepsilon_B}(\overrightarrow{\partial^l_B}+F_B)E^{BA} + (-1)^{\varepsilon_A}\Gamma^A{}_{BC}E^{CB} \;=\; 0 \qquad \text{(C.8)}$$

is used in the second equalities of eqs. (C.3) and (C.6). Inserting back in eq. (C.1), one finds that

$$R \;=\; -8\nu^{(0)}_F - \nu^{(1)} - \frac{1}{2}R_{VI} \;, \qquad \text{(C.9)}$$

where $\nu^{(0)}_F$ and $\nu^{(1)}$ are given in eqs. (4.6) and (1.8). Now it remains to eliminate R_{VI} from eq. (C.9). Note that R_{VI} only depends on the torsion-free part of the connection $\Gamma^A{}_{BC}$, so one does in principle not need the torsion-free condition (10.9) from now on. One calculates that

$$\begin{aligned}
\frac{1}{2}R_{VI} &= -\frac{1}{2}(-1)^{\varepsilon_A(\varepsilon_D+1)}\Gamma_B{}^A{}_C E^{CD}(\overrightarrow{\partial^l_A}E_{DF})E^{FB} \;=\; -(-1)^{\varepsilon_A}\Gamma_B{}^A{}_C E^{CD}(\overrightarrow{\partial^l_D}E_{AF})E^{FB} \\
&= \Gamma^A{}_{BC}E^{CD}(\overrightarrow{\partial^l_D}E^{BF})E_{FA} \;=\; -\nu^{(2)} - R_{VI} \;. && \text{(C.10)}
\end{aligned}$$

Here, the closeness relation (3.4) is used in the second equality and the antisymplectic condition (10.6) in the fourth equality. In other words,

$$R_{VI} \;=\; -\frac{2}{3}\nu^{(2)} \;. \qquad \text{(C.11)}$$

Combining eqs. (C.9) and (C.11) yields the main result of Proposition 11.1:

$$R \;=\; -8\nu_F \;. \qquad \text{(C.12)}$$

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Paper IV

Odd Scalar Curvature in Anti-Poisson Geometry

BY

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Phys. Lett. **B663** (2008) 132

arXiv:0712.3699

Odd Scalar Curvature in Anti-Poisson Geometry

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January 31, 2017

Abstract

Recent works have revealed that the recipe for field-antifield quantization of Lagrangian gauge theories can be considerably relaxed when it comes to choosing a path integral measure ρ if a zero-order term ν_{ρ} is added to the Δ operator. The effects of this odd scalar term ν_{ρ} become relevant at two-loop order. We prove that ν_{ρ} is essentially the odd scalar curvature of an arbitrary torsion-free connection that is compatible with both the anti-Poisson structure E and the density ρ. This extends a previous result for non-degenerate antisymplectic manifolds to degenerate anti-Poisson manifolds that admit a compatible two-form.

PACS number(s): 02.40.-k; 03.65.Ca; 04.60.Gw; 11.10.-z; 11.10.Ef; 11.15.Bt.
Keywords: BV Field-Antifield Formalism; Odd Laplacian; Anti-Poisson Geometry; Semi-density; Connection; Odd Scalar Curvature.

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1 Introduction

The main purpose of this Letter is to report on new geometric insights into the field-antifield formalism. In general, the field-antifield formalism [1, 2, 3] is a recipe for constructing Feynman rules for Lagrangian field theories with gauge symmetries. The field-antifield formalism is in principle able to handle the most general gauge algebra, *i.e.* open gauge algebras of reducible type. The input is usually a local relativistic field theory, formulated via a classical action principle in a geometric configuration space. In the field-antifield scheme, the original field variables are extended with various stages of ghosts, antighosts and Lagrange multipliers — all of which are then further extended with corresponding antifields; the gauge symmetries are encoded in a nilpotent Fermionic BRST symmetry [4, 5]; and the original action is deformed into a BRST-invariant master action, whose Hessian has the maximal allowed rank. The full quantum master action

$$W \;=\; S+\sum_{n=1}^{\infty}\hbar^{n}M_{n} \tag{1.1}$$

is determined recursively order by order in $\hbar$ from a consistent set of quantum master equations

$$(S,S) \;=\; 0\ , \tag{1.2}$$
$$(M_1,S) \;=\; i(\Delta_{\rho}S)\ , \tag{1.3}$$
$$(M_2,S) \;=\; i(\Delta_{\rho}M_1)+\nu_{\rho}-\frac{1}{2}(M_1,M_1)\ , \tag{1.4}$$
$$(M_n,S) \;=\; i(\Delta_{\rho}M_{n-1})-\frac{1}{2}\sum_{r=1}^{n-1}(M_r,M_{n-r})\ , \qquad n\geq 3\ . \tag{1.5}$$

Here $(\cdot,\cdot)$ is the antibracket (or anti-Poisson structure), Δ_{ρ} is the odd Laplacian and ν_{ρ} is an odd scalar, which become relevant in perturbation theory at loop order 0, 1, and 2, respectively. It has only recently been realized that the field-antifield formalism can consistently accommodate a non-zero ν_{ρ} term, thereby providing a more flexible framework for field-antifield quantization [6, 7, 8].

The classical master equation (1.2) is a generalization of Zinn-Justin's equation [9], which allows to set up consistent renormalization (if the field theory is renormalizable). If the theory is not anomalous at the one-loop level, there will exist a local solution M_1 to the next equation (1.3), and so forth. Although the field-antifield formalism in its basic form is only a formal scheme — *i.e.* particularly, it assumes that results from finite dimensional analysis are directly applicable to field theory, which has infinitely many degrees of freedom — it has nevertheless been successfully applied to a large variety of physical models. It has mainly been used in a truncated form of the full set of quantum master eqs. (1.2) – (1.5), where all the following quantities

$$(S,S),\ (\Delta_{\rho}S),\ \nu_{\rho},\ M_1,\ M_2,\ M_3,\ldots, \tag{1.6}$$

are set identically equal to zero. One can for instance mention the AKSZ paradigm [10, 11] as a broad example that uses the truncated field-antifield formalism (1.6) to quantize supersymmetric topological field theories [12, 13, 14, 15]. Currently, very few scientific works describe solutions with non-zero M_n's, primarily due to the singular nature of the odd Laplacian Δ_{ρ} in field theory (again because of the infinitely many degrees of freedom). Nevertheless, it should be fruitful to study generic solutions of the full quantum master equation. See the original paper [1] for an interesting solution with $M_1\neq 0$. Finally, it has in many cases been explicitly checked that the field-antifield formalism produces the same result as the Hamiltonian formulation [16, 17, 18]. The formalism has also influenced work in closed string field theory [19] and several branches of mathematics. The geometry behind the field-antifield formalism was further clarified in Ref. [20, 21, 22, 23].

In this Letter we shall only explicitly consider the case of finitely many variables. Our main result concerns the odd scalar ν_{ρ}, which is a certain function of the anti-Poisson structure E^{AB} and the density ρ, cf. eq. (6.1) below. It turns out that ν_{ρ} has a geometric interpretation as (minus 1/8 times) the odd scalar curvature R of any connection ∇ that satisfies three conditions; namely that ∇ is 1) anti-Poisson, 2) torsion-free and 3) ρ-compatible. This is a rather robust conclusion as we shall prove in this Letter that it even holds for degenerate antibrackets. (Degenerate anti-Poisson structures appear naturally from for instance the Dirac antibracket construction for antisymplectic second-class constraints [7, 21, 24, 25].)

2 Anti-Poisson structure E^{AB}

An *anti-Poisson* structure is by definition a possibly degenerate $(2,0)$ tensor field E^{AB} with upper indices that is Grassmann-odd

$$\varepsilon(E^{AB}) \;=\; \varepsilon_A+\varepsilon_B+1 \;, \tag{2.1}$$

that is skewsymmetric

$$E^{AB} \;=\; -\,(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}E^{BA} \;, \tag{2.2}$$

and that satisfies the Jacobi identity

$$\sum_{\text{cycl. } A,B,C}(-1)^{(\varepsilon_A+1)(\varepsilon_C+1)}E^{AD}(\overrightarrow{\partial^{\ell}_D}E^{BC}) \;=\; 0 \;. \tag{2.3}$$

3 Compatible two-form E_{AB}

In general, an anti-Poisson manifold could have singular points where the rank of E^{AB} jumps, and it is necessary to impose a regularity criterion to proceed. We shall here assume that the anti-Poisson structure E^{AB} admits a compatible two-form field E_{AB}, *i.e.* that there exists a two-form field E_{AB} with lower indices that is Grassmann-odd

$$\varepsilon(E_{AB}) \;=\; \varepsilon_A+\varepsilon_B+1 \;, \tag{3.1}$$

that is skewsymmetric

$$E_{AB} \;=\; -\,(-1)^{\varepsilon_A\varepsilon_B}E_{BA} \;, \tag{3.2}$$

and that is *compatible* with the anti-Poisson structure in the sense that

$$E^{AB}E_{BC}E^{CD} \;=\; E^{AD} \;, \tag{3.3}$$

$$E_{AB}E^{BC}E_{CD} \;=\; E_{AD} \;. \tag{3.4}$$

This is a relatively mild requirement, which is always automatically satisfied for a Dirac antibracket on antisymplectic manifolds with antisymplectic second-class constraints [7, 21, 24, 25]. Note that the two-form E_{AB} is neither unique nor necessarily closed. One can define a $(1,1)$ tensor field as

$$P^{A}{}_{C} \;\equiv\; E^{AB}E_{BC} \;, \tag{3.5}$$

or equivalently,

$$P_{A}{}^{C} \;\equiv\; E_{AB}E^{BC} \;=\; (-1)^{\varepsilon_A(\varepsilon_C+1)}P^{C}{}_{A} \;. \tag{3.6}$$

It then follows from either of the compatibility relations (3.3) and (3.4) that $P^{A}{}_{B}$ is an idempotent

$$P^{A}{}_{B}P^{B}{}_{C} \;=\; P^{A}{}_{C} \;. \tag{3.7}$$

4 The Δ_E Operator

An anti-Poisson structure with a compatible two-form field E_{AB} gives rise to a Grassmann-odd, second-order Δ_E operator that takes semidensities to semidensities. It is defined in arbitrary coordinates as [7]

$$\Delta_E \equiv \Delta_1 + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{8} - \frac{\nu^{(3)}}{24} + \frac{\nu^{(4)}}{24} + \frac{\nu^{(5)}}{12} , \tag{4.1}$$

where Δ_1 is the odd Laplacian

$$\Delta_\rho \equiv \frac{(-1)^{\varepsilon_A}}{2\rho} \overrightarrow{\partial^\ell_A} \rho E^{AB} \overrightarrow{\partial^\ell_B} , \tag{4.2}$$

with $\rho = 1$, and where

$$\nu^{(1)} \equiv (-1)^{\varepsilon_A} (\overrightarrow{\partial^\ell_B} \overrightarrow{\partial^\ell_A} E^{AB}) , \tag{4.3}$$

$$\nu^{(2)} \equiv (-1)^{\varepsilon_A \varepsilon_C} (\overrightarrow{\partial^\ell_D} E^{AB}) E_{BC} (\overrightarrow{\partial^\ell_A} E^{CD}) , \tag{4.4}$$

$$\nu^{(3)} \equiv (-1)^{\varepsilon_B} (\overrightarrow{\partial^\ell_A} E_{BC}) E^{CD} (\overrightarrow{\partial^\ell_D} E^{BA}) , \tag{4.5}$$

$$\nu^{(4)} \equiv (-1)^{\varepsilon_B} (\overrightarrow{\partial^\ell_A} E_{BC}) E^{CD} (\overrightarrow{\partial^\ell_D} E^{BF}) P_F{}^A , \tag{4.6}$$

$$\begin{aligned} \nu^{(5)} &\equiv (-1)^{\varepsilon_A \varepsilon_C} (\overrightarrow{\partial^\ell_D} E^{AB}) E_{BC} (\overrightarrow{\partial^\ell_A} E^{CF}) P_F{}^D \\ &= (-1)^{(\varepsilon_A+1)\varepsilon_B} E^{AD} (\overrightarrow{\partial^\ell_D} E^{BC}) (\overrightarrow{\partial^\ell_C} E_{AF}) P^F{}_B . \end{aligned} \tag{4.7}$$

It is shown in Ref. [7] that the Δ_E operator defined in eq. (4.1) does not depend on the choice of local coordinates, it does not depend on the choice of compatible two-form field E_{AB}, and it does map semidensities into semidensities. Moreover, the Jacobi identity (2.3) precisely ensures that Δ_E is nilpotent

$$\Delta_E^2 = \frac{1}{2}[\Delta_E, \Delta_E] = 0 . \tag{4.8}$$

Earlier works on the Δ_E operator include Ref. [6, 25, 26, 27, 28, 29].

5 The Δ Operator

Classically, the field-antifield formalism is governed by the anti-Poisson structure E^{AB}, or equivalently, the antibracket

$$(f, g) \equiv (f \overleftarrow{\partial^r_A}) E^{AB} (\overrightarrow{\partial^\ell_B} g) = -(-1)^{(\varepsilon_f+1)(\varepsilon_g+1)} (g, f) . \tag{5.1}$$

Quantum mechanically, the field-antifield recipe instructs one to choose an arbitrary path integral measure ρ, and to use it to build a nilpotent, Grassmann-odd, second-order Δ operator that takes scalar functions into scalar functions. It is natural to build the Δ operator by conjugating the Δ_E operator (4.1) with appropriate square roots of the density ρ as follows:

$$\Delta \equiv \frac{1}{\sqrt{\rho}} \Delta_E \sqrt{\rho} . \tag{5.2}$$

In this way the Δ operator trivially inherits the nilpotency property from the Δ_E operator,

$$\Delta^2 = \frac{1}{\sqrt{\rho}} \Delta_E^2 \sqrt{\rho} = 0 . \tag{5.3}$$

In physical applications the nilpotency (5.3) of Δ is important for the underlying BRST symmetry of the theory.

6 The Odd Scalar ν_ρ

The odd scalar function ν_ρ is defined as

$$\nu_\rho \;\equiv\; (\Delta 1) \;=\; \frac{1}{\sqrt{\rho}}(\Delta_E\sqrt{\rho}) \;=\; \nu_\rho^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{8} - \frac{\nu^{(3)}}{24} + \frac{\nu^{(4)}}{24} + \frac{\nu^{(5)}}{12} \;, \tag{6.1}$$

where $\nu^{(1)}$, $\nu^{(2)}$, $\nu^{(3)}$, $\nu^{(4)}$, $\nu^{(5)}$ are given in eqs. (4.3)–(4.7), and the quantity $\nu_\rho^{(0)}$ is given as

$$\nu_\rho^{(0)} \;\equiv\; \frac{1}{\sqrt{\rho}}(\Delta_1\sqrt{\rho}) \;. \tag{6.2}$$

The second-order Δ operator (5.2) decomposes as

$$\Delta \;=\; \Delta_\rho + \nu_\rho \;, \tag{6.3}$$

where Δ_ρ is the odd Laplacian (4.2). The nilpotency of Δ implies that

$$\Delta_\rho^2 \;=\; (\nu_\rho, \cdot) \;, \tag{6.4}$$

$$(\Delta_\rho \nu_\rho) \;=\; 0 \;. \tag{6.5}$$

The possibility of a non-trivial ν_ρ has only recently been observed, cf. Ref. [6, 7, 8]. In the past, the odd scalar term ν_ρ was not present due to a certain compatibility relation between E and ρ, which was unnecessarily imposed, and which (using our new terminology) made ν_ρ vanish. In terms of the quantum master equation

$$\Delta e^{\frac{i}{\hbar}W} \;=\; 0 \;, \tag{6.6}$$

the odd scalar ν_ρ enters at the two-loop order $\mathcal{O}(\hbar^2)$

$$\frac{1}{2}(W,W) \;=\; i\hbar\Delta_\rho W + \hbar^2\nu_\rho \;, \tag{6.7}$$

which in turn leads to the set of eqs. (1.2) – (1.5).

7 Connection

In the next two Sections 7 and 8 we will briefly state our sign conventions and definitions for the covariant derivative and the curvature in the presence of Fermionic degrees of freedom. A more complete treatment can be found in Ref. [8, 30]. Other references include Ref. [31]. Our convention for the left covariant derivative $(\nabla_A X)^B$ of a left vector field X^A is [30]

$$(\nabla_A X)^B \;\equiv\; (\overrightarrow{\partial^\ell_A} X^B) + (-1)^{\varepsilon_X(\varepsilon_B+\varepsilon_C)}\Gamma_A{}^B{}_C X^C \;, \qquad \varepsilon(X^A) \;=\; \varepsilon_X + \varepsilon_A \;. \tag{7.1}$$

A connection $\Gamma_A{}^B{}_C$ is called *anti-Poisson* if it preserves the anti-Poisson structure E^{AB}, *i.e.*

$$0 \;=\; (\nabla_A E)^{BC} \;\equiv\; (\overrightarrow{\partial^\ell_A} E^{BC}) + \Big(\Gamma_A{}^B{}_D E^{DC} - (-1)^{(\varepsilon_B+1)(\varepsilon_C+1)}(B \leftrightarrow C)\Big) \;. \tag{7.2}$$

It is useful to define a reordered Christoffel symbol $\Gamma^A{}_{BC}$ as

$$\Gamma^A{}_{BC} \;\equiv\; (-1)^{\varepsilon_A\varepsilon_B}\Gamma_B{}^A{}_C \;. \tag{7.3}$$

A *torsion-free* connection $\Gamma^A{}_{BC}$ has the following symmetry in the lower indices:

$$\Gamma^A{}_{BC} \;=\; -(-1)^{(\varepsilon_B+1)(\varepsilon_C+1)}\Gamma^A{}_{CB} \; . \tag{7.4}$$

A connection $\Gamma^A{}_{BC}$ is called ρ-*compatible* if

$$\Gamma^B{}_{BA} \;=\; (\ln\rho \overleftarrow{\partial^r_A}) \; . \tag{7.5}$$

There are in principle two definitions for the divergence $\mathrm{div}X$ of a Bosonic vector field X with $\varepsilon_X = 0$. The first divergence definition depends on the density ρ

$$\mathrm{div}_\rho X \;\equiv\; \frac{(-1)^{\varepsilon_A}}{\rho}\overrightarrow{\partial^\ell_A}(\rho X^A) \; , \tag{7.6}$$

while the second definition depends on the connection ∇

$$\mathrm{div}_\nabla X \;\equiv\; \mathrm{str}(\nabla X) \;\equiv\; (-1)^{\varepsilon_A}(\nabla_A X)^A \;=\; ((-1)^{\varepsilon_A}\overrightarrow{\partial^\ell_A} + \Gamma^B{}_{BA})X^A \; . \tag{7.7}$$

The ρ-compatibility condition (7.5) precisely ensures that the two definitions (7.6) and (7.7) coincide, and hence that there is a unique notion of volume [32]. We shall only consider torsion-free connections ∇ that are anti-Poisson and ρ-compatible, *i.e.* connections that satisfy the above three conditions (7.2), (7.4) and (7.5). Then the odd Laplacian Δ_ρ can be written on a manifestly covariant form

$$\Delta_\rho \;=\; \frac{(-1)^{\varepsilon_A}}{2}\nabla_A E^{AB}\nabla_B \;=\; \frac{(-1)^{\varepsilon_B}}{2}E^{BA}\nabla_A\nabla_B \; . \tag{7.8}$$

8 Curvature

The Riemann curvature tensor is

$$R^A{}_{BCD} \;\equiv\; (-1)^{\varepsilon_A\varepsilon_B}(\overrightarrow{\partial^\ell_B}\Gamma^A{}_{CD}) + \Gamma^A{}_{BE}\Gamma^E{}_{CD} - (-1)^{\varepsilon_B\varepsilon_C}(B \leftrightarrow C) \; . \tag{8.1}$$

(Note that the ordering of indices on the Riemann curvature tensor is slightly non-standard to minimize appearances of sign factors.) The Ricci tensor is

$$R_{AB} \;\equiv\; R^C{}_{CAB} \;=\; \frac{(-1)^{\varepsilon_C}}{\rho}(\overrightarrow{\partial^\ell_C}\rho\Gamma^C{}_{AB}) - (\overrightarrow{\partial^\ell_A}\ln\rho\overleftarrow{\partial^r_B}) - \Gamma_A{}^C{}_D\Gamma^D{}_{CB} \;=\; -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}R_{BA} \; . \tag{8.2}$$

9 Odd Scalar Curvature

The odd scalar curvature R is defined as the Ricci tensor R_{AB} contracted with the anti-Poisson tensor E^{AB},

$$R \;\equiv\; R_{AB}E^{BA} \;=\; E^{AB}R_{BA} \; , \qquad\qquad \varepsilon(R) \;=\; 1 \; . \tag{9.1}$$

We now assert that the odd scalar curvature

$$R \;=\; -8\nu_\rho \tag{9.2}$$

of an arbitrary connection ∇ that is anti-Poisson, torsion-free and ρ-compatible, is equal to (minus eight times) the odd scalar ν_ρ. In particular one sees that the odd scalar curvature R carries no information about the connection ∇ used, and it depends only on E and ρ. Equation (9.2) was proven for the non-degenerated case in Ref. [8]. The degenerated case is proven in Appendix A.

Acknowledgement: We would like to thank P.H. Damgaard for discussions. K.B. thanks the Lebedev Physics Institute and the Niels Bohr Institute for warm hospitality. The work of I.A.B. is supported by grants RFBR 05-01-00996, RFBR 05-02-17217 and LSS-4401.2006.2. The work of K.B. is supported by the Ministry of Education of the Czech Republic under the project MSM 0021622409.

A Proof of the Main Eq. (9.2)

Equation (C.9) in Ref. [8] yields that the odd scalar curvature R can be written as

$$R \;=\; -8\nu_{\rho}^{(0)} - \nu^{(1)} - \frac{1}{2}R_I \;, \tag{A.1}$$

where $\nu_{\rho}^{(0)}$, $\nu^{(1)}$ and R_I are defined in eqs. (6.2), (4.3) and (A.2), respectively. Since the expression (A.2) below for R_I only depends on the torsion-free part of the connection, one does in principle not need the torsion-free condition (7.4) from now on. The heart of the proof consists of the following ten "one-line calculations":

$$\begin{aligned}
R_I &\equiv \Gamma^A{}_{BC}(E^{CB}\overleftarrow{\partial^r_A}) \;=\; \Gamma^A{}_{BC}((E^{CD}E_{DF}E^{FB})\overleftarrow{\partial^r_A}) \;=\; 2R_{II}+R_{III} \;, && \text{(A.2)}\\
R_{II} &\equiv \Gamma^A{}_{BC}P^C{}_D(E^{DB}\overleftarrow{\partial^r_A}) \;=\; -R_{IV}-\nu^{(2)} \;, && \text{(A.3)}\\
R_{III} &\equiv (-1)^{\varepsilon_A(\varepsilon_C+1)}\Gamma_F{}^A{}_B E^{BC}(\overrightarrow{\partial^\ell_A}E_{CD})E^{DF} \;=\; 2R_{III}+R_V \;, && \text{(A.4)}\\
R_{IV} &\equiv \Gamma^A{}_{BC}E^{CD}(\overrightarrow{\partial^\ell_D}E^{BF})E_{FA} \;=\; R_{VI}-R_{IV} \;, && \text{(A.5)}\\
R_V &\equiv (-1)^{\varepsilon_A\varepsilon_C}\Gamma_F{}^A{}_B P^B{}_C(\overrightarrow{\partial^\ell_A}E^{CD})P_D{}^F \;=\; R_{VII}-\nu^{(5)} \;, && \text{(A.6)}\\
R_{VI} &\equiv \Gamma^A{}_{BC}(E^{CB}\overleftarrow{\partial^r_D})P^D{}_A \;=\; 2R_{VIII}+R_{IX} \;, && \text{(A.7)}\\
R_{VII} &\equiv (-1)^{(\varepsilon_A+1)(\varepsilon_C+1)}E_{AB}\Gamma^B{}_{CD}E^{DF}(\overrightarrow{\partial^\ell_F}E^{AG})P_G{}^C \;=\; R_{IV}-R_{VIII} \;, && \text{(A.8)}\\
R_{VIII} &\equiv \Gamma^A{}_{BC}P^C{}_D(E^{DB}\overleftarrow{\partial^r_F})P^F{}_A \;=\; -R_{IV}-\nu^{(5)} \;, && \text{(A.9)}\\
R_{IX} &\equiv (-1)^{\varepsilon_A(\varepsilon_C+1)}\Gamma_G{}^A{}_B E^{BC}P_A{}^D(\overrightarrow{\partial^\ell_D}E_{CF})E^{FG} \;=\; -R_X-\nu^{(4)} \;, && \text{(A.10)}\\
R_X &\equiv (-1)^{\varepsilon_A}\Gamma_F{}^A{}_B E^{BC}(\overrightarrow{\partial^\ell_C}E_{AD})E^{DF} \;=\; -R_{III}-\nu^{(3)} \;. && \text{(A.11)}
\end{aligned}$$

Here we have used the upper compatibility relation (3.3) for the two-form E_{AB} in the second equality of eqs. (A.2), (A.7), (A.8), (A.9) and (A.10); the lower compatibility relation (3.4) for the two-form E_{AB} in the second equality of eq. (A.4); the anti-Poisson property (7.2) for the connection ∇ in the second equality of eqs. (A.3), (A.6), (A.9), (A.10) and (A.11); and the Jacobi identity (2.3) in the second equality of eqs. (A.5) and (A.8). From these ten relations (A.2)–(A.11), the quantity R_{III} can be determined as follows:

$$\begin{aligned}
-R_{III} &= R_V \;=\; R_{VII}-\nu^{(5)} \;=\; (R_{IV}-R_{VIII})+(R_{IV}+R_{VIII}) \;=\; 2R_{IV}\\
&= R_{VI} \;=2R_{VIII}+R_{IX} \;=\; -2(R_{IV}+\nu^{(5)})+(R_{III}+\nu^{(3)}-\nu^{(4)})\\
&= 2R_{III}+(\nu^{(3)}-\nu^{(4)}-2\nu^{(5)}) \;,
\end{aligned} \tag{A.12}$$

so that

$$R_{III} \;=\; \frac{1}{3}(-\nu^{(3)}+\nu^{(4)}+2\nu^{(5)}) \;. \tag{A.13}$$

Next, R_I can be expressed in terms of R_{III}:

$$\frac{1}{2}R_I \;=\; R_{II}+\frac{1}{2}R_{III} \;=\; -(R_{IV}+\nu^{(2)})+\frac{1}{2}R_{III} \;=\; R_{III}-\nu^{(2)} \;. \tag{A.14}$$

Inserting eqs. (A.13) and (A.14) into eq. (A.1) yields the main eq. (9.2):

$$R \;=\; -8\nu_{\rho}^{(0)}-\nu^{(1)}-\frac{1}{2}R_{I} \;=\; -8\nu_{\rho}^{(0)}-\nu^{(1)}+\nu^{(2)}+\frac{1}{3}(\nu^{(3)}-\nu^{(4)}-2\nu^{(5)}) \;=\; -8\nu_{\rho}\;. \qquad (A.15)$$

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Paper V

A Comparative Study of Laplacians and Schroedinger-Lichnerowicz-Weitzenboeck Identities in Riemannian and Antisymplectic Geometry

BY

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J. Math. Phys. **50** (2009) 073504

arXiv:0809.4269

A Comparative Study of Laplacians and Schrödinger–Lichnerowicz–Weitzenböck Identities in Riemannian and Antisymplectic Geometry

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June 14, 2013

Abstract

We introduce an antisymplectic Dirac operator and antisymplectic gamma matrices. We explore similarities between, on one hand, the Schrödinger–Lichnerowicz formula for spinor bundles in Riemannian spin geometry, which contains a zeroth–order term proportional to the Levi–Civita scalar curvature, and, on the other hand, the nilpotent, Grassmann–odd, second–order Δ operator in antisymplectic geometry, which in general has a zeroth–order term proportional to the odd scalar curvature of an arbitrary antisymplectic and torsionfree connection that is compatible with the measure density. Finally, we discuss the close relationship with the two–loop scalar curvature term in the quantum Hamiltonian for a particle in a curved Riemannian space.

MSC number(s): 15A66; 53A55; 53B20; 58A50; 58C50.
Keywords: Dirac Operator; Spin Representations; BV Field–Antifield Formalism; Antisymplectic Geometry; Odd Laplacian.

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Contents

1 Introduction

What do Riemannian and antisymplectic geometry have in common? The short answer is that out of the $2 \times 2 = 4$ classical classes of even and odd, Riemannian and symplectic geometries, they are the only two possibilities that possess non–trivial Laplacians, scalar curvatures and Weitzenböck–type identities, cf. Table 1. Our present investigation is partly spurred by the following remarkable fact. On one hand, one has the nilpotent, Grassmann–odd Δ operator, which plays a fundamental rôle in antisymplectic geometry, and which helps encode the BRST symmetry in the field–antifield formalism [1, 2, 3]. It can be written as [4]

$$2\Delta \;=\; 2\Delta_{\rho} - \frac{R}{4} \qquad \text{(antisymplectic)} \qquad (1.0.1)$$

where Δ_{ρ} is the odd Laplacian, and R is the odd scalar curvature of an arbitrary antisymplectic, torsionfree and ρ–compatible connection $\nabla^{(\Gamma)} = d + \Gamma$. On the other hand, on a Riemannian spin manifold, one has the Schrödinger–Lichnerowicz formula [5, 6]

$$D^{(\sigma)} D^{(\sigma)} \;=\; \Delta^{(\sigma)}_{\rho_g} - \frac{R}{4} \qquad \text{(Riemannian)} \qquad (1.0.2)$$

where $D^{(\sigma)}$ is the Dirac operator, $\Delta^{(\sigma)}_{\rho_g}$ is the spinor Laplacian, and R is the scalar Levi–Civita curvature. The formula (1.0.1) has been multiplied with a factor of 2 to ease comparison with formula (1.0.2) because of the standard practice to normalize odd Laplacians with an internal factor 1/2. In both formulas (1.0.1) and (1.0.2), the coefficient in front of the zeroth–order scalar curvature term is exactly the same, namely minus a quarter! Of course, there are crucial differences between eqs. (1.0.1) and (1.0.2). The second–order operators in eq. (1.0.1) act on scalar functions, while the Dirac operator $D^{(\sigma)}$ and the Laplacian $\Delta^{(\sigma)}_{\rho_g}$ in eq. (1.0.2) act on spinors, as the index "σ" is meant to indicate. (The subscript $\rho_g \equiv \sqrt{g}$ refers to the canonical Riemannian density.)

Our investigation can roughly be divided in three parts. The first part (which is mainly covered in Subsections 3.1–3.5, 3.9 and 4.1–4.4) is to define a Grassmann–even Riemannian analogue of the odd Δ operator (1.0.1), that takes scalars in scalars:

$$\Delta_{\rho_g} - \frac{R}{4} \qquad \text{(Riemannian)} \ . \tag{1.0.3}$$

Here Δ_{ρ_g} is the Laplace–Beltrami operator and R is the Levi–Civita scalar curvature. The zeroth–order term $-R/4$ in the even operator (1.0.3) is special in several ways (as compared to other choices of the zeroth–order term). For instance, the even operator (1.0.3) with this particular zeroth–order term $-R/4$ is closely related to the quantum Hamiltonian $\hat{H}$ for a particle moving in the Riemannian manifold [7, 8, 9, 10, 11, 12, 13, 14, 15, 16], cf. Subsection 3.10. Central to our investigation is the fact that the zeroth–order term $-R/4$ also possesses a special mathematical property. To see this property, one notes that it is possible to uniquely identify how all zeroth–order terms depend on the canonical Riemannian density ρ_g, due to a classification of scalar invariants, see Proposition 3.2. Therefore it is possible to consistently replace all the appearances of ρ_g with an arbitrary density ρ. One may now show that the ρ–lifted version of the operator (1.0.3) is the unique operator such that the $\sqrt{\rho}$–conjugated operator is independent of ρ. That's the special property. This has parallels to antisymplectic geometry, where the odd Δ operator (1.0.1) shares a similar characterization. In antisymplectic geometry, the $\sqrt{\rho}$–conjugated operator

$$\Delta_E \ = \ \sqrt{\rho}\Delta\frac{1}{\sqrt{\rho}} \qquad \text{(antisymplectic)} \tag{1.0.4}$$

is precisely Khudaverdian's Δ_E operator [17, 18, 19, 20, 21, 22, 23]. The Δ_E operator (1.0.4) is distinguished by being nilpotent and independent of ρ. In fact, when one tracks the equations in detail, it is possible to see that the same coefficient $-1/4$ in front of the odd and even scalar curvature terms in eqs. (1.0.1) and (1.0.3) is not a coincidence, but indeed follows from the same underlying principle of ρ–independence. Thus it establishes a bridge between the odd and even operators (1.0.1) and (1.0.3).

We should also mention that the even operator (1.0.3) is often compared with the conformally covariant Laplacian

$$\Delta_{\rho_g} - \frac{(N-2)R}{(N-1)4} \qquad \text{(Riemannian)} \tag{1.0.5}$$

where $N = \dim(M)$ is the dimension of the Riemannian manifold M. The zeroth–order term $-R/4$ corresponds to $N = \infty$.

The second part (which is covered in Subsections 6.4–6.10) is to check within Riemannian geometry, if there is a bridge between the even operator (1.0.3) that acts on scalar functions, and the square of the Dirac operator (1.0.2) that acts on the spinor bundle $\mathcal{S}$. There is a well–defined group–theoretical procedure how to compare scalars and spinors. Firstly, the Dirac operator is extended to a Dirac operator that acts on the bispinor bundle $\mathcal{S} \otimes \mathcal{S}^T$. The Clebsch–Gordan decomposition $\mathcal{S} \otimes \mathcal{S}^T = \underline{\mathbf{1}} \oplus \ldots$, in turn, contains a singlet representation, *i.e.*, a scalar invariant, which is denoted as $||s\rangle\rangle$. Thus one just has to project the square of the bispinor Dirac operator to the singlet representation to obtain an operator that acts on scalars. Somewhat surprisingly, the operator turns out to be just the bare Laplace–Beltrami operator Δ_{ρ_g} with *no* zeroth–order term at all, cf. Theorem 6.6. Roughly speaking, after the projection to the singlet state $||s\rangle\rangle$, the $-R/4$ curvature term in the spinor sector $\mathcal{S}$ is canceled by an opposite amount $+R/4$ in the transposed spinor sector $\mathcal{S}^T$. So we have to conclude for the second part, that the above group–theoretical procedure yields *no* relation between the even operator (1.0.3) that acts on scalar functions, and the square of the Dirac operator (1.0.2), despite the fact that they both contain the same $-R/4$ term!

The third part develops the antisymplectic side. It is spurred by the following questions.

1. Do there exist antisymplectic Clifford algebras and spinors?
2. Does there exists a natural spinor generalization $\Delta^{(\sigma)}$ of the odd Δ operator (1.0.1), which takes antisymplectic spinors to antisymplectic spinors?
3. Can the odd $\Delta^{(\sigma)}$ operator from question 2 be written as a square

$$\Delta^{(\sigma)} \stackrel{?}{=} D^{(\sigma)} \star D^{(\sigma)} \qquad \text{(antisymplectic)} \tag{1.0.6}$$

   of an antisymplectic Dirac operator $D^{(\sigma)} = \gamma^A \nabla^{(\sigma)}_A$, where "$\star$" is a Fermionic multiplication, $\varepsilon(\star) = 1$, and γ^A are antisymplectic γ matrices?

The answers, which will be derived in detail in Sections 4 and 7, are, by most standards, "*no*" to question 3, and "*yes, there exists a first–order formalism, but there is no second–order formalism*" to question 1 and 2. Here the first– and second–order formalism refer to the realizations of the Lie–algebras of infinitesimal frame and coordinate changes in terms of first– and second–order differential operators, respectively. The obstacle in eq. (1.0.6) lies in the definition of the $\star$ multiplication. We shall, however, introduce a Fermionic nilpotent parameter θ that can be though of as the inverse $\star^{-1}$, but since such θ parameter by definition is not invertible, the $\star$ multiplication itself becomes meaningless. The trick is therefore, roughly speaking, to multiply both side of eq. (1.0.6) with $\theta \equiv \star^{-1}$, cf. Theorem 4.4 and Theorem 7.1.

At the coarsest level, the main text is organized into $3 \times 2 = 6$ sections. The three Sections 2–4 are devoted to general (=not necessarily spin) manifolds, while the next three Sections 5–7 deal exclusively with spin manifolds. Sections 3 and 6 consider the Riemannian case, and Sections 4 and 7 consider the antisymplectic case, while Sections 2 and 5 consider the general theory that is common for both Riemannian and antisymplectic case. The general theory Sections 2 and 5 explain differential geometry, such as, connections, torsion tensors, vielbeins, flat and curved exterior forms, etc., in the context of supermanifolds, where sign factors are important. The Riemannian curvature tensor, the Ricci tensor and the scalar curvature are considered in Subsections 2.4–2.6, 3.7–3.8 and 4.6–4.7. Finally, Section 8 has our conclusions.

1.1 General Remarks About Notation

Adjectives from supermathematics such as "graded", "super", etc., are implicitly implied. The sign conventions are such that two exterior forms ξ and η, of Grassmann–parity ε_ξ, ε_η and of form–degree p_ξ, p_η, commute in the following graded sense:

$$\eta \wedge \xi \;=\; (-1)^{\varepsilon_\xi \varepsilon_\eta + p_\xi p_\eta} \xi \wedge \eta \;\equiv\; (-1)^{\vec{\varepsilon}_\xi \cdot \vec{\varepsilon}_\eta} \xi \wedge \eta \tag{1.1.1}$$

inside the exterior algebra. The pair (ε, p) acts as a 2–dimensional vector–valued Grassmann–parity

$$\vec{\varepsilon} \;:=\; \begin{bmatrix} \varepsilon \\ p \pmod 2 \end{bmatrix} , \tag{1.1.2}$$

Table 1: The $2 \times 2 = 4$ classical geometries and their symmetries [18]. Only even Riemannian and antisymplectic geometries have non–trivial Laplacians, scalar curvatures and Weitzenböck–type identities.

	Even Geometry	Odd Geometry
Riemannian Covariant Metric	$g = Y^A g_{AB} \vee Y^B$ $\varepsilon(g_{AB}) = \varepsilon_A + \varepsilon_B$ $g_{BA} = -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} g_{AB}$ Symmetric No Closeness Relation	$g = Y^A g_{AB} \vee Y^B$ $\varepsilon(g_{AB}) = \varepsilon_A + \varepsilon_B + 1$ $g_{BA} = (-1)^{\varepsilon_A \varepsilon_B} g_{AB}$ Symmetric No Closeness Relation
Inverse Riemannian Contravariant Metric	$\varepsilon(g^{AB}) = \varepsilon_A + \varepsilon_B$ $g^{BA} = (-1)^{\varepsilon_A \varepsilon_B} g^{AB}$ Symmetric Even Laplacian	$\varepsilon(g^{AB}) = \varepsilon_A + \varepsilon_B + 1$ $g^{BA} = (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} g^{AB}$ Skewsymmetric No Laplacian
Symplectic Covariant Two–Form	$\omega = \frac{1}{2} C^A \omega_{AB} \wedge C^B$ $\varepsilon(\omega_{AB}) = \varepsilon_A + \varepsilon_B$ $\omega_{BA} = (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} \omega_{AB}$ Skewsymmetric Closeness Relation	$E = \frac{1}{2} C^A E_{AB} \wedge C^B$ $\varepsilon(E_{AB}) = \varepsilon_A + \varepsilon_B + 1$ $E_{BA} = -(-1)^{\varepsilon_A \varepsilon_B} E_{AB}$ Skewsymmetric Closeness Relation
Inverse Symplectic Contravariant Tensor	$\varepsilon(\omega^{AB}) = \varepsilon_A + \varepsilon_B$ $\omega^{BA} = -(-1)^{\varepsilon_A \varepsilon_B} \omega^{AB}$ Skewsymmetric No Laplacian	$\varepsilon(E^{AB}) = \varepsilon_A + \varepsilon_B + 1$ $E^{BA} = -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} E^{AB}$ Symmetric Odd Laplacian

as indicated in the second equality of eq. (1.1.1). The first component carries ordinary Grassmann–parity ε, while the second component carries form–parity, *i.e.*, form degree modulo two. The exterior wedge symbol "$\wedge$" is often not written explicitly, as it is redundant information that can be deduced from the Grassmann– and form–parity. The commutator $[F, G]$ and anticommutator $\{F, G\}_+$ of two operators F and G are

$$[F, G] \;:=\; FG - (-1)^{\varepsilon_F \varepsilon_G + p_F p_G} GF \;\equiv\; FG - (-1)^{\vec{\varepsilon}_F \cdot \vec{\varepsilon}_G} GF \;, \qquad (1.1.3)$$

$$\{F, G\}_+ \;:=\; FG + (-1)^{\varepsilon_F \varepsilon_G + p_F p_G} GF \;\equiv\; FG + (-1)^{\vec{\varepsilon}_F \cdot \vec{\varepsilon}_G} GF \;. \qquad (1.1.4)$$

The commutator (1.1.3) fulfills the Jacobi identity

$$\sum_{\text{cycl. } F,G,H} (-1)^{\vec{\varepsilon}_F \cdot \vec{\varepsilon}_H} [F, [G, H]] \;=\; 0 \;. \qquad (1.1.5)$$

The transposed of a product of operators is:

$$(FG)^T \;=\; (-1)^{\varepsilon_F \varepsilon_G + p_F p_G} G^T F^T \;\equiv\; (-1)^{\vec{\varepsilon}_F \cdot \vec{\varepsilon}_G} G^T F^T \;. \qquad (1.1.6)$$

Covariant and exterior derivatives will always be from the left, while partial derivatives can be from either left or right. We shall sometimes use round parenthesis "()" to indicate how far derivatives act, see *e.g.*, eqs. (2.3.3), (3.3.2), (3.4.2) and (3.4.3) below.

2 General Theory

2.1 Connection $\nabla^{(\Gamma)} = d + \Gamma$

Let there be given a manifold M with local coordinates z^A of Grassmann–parity $\varepsilon(z^A) = \varepsilon_A$ (and form–degree $p(z^A) = 0$). Assume that M is endowed with a measure density ρ. Let $\Gamma(TM)$ denote the set of sections in the tangent bundle TM, *i.e.*, the set of vector fields on M. Let M be endowed with a tangent bundle connection $\nabla^{(\Gamma)} = d + \Gamma = dz^A \otimes \nabla^{(\Gamma)}_A \ : \ \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$

$$\nabla^{(\Gamma)}_A \ = \ \frac{\overrightarrow{\partial^{\ell}}}{\partial z^A} + \partial^r_B \ \Gamma^B{}_{AC} \ \overrightarrow{dz^C} \ . \qquad (2.1.1)$$

Here $\partial^r_A \equiv (-1)^{\varepsilon_A} \partial^{\ell}_A$ are not usual partial derivatives. In particular, they do not act on the Christoffel symbols $\Gamma^B{}_{AC}$ in eq. (2.1.1). Rather they are a dual basis to the one–forms $\overrightarrow{dz^A}$:

$$\overrightarrow{dz^A} \ (\partial^r_B) \ = \ \delta^A_B \ , \qquad \varepsilon(\overrightarrow{dz^A}) \ = \ \varepsilon_A \ = \ \varepsilon(\partial^r_A) \ . \qquad (2.1.2)$$

Phrased differently, the ∂^r_A are merely bookkeeping devices, that transform as right partial derivatives under general coordinate transformations. (To be able to distinguish them from true partial derivatives, the differentiation variable z^A on a true partial derivative $\partial/\partial z^A$ is written explicitly.) For fixed index "A" in eq. (2.1.1), the Christoffel symbol $\Gamma^B{}_{AC}$ is a matrix with respect to index "B" and index "C", and $\partial^r_B \ \Gamma^B{}_{AC} \ \overrightarrow{dz^C}$ is the corresponding linear operator: $TM \to TM$. (We shall often refer to a linear operator by its matrix, and vice–versa.)

The form–parities $p(\overrightarrow{dz^A}) = p(\partial^r_A)$ are either all 0 or all 1, depending on applications, whereas a 1–form dz^A with no arrow "$\rightarrow$" always carries odd form–parity $p(dz^A) = 1$ (and Grassmann–parity $\varepsilon(dz^A) = \varepsilon_A$).

2.2 Torsion

The torsion tensor $T^{(\Gamma)} : \ \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ is defined as

$$\begin{aligned} T^{(\Gamma)} \ &\equiv \ \frac{1}{2} dz^A \wedge \partial^r_B \ T^{(\Gamma)B}{}_{AC} \ dz^C \ := \ [\nabla^{(\Gamma)} \overset{\wedge}{,} \mathrm{Id}] \\ &= \ [dz^A \frac{\overrightarrow{\partial^{\ell}}}{\partial z^A} + dz^A \ \partial^r_B \ \Gamma^B{}_{AD} \ \overrightarrow{dz^D} \overset{\wedge}{,} \partial^r_C \ dz^C] \ = \ dz^A \wedge \partial^r_B \ \Gamma^B{}_{AC} \ dz^C \ . \end{aligned} \qquad (2.2.1)$$

where it is implicitly understood that there are no contractions with base manifold indices, in this case index "A" and index "C". As expected, the torsion tensor is just an antisymmetrization of the Christoffel symbol $\Gamma^B{}_{AC}$ with respect to the lower indices,

$$T^{(\Gamma)A}{}_{BC} \ := \ \Gamma^A{}_{BC} + (-1)^{(\varepsilon_B+1)(\varepsilon_C+1)} (B \leftrightarrow C) \ . \qquad (2.2.2)$$

In particular, the Christoffel symbol

$$\Gamma^A{}_{BC} \ = \ -(-1)^{(\varepsilon_B+1)(\varepsilon_C+1)} (B \leftrightarrow C) \qquad (2.2.3)$$

is symmetric with respect to the lower indices when the connection is torsionfree.

2.3 Divergence

A connection $\nabla^{(\Gamma)}$ can be used to define a divergence of a Bosonic vector field X^A as

$$\mathrm{str}(\nabla^{(\Gamma)}X) \;\equiv\; (-1)^{\varepsilon_A}(\nabla^{(\Gamma)}_A X)^A \;=\; ((-1)^{\varepsilon_A}\frac{\overrightarrow{\partial^\ell}}{\partial z^A}+\Gamma^B{}_{BA})X^A \;, \qquad \varepsilon_X \;=\; 0 \;. \tag{2.3.1}$$

On the other hand, the divergence is defined in terms of ρ as

$$\mathrm{div}_\rho X \;:=\; \frac{(-1)^{\varepsilon_A}}{\rho}\frac{\overrightarrow{\partial^\ell}}{\partial z^A}(\rho X^A) \;. \tag{2.3.2}$$

See Ref. [24] for a mathematical exposition of divergence operators on supermanifolds. The $\nabla^{(\Gamma)}$ connection is called compatible with the measure density ρ if

$$\Gamma^B{}_{BA} \;=\; (-1)^{\varepsilon_A}(\frac{\overrightarrow{\partial^\ell}}{\partial z^A}\ln\rho) \;. \tag{2.3.3}$$

In this case, the two definitions (2.3.1) and (2.3.2) of divergence agree, cf. Ref. [4].

2.4 The Riemann Curvature

We discuss in this Subsection 2.4 the Riemann curvature tensor on a supermanifold [25]. See Ref. [12] and Ref. [26] for related discussions. The Riemann curvature $R^{(\Gamma)}$ is defined as (half) the commutator of the $\nabla^{(\Gamma)}$ connection (2.1.1),

$$\begin{aligned} R^{(\Gamma)} \;&=\; \frac{1}{2}[\nabla^{(\Gamma)} \overset{\wedge}{,} \nabla^{(\Gamma)}] \;=\; -\frac{1}{2}dz^B\wedge dz^A \;\otimes\; [\nabla^{(\Gamma)}_A, \nabla^{(\Gamma)}_B] \\ &=\; -\frac{1}{2}dz^B\wedge dz^A \;\otimes\; \partial^r_D\; R^D{}_{ABC}\;\; \overrightarrow{dz^C} \;, \end{aligned} \tag{2.4.1}$$

where it is implicitly understood that there are no contractions with base manifold indices, in this case index "A" and index "B". (For a torsionfree connection such contractions vanish, and there is no ambiguity.)

$$\begin{aligned} R^D{}_{ABC} \;&=\; \overrightarrow{dz^D}\left([\nabla^{(\Gamma)}_A, \nabla^{(\Gamma)}_B]\partial^r_C\right) \\ &=\; (-1)^{\varepsilon_D\varepsilon_A}(\frac{\overrightarrow{\partial^\ell}}{\partial z^A}\Gamma^D{}_{BC}) + \Gamma^D{}_{AE}\;\Gamma^E{}_{BC} - (-1)^{\varepsilon_A\varepsilon_B}(A\leftrightarrow B) \;, \end{aligned} \tag{2.4.2}$$

Note that the order of indices in the Riemann curvature tensor $R^D{}_{ABC}$ is non–standard. This is to minimize appearances of Grassmann sign factors. Alternatively, the Riemann curvature tensor may be defined as

$$R(X,Y)Z \;=\; \left([\nabla^{(\Gamma)}_X, \nabla^{(\Gamma)}_Y] - \nabla^{(\Gamma)}_{[X,Y]}\right)Z \;=\; Y^B X^A R_{AB}{}^D{}_C Z^C\;\partial^\ell_D \;, \tag{2.4.3}$$

where $X = X^A\partial^\ell_A$, $Y = Y^B\partial^\ell_B$ and $Z = Z^C\partial^\ell_C$ are left vector field of even Grassmann– and form–parity. The Riemann curvature tensor $R_{AB}{}^D{}_C$ reads in local coordinates

$$R_{AB}{}^D{}_C \;=\; (-1)^{\varepsilon_D(\varepsilon_A+\varepsilon_B)}R^D{}_{ABC} \;=\; (\frac{\overrightarrow{\partial^\ell}}{\partial z^A}\Gamma_B{}^D{}_C) + (-1)^{\varepsilon_B\varepsilon_D}\Gamma_A{}^D{}_E\;\Gamma^E{}_{BC} - (-1)^{\varepsilon_A\varepsilon_B}(A\leftrightarrow B) \;. \tag{2.4.4}$$

Here we have introduced a reordered Christoffel symbol

$$\Gamma_{A}{}^{B}{}_{C} \ := \ (-1)^{\varepsilon_{A}\varepsilon_{B}}\Gamma^{B}{}_{AC} \ . \tag{2.4.5}$$

It is sometimes useful to reorder the indices in the Riemann curvature tensors as

$$R_{ABC}{}^{D} \ = \ ([\nabla_{A},\nabla_{B}]\partial^{\ell}_{C})^{D} \ = \ (-1)^{\varepsilon_{C}(\varepsilon_{D}+1)}R_{AB}{}^{D}{}_{C} \ . \tag{2.4.6}$$

Note that all expressions (2.4.2), (2.4.4) and (2.4.6) of Riemann curvature tensor are antisymmetric under an $(A\leftrightarrow B)$ exchange of index "A" and "B". The first Bianchi identity reads (in the torsionfree case):

$$0 \ = \ \sum_{\text{cycl. } A,B,C}(-1)^{\varepsilon_{A}\varepsilon_{C}}R_{ABC}{}^{D} \ . \tag{2.4.7}$$

We have exceptionally used the convention $p(\partial^{\ell}_{A}){=}0$ in eqs. (2.4.3) and (2.4.6).

2.5 The Ricci Tensor

The Ricci tensor is defined as

$$R_{AB} \ := \ R^{C}{}_{CAB} \ . \tag{2.5.1}$$

The Ricci tensor becomes symmetric

$$\begin{aligned} R_{AB} \ &= \ \frac{(-1)^{\varepsilon_{C}}}{\rho}\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{C}}(\rho\Gamma^{C}{}_{AB}) - (\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}\ln\rho\frac{\overleftarrow{\partial^{r}}}{\partial z^{B}}) - \Gamma_{A}{}^{D}{}_{C}\ \Gamma^{C}{}_{DB} \\ &= \ -(-1)^{(\varepsilon_{A}+1)(\varepsilon_{B}+1)}(A\leftrightarrow B) \ , \end{aligned} \tag{2.5.2}$$

when the $\nabla^{(\Gamma)}$ connection is torsionfree $T^{(\Gamma)}{=}0$ and ρ–compatible (2.3.3).

2.6 The Ricci Two–Form

The Ricci two–form is defined as

$$\mathcal{R}_{AB} \ := \ R_{AB}{}^{C}{}_{C}(-1)^{\varepsilon_{C}} \ = \ -(-1)^{\varepsilon_{A}\varepsilon_{B}}(A\leftrightarrow B) \ . \tag{2.6.1}$$

The Ricci two–form vanishes

$$\mathcal{R}_{AB} \ = \ 0 \ , \tag{2.6.2}$$

when the $\nabla^{(\Gamma)}$ connection is torsionfree $T^{(\Gamma)}{=}0$ and ρ–compatible (2.3.3).

2.7 Covariant Tensors

Let

$$\Omega_{mn}(M) \ := \ \Gamma\left(\bigwedge{}^{m}(T^{*}M)\otimes\bigvee{}^{n}(T^{*}M)\right) \tag{2.7.1}$$

be the vector space of $(0,m{+}n)$–tensors $\eta_{A_{1}\cdots A_{m}B_{1}\cdots B_{n}}(z)$ that are antisymmetric with respect to the first m indices $A_{1}\ldots A_{m}$, and symmetric with respect to the last n indices $B_{1}\ldots B_{n}$. As usual, it is practical to introduce a coordinate–free notation

$$\eta(z;C;Y) \ = \ \frac{1}{m!n!}C^{A_{m}}\wedge\cdots\wedge C^{A_{1}}\ \eta_{A_{1}\cdots A_{m}B_{1}\cdots B_{n}}(z)\otimes Y^{B_{n}}\vee\cdots\vee Y^{B_{1}} \ . \tag{2.7.2}$$

Here the variables Y^A are symmetric counterparts to the one–form basis $C^A \equiv dz^A$.

$$\begin{array}{rclcrclcrcl} C^A \wedge C^B &=& -(-1)^{\varepsilon_A\varepsilon_B} C^B \wedge C^A \ , && \varepsilon(C^A) &=& \varepsilon_A \ , && p(C^A) &=& 1 \ , \\ Y^A \vee Y^B &=& (-1)^{\varepsilon_A\varepsilon_B} Y^B \vee Y^A \ , && \varepsilon(Y^A) &=& \varepsilon_A \ , && p(Y^A) &=& 0 \ . \end{array} \tag{2.7.3}$$

The covariant derivative can be realized on covariant tensors $\eta \in \Omega_{mn}(M)$ by a linear differential operator

$$\nabla^{(T)}_A \ = \ \frac{\overrightarrow{\partial^\ell}}{\partial z^A} - \Gamma_A{}^B{}_C \ T^C{}_B \ , \tag{2.7.4}$$

where

$$T^A{}_B \ := \ C^A \frac{\overrightarrow{\partial^\ell}}{\partial C^B} + Y^A \frac{\overrightarrow{\partial^\ell}}{\partial Y^B} \tag{2.7.5}$$

are themselves linear differential operators. They are generators of the general linear ($= gl$) Lie–algebra,

$$[T^A{}_B, T^C{}_D] \ = \ \delta^C_B \ T^A{}_D - (-1)^{(\varepsilon_A+\varepsilon_B)(\varepsilon_C+\varepsilon_D)} \delta^A_D \ T^C{}_B \ . \tag{2.7.6}$$

It is important for the implementation (2.7.4) to make sense that η carries no explicit indices, *i.e.*, all indices should be paired as indicated in eq. (2.7.2). The Lie–algebra (2.7.6) reflects infinitesimal coordinate transformation, *i.e.*, diffeomorphism invariance.

2.8 Coordinate Transformations

Consider for simplicity a one–form $\eta = \eta_A(z) C^A \in \Omega_{10}(M)$. The covariant derivative reads

$$(\nabla_A \eta)_C \ = \ (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} \eta_C) - \eta_B \ \Gamma^B{}_{AC} \ . \tag{2.8.1}$$

Under a coordinate transformation $z^A \to z'^A$ one has

$$\eta_A \ = \ \eta'_B (z'^B \frac{\overleftarrow{\partial^r}}{\partial z^A}) \ , \tag{2.8.2}$$

$$C'^A \ = \ (z'^A \frac{\overleftarrow{\partial^r}}{\partial z^B}) C^B \ = \ C^B (\frac{\overrightarrow{\partial^\ell}}{\partial z^B} z'^A) \ , \tag{2.8.3}$$

$$(-1)^{\varepsilon_A\varepsilon_B} (z'^B \frac{\overleftarrow{\partial^r}}{\partial z^D}) \Gamma^D{}_{AC} \ = \ (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} z'^B \frac{\overleftarrow{\partial^r}}{\partial z^C}) + (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} z'^D) \Gamma'_D{}^B{}_E (z'^E \frac{\overleftarrow{\partial^r}}{\partial z^C}) \ , \tag{2.8.4}$$

so that the covariant derivative transforms covariantly,

$$(\nabla_A \eta)_D \ = \ (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} z'^B)(\nabla_{'B} \eta')_C (z'^C \frac{\overleftarrow{\partial^r}}{\partial z^D}) \ . \tag{2.8.5}$$

3 Riemannian Geometry

3.1 Metric

Let there be given a (pseudo) Riemannian metric, *i.e.*, a covariant symmetric $(0,2)$ tensor field

$$g \ = \ Y^A \ g_{AB} \vee Y^B \ \in \ \Omega_{02}(M) \ , \tag{3.1.1}$$

of Grassmann–parity

$$\varepsilon(g_{AB}) \;=\; \varepsilon_A+\varepsilon_B \;, \qquad \varepsilon(g) \;=\; 0 \;, \qquad p(g_{AB}) \;=\; 0 \;, \tag{3.1.2}$$

and of symmetry

$$g_{BA} \;=\; -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}g_{AB} \;. \tag{3.1.3}$$

We shall not need nor discuss positivity/reality/Hermiticity–conditions in this paper (except for the application to a particle in a curved space, cf. Subsection 3.10). The symmetry (3.1.3) becomes more transparent if one reorders the Riemannian metric as

$$g \;=\; Y^B \vee Y^A \tilde{g}_{AB} \;, \tag{3.1.4}$$

where

$$\tilde{g}_{AB} \;:=\; g_{AB}(-1)^{\varepsilon_B} \;. \tag{3.1.5}$$

Then the symmetry (3.1.3) simply reads

$$\tilde{g}_{BA} \;=\; (-1)^{\varepsilon_A\varepsilon_B}\tilde{g}_{AB} \;. \tag{3.1.6}$$

The Riemannian metric g_{AB} is assumed to be non–degenerate, *i.e.*, there exists an inverse contravariant symmetric $(2,0)$ tensor field g^{AB} such that

$$g_{AB}\; g^{BC} \;=\; \delta_A^C \;. \tag{3.1.7}$$

The inverse g^{AB} has Grassmann–parity

$$\varepsilon(g^{AB}) \;=\; \varepsilon_A+\varepsilon_B \;, \tag{3.1.8}$$

and symmetry

$$g^{BA} \;=\; (-1)^{\varepsilon_A\varepsilon_B}g^{AB} \;. \tag{3.1.9}$$

The canonical density on a Riemannian manifold is

$$\rho_g \;:=\; \sqrt{g} \;:=\; \sqrt{\mathrm{sdet}(g_{AB})} \;. \tag{3.1.10}$$

This should be compared with the antisymplectic case, where the density ρ is kept arbitrary, since there is no canonical choice [25]. To ease comparison, we shall temporarily allow for arbitrary densities ρ in the Riemannian case as well.

3.2 Laplacian Δ_ρ

A Laplacian Δ_ρ, which takes scalar functions to scalar functions, can be constructed from the inverse metric g^{AB} and a (not necessarily canonical) density ρ,

$$\Delta_\rho \;:=\; \frac{(-1)^{\varepsilon_A}}{\rho}\frac{\overrightarrow{\partial^\ell}}{\partial z^A}\rho g^{AB}\frac{\overrightarrow{\partial^\ell}}{\partial z^B} \;, \qquad \varepsilon(\Delta_\rho) \;=\; 0 \;, \qquad p(\Delta_\rho) \;=\; 0 \;. \tag{3.2.1}$$

A metric bracket (f,g) of two functions $f\!=\!f(z)$ and $g\!=\!g(z)$ can be defined via a double commutator with the Laplacian, acting on the constant unit function 1,

$$\begin{aligned}(f,g) \;&:=\; \frac{1}{2}[[\overrightarrow{\Delta}_\rho,f],g]1 \;\equiv\; \frac{1}{2}\Delta_\rho(fg)-\frac{1}{2}(\Delta_\rho f)g-\frac{1}{2}f(\Delta_\rho g)+\frac{1}{2}fg(\Delta_\rho 1)\\ &=\; (f\frac{\overleftarrow{\partial^r}}{\partial z^A})g^{AB}(\frac{\overrightarrow{\partial^\ell}}{\partial z^B}g) \;=\; (-1)^{\varepsilon_f\varepsilon_g}(g,f) \;.\end{aligned} \tag{3.2.2}$$

There are *no* closeness relations (resp. Jacobi identities) associated with the Riemannian g_{AB} metric (3.1.4) (resp. metric $(\cdot,\cdot)$ bracket (3.2.2)) in contrast to symplectic situations. In fact, even if such closeness relations and Jacobi identities were to be artificially enforced in one coordinate patch, they would not transform covariantly under general coordinate transformations $z^A\to z'^B$. See also Subsection 3.1 in Ref. [27].

3.3 Two–cocycle $\nu(\rho';\rho,g)$

It is possible to introduce a Riemannian analogue of the two–cocycle of Khudaverdian and Voronov [18, 21, 4]. The two–cocycle $\nu(\rho';\rho,g)$ is a function of a measure density ρ' with respect to a reference system (ρ,g),

$$\nu(\rho';\rho,g) \;:=\; \sqrt{\frac{\rho}{\rho'}}(\Delta_{\rho}\sqrt{\frac{\rho'}{\rho}}) \;=\; \nu^{(0)}_{\rho'}-\nu^{(0)}_{\rho} \;, \tag{3.3.1}$$

where

$$\nu^{(0)}_{\rho} \;:=\; \frac{1}{\sqrt{\rho}}(\Delta_{1}\sqrt{\rho}) \;=\; -\sqrt{\rho}(\Delta_{\rho}\frac{1}{\sqrt{\rho}}) \;=\; (\Delta_{1}\ln\sqrt{\rho})+(\ln\sqrt{\rho},\ln\sqrt{\rho}) \;. \tag{3.3.2}$$

Here Δ_1 is the Laplacian (3.2.1) with $\rho=1$. The expression (3.3.1) acts as a scalar under general coordinate transformations, and satisfies the following two–cocycle condition:

$$\nu(\rho_1;\rho_2,g)+\nu(\rho_2;\rho_3,g)+\nu(\rho_3;\rho_1,g) \;=\; 0 \;. \tag{3.3.3}$$

In fact, it is a two–coboundary, because we shall prove in the next Subsection 3.4, that there exists a scalar ν_{ρ}, such that

$$\nu(\rho';\rho,g) \;=\; \nu_{\rho'}-\nu_{\rho} \;. \tag{3.3.4}$$

3.4 Scalar ν_{ρ}

A Grassmann–even function ν_{ρ} can be constructed from the metric g and a (not necessarily canonical) density ρ as

$$\nu_{\rho} \;:=\; \nu^{(0)}_{\rho}+\frac{\nu^{(1)}}{4}-\frac{\nu^{(2)}}{8}-\frac{\nu^{(3)}}{16} \;, \tag{3.4.1}$$

where $\nu^{(0)}_{\rho}$ is given by eq. (3.3.2), and

$$\nu^{(1)} \;:=\; (-1)^{\varepsilon_A}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^A}g^{AB}\frac{\overleftarrow{\partial^{r}}}{\partial z^B})(-1)^{\varepsilon_B} \;, \tag{3.4.2}$$

$$\begin{aligned}\nu^{(2)} \;&:=\; -(-1)^{\varepsilon_C}(z^C,(z^B,z^A))(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^A}g_{BC}) \\ &=\; -(-1)^{(\varepsilon_A+1)(\varepsilon_D+1)}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^D}g^{AB})g_{BC}(g^{CD}\frac{\overleftarrow{\partial^{r}}}{\partial z^A}) \;, \end{aligned} \tag{3.4.3}$$

$$\nu^{(3)} \;:=\; (-1)^{\varepsilon_A}(g_{AB},g^{BA}) \;. \tag{3.4.4}$$

Here $(\cdot,\cdot)$ is the metric bracket (3.2.2).

Lemma 3.1 *The even quantity ν_{ρ} is a scalar, i.e., it does not depend on the coordinate system.*

PROOF OF LEMMA 3.1: Under an arbitrary infinitesimal coordinate transformation $\delta z^A = X^A$, one calculates (by using methods similar to the antisymplectic case [22])

$$\delta\nu^{(0)}_{\rho} \;=\; -\frac{1}{2}\Delta_{1}\mathrm{div}_{1}X \;, \tag{3.4.5}$$

$$\delta\nu^{(1)} \;=\; 2\Delta_{1}\mathrm{div}_{1}X+(-1)^{\epsilon_C}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^C}g^{AB})(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^B}\frac{\overrightarrow{\partial^{\ell}}}{\partial z^A}X^C) \;, \tag{3.4.6}$$

$$\delta\nu^{(2)} \;=\; 2(-1)^{\epsilon_C}(\frac{\overrightarrow{\partial^\ell}}{\partial z^C}g^{AB})(\frac{\overrightarrow{\partial^\ell}}{\partial z^B}\frac{\overrightarrow{\partial^\ell}}{\partial z^A}X^C) + 2(-1)^{\epsilon_A}g_{AB}(g^{BC},\frac{\overrightarrow{\partial^\ell}}{\partial z^C}X^A)\;, \tag{3.4.7}$$

$$\delta\nu^{(3)} \;=\; -4(-1)^{\epsilon_A}g_{AB}(g^{BC},\frac{\overrightarrow{\partial^\ell}}{\partial z^C}X^A)\;. \tag{3.4.8}$$

One easily sees that while the four constituents $\nu_\rho^{(0)}$, $\nu^{(1)}$, $\nu^{(2)}$ and $\nu^{(3)}$ separately have non–trivial transformation properties, the linear combination ν_ρ in eq. (3.4.1) is indeed a scalar.

□

Spurred by what happens in the antisymplectic case [4], we would like to classify which zeroth–order term ν one could add to the Laplacian (3.2.1). The following Proposition 3.2 is designed to answer this question.

Proposition 3.2 (Classification of 2–order differential invariants) *If a function $\nu = \nu(z)$ has the following properties:*

1. *The function ν is a scalar.*
2. *$\nu(z)$ is a polynomial of the metric $g_{AB}(z)$, the density $\rho(z)$, their inverses, and z–derivatives thereof in the point z.*
3. *ν is invariant under constant rescaling of the density $\rho \to \lambda\rho$, where λ is a z–independent parameter.*
4. *ν scales as $\nu \to \lambda\nu$ under constant Weyl scaling $g^{AB} \to \lambda g^{AB}$, where λ is a z–independent parameter.*
5. *Each term in ν contains precisely two z–derivatives.*

Then ν is of the form

$$\nu \;=\; \alpha\,\nu_\rho + \beta\,\nu_{\rho_g} + \gamma\,(\ln\frac{\rho}{\rho_g},\ln\frac{\rho}{\rho_g})\;, \tag{3.4.9}$$

where α, β and γ are three arbitrary z–independent parameters.

Remarks: Conditions 1–5 are imposed, because the Laplacian (3.2.1) has these properties. Note that if one collects the ρ–dependence into a function of $\ln\rho$ and its z–derivatives, the conditions 2 and 3 both exclude undifferentiated $\ln\rho$–dependence (because $\ln\rho$ is not a finite polynomial in ρ and ρ^{-1}, and because $\ln\rho \to \ln\rho + \ln\lambda$ is not invariant, respectively). So scalars like $\nu_\rho \ln(\rho/\rho_g)$ are excluded from our considerations.

Sketched proof of Proposition 3.2: The first idea of the proof is to replace condition 1 with a weaker condition

1′. *The function ν is invariant under affine coordinate transformations $z^A \to z'^B = \Lambda^B{}_A z^A + \lambda^B$.*

Secondly, recall that every polynomial is a finite linear combinations of monomials. One can argue that if $\nu(z)$ is a polynomial that satisfy condition 1′ plus conditions 2–5 of Proposition 3.2, then each of its constituent monomials (that contributes nontrivially) must by themselves satisfy condition 1′ plus conditions 2–5. Thus one can limit the search (for a linear basis) to monomials. It follows from lengthy but straightforward combinatorial arguments that a basis for the polynomials ν that satisfy condition $1'$ plus conditions 2–5 is:

$$\nu_{\rho}^{(0)}\ ,\quad \nu_{\rho_g}^{(0)}\ ,\quad \nu^{(1)}\ ,\quad \nu^{(2)}\ ,\quad \nu^{(3)}\ ,\quad \nu^{(4)}\ ,\quad \nu_{\rho}^{(5)}\ ,\quad \nu_{\rho_g}^{(5)}\ ,\quad \nu_{\rho}^{(6)}\ ,\quad \nu_{\rho_g}^{(6)}\ ,\quad \nu_{\rho}^{(7)}\ , \tag{3.4.10}$$

where $\nu_{\rho}^{(0)}$, $\nu^{(1)}$, $\nu^{(2)}$, $\nu^{(3)}$ were defined above, and

$$\begin{aligned}
\nu^{(4)} &:= (-1)^{\varepsilon_A}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^A}g^{AB})g_{BC}(g^{CD}\frac{\overleftarrow{\partial^{r}}}{\partial z^D})(-1)^{\varepsilon_D}\ , && (3.4.11)\\
\nu_{\rho}^{(5)} &:= (-1)^{\varepsilon_A}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^A}g^{AB})(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^B}\ln\rho)\ , && (3.4.12)\\
\nu_{\rho}^{(6)} &:= (\ln\rho,\ln\rho)\ , && (3.4.13)\\
\nu_{\rho}^{(7)} &:= (\ln\rho,\ln\rho_g)\ . && (3.4.14)
\end{aligned}$$

Thirdly, under an arbitrary infinitesimal coordinate transformation $\delta z^A = X^A$, one calculates

$$\begin{aligned}
\delta\nu^{(4)} &= 2(-1)^{\varepsilon_A}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^A}g^{AB})(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^B}\mathrm{div}_1 X)\\
&\quad +2g^{AB}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^B}\frac{\overrightarrow{\partial^{\ell}}}{\partial z^A}X^C)g_{CD}(g^{DE}\frac{\overleftarrow{\partial^{r}}}{\partial z^E})(-1)^{\varepsilon_E}\ , && (3.4.15)\\
\delta\nu_{\rho}^{(5)} &= (\ln\rho,\mathrm{div}_1 X) - (-1)^{\varepsilon_A}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^A}g^{AB})(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^B}\mathrm{div}_1 X)\\
&\quad +g^{AB}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^B}\frac{\overrightarrow{\partial^{\ell}}}{\partial z^A}X^C)(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^C}\ln\rho)\ , && (3.4.16)\\
\delta\nu_{\rho}^{(6)} &= -2(\ln\rho,\mathrm{div}_1 X)\ , && (3.4.17)\\
\delta\nu_{\rho}^{(7)} &= -(\ln(\rho_g\rho),\mathrm{div}_1 X)\ . && (3.4.18)
\end{aligned}$$

It is easy to check that the only linear combinations of the basis elements (3.4.10) that satisfy condition 1, are given by formula (3.4.9).

□

3.5 Δ And Δ_g

The Riemannian analogue Δ_g of Khudaverdian's Δ_E operator [17, 18, 19, 20, 21, 22, 23] is defined as

$$\Delta_g \ :=\ \Delta_1 + \frac{\nu^{(1)}}{4} - \frac{\nu^{(2)}}{8} - \frac{\nu^{(3)}}{16}\ . \tag{3.5.1}$$

We will prove below that the Δ_g operator (3.5.1) takes semidensities to semidensities. It is obviously manifestly independent of ρ. Next, we define a Riemannian analogue of the Grassmann–odd nilpotent Δ operator in antisymplectic geometry [4]. The even Δ operator, which takes scalar functions to scalar functions, is defined for arbitrary ρ as

$$\Delta \ :=\ \Delta_{\rho} + \nu_{\rho}\ . \tag{3.5.2}$$

This Δ operator (3.5.2) is well–defined, because of Lemma 3.1. One may prove (by using methods similar to the antisymplectic case [22, 4]), that the two operators Δ and Δ_g are related via a similarity–transformation with $\sqrt{\rho}$,

$$\Delta_g \ =\ \sqrt{\rho}\Delta\frac{1}{\sqrt{\rho}}\ . \tag{3.5.3}$$

PROOF OF EQ. (3.5.3): Let σ denote an arbitrary argument for the Δ_g operator. (The argument σ is a semidensity, but we shall not use this fact.) Then, it follows from the explicit ν_ρ formula (3.4.1) that

$$\begin{aligned}(\Delta_g\sigma) &= (\Delta_1\sigma)+(\frac{\nu^{(1)}}{4}-\frac{\nu^{(2)}}{8}-\frac{\nu^{(3)}}{16})\sigma \;=\; (\Delta_1\sigma)-(\Delta_1\sqrt{\rho})\frac{\sigma}{\sqrt{\rho}}+\nu_\rho\sigma \\ &= \sqrt{\rho}(\Delta_1\frac{\sigma}{\sqrt{\rho}})+2(\sqrt{\rho},\frac{\sigma}{\sqrt{\rho}})+\nu_\rho\sigma \;=\; \sqrt{\rho}(\Delta_\rho\frac{\sigma}{\sqrt{\rho}})+\nu_\rho\sigma \;=\; \sqrt{\rho}(\Delta\frac{\sigma}{\sqrt{\rho}}) \ . \end{aligned} \tag{3.5.4}$$

□

Eq. (3.5.3) shows that the Δ_g operator (3.5.1) takes semidensities to semidensities. The Δ operator (3.5.2) has, in turn, the remarkable property that the $\sqrt{\rho}$–conjugated operator $\sqrt{\rho}\Delta\frac{1}{\sqrt{\rho}}$ is independent of ρ. This is strikingly similar to what happens in the antisymplectic case, cf. Subsection 4.4. It is interesting to investigate how unique this property is? Consider a primed operator

$$\Delta' \;:=\; \Delta+\nu=\Delta_\rho+\nu_\rho+\nu \ , \tag{3.5.5}$$

where ν is a most general zeroth–order term. (We will in this paper not consider the possibility of changing second– and first–order parts of Laplace operators, *i.e.*, we will only consider changes to the zeroth–order term for simplicity.) It is easy to see from eqs. (3.5.3) and (3.5.5) that the corresponding $\sqrt{\rho}$–conjugated operator $\sqrt{\rho}\Delta'\frac{1}{\sqrt{\rho}}$ is independent of ρ if and only if the shift term ν is ρ–independent. On the other hand, by invoking Proposition 3.2, one sees that ν is ρ–independent if and only if $\nu=\beta\nu_{\rho_g}$ is proportional to ν_{ρ_g}. So an operator of the form $\Delta'=\Delta+\beta\nu_{\rho_g}$, for arbitrary coefficient β, is the most general operator with this property. This is the minimal answer one could possibly have hoped for, since a ρ–independence argument will never be able to detect the presence of a ρ–independent shift term like $\beta\nu_{\rho_g}$.

3.6 Levi–Civita Connection

A connection $\nabla^{(\Gamma)}$ is called *metric*, if it preserves the metric g,

$$0 \;=\; (\nabla^{(\Gamma)}_A\tilde{g})_{BC} \;=\; (\frac{\overrightarrow{\partial^\ell}}{\partial z^A}\tilde{g}_{BC})-((-1)^{\varepsilon_A\varepsilon_B}\Gamma_{BAC}+(-1)^{\varepsilon_B\varepsilon_C}(B\leftrightarrow C)) \ . \tag{3.6.1}$$

Here we have lowered the Christoffel symbol with the metric

$$\Gamma_{ABC} \;:=\; g_{AD}\Gamma^D{}_{BC}(-1)^{\varepsilon_C} \ . \tag{3.6.2}$$

The metric condition (3.6.1) reads in terms of the contravariant inverse metric

$$0 \;=\; (\nabla^{(\Gamma)}_A g)^{BC} \;\equiv\; (\frac{\overrightarrow{\partial^\ell}}{\partial z^A}g^{BC})+\Big(\Gamma_A{}^B{}_D g^{DC}+(-1)^{\varepsilon_B\varepsilon_C}(B\leftrightarrow C)\Big) \ . \tag{3.6.3}$$

The Levi–Civita connection is the unique connection $\nabla^{(\Gamma)}$ that is both torsionfree $T^{(\Gamma)}=0$ and metric (3.6.1). The Levi–Civita formula for the lowered Christoffel symbol in terms of derivatives of the metric reads

$$2\Gamma_{CAB} \;=\; (-1)^{\varepsilon_A\varepsilon_C}(\frac{\overrightarrow{\partial^\ell}}{\partial z^A}\tilde{g}_{CB})+(-1)^{(\varepsilon_A+\varepsilon_C)\varepsilon_B}(\frac{\overrightarrow{\partial^\ell}}{\partial z^B}\tilde{g}_{CA})-(\frac{\overrightarrow{\partial^\ell}}{\partial z^C}\tilde{g}_{AB}) \ . \tag{3.6.4}$$

A density ρ is compatible (2.3.3) with the Levi–Civita Christoffel symbol (3.6.4) if and only if ρ is proportional to the canonical density (3.1.10).

3.7 The Riemann Curvature

For a metric connection $\nabla^{(\Gamma)}$, we prefer to work with a $(0,4)$ Riemann tensor (as opposed to a $(1,3)$ tensor) by lowering the upper index with the metric (3.1.1). In terms of Christoffel symbols it is easiest to work with expression (2.4.2):

$$\begin{aligned} R_{D,ABC} &:= g_{DE}R^{E}{}_{ABC}(-1)^{\varepsilon_C} \\ &= (-1)^{\varepsilon_A\varepsilon_D}\left(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}\Gamma_{DBC} + (-1)^{\varepsilon_E(\varepsilon_A+\varepsilon_D+1)+\varepsilon_C}\Gamma_{EAD}\Gamma^{E}{}_{BC}\right) \\ &\quad -(-1)^{\varepsilon_A\varepsilon_B}(A\leftrightarrow B)\ . \end{aligned} \tag{3.7.1}$$

In the second equality of eq. (3.7.1) is used the metric condition (3.6.1). If the metric condition (3.6.1) is used one more time on the first term in eq. (3.7.1), one derives the following skewsymmetry

$$R_{D,ABC} = -(-1)^{(\varepsilon_A+\varepsilon_B)(\varepsilon_C+\varepsilon_D)+\varepsilon_C\varepsilon_D}(C\leftrightarrow D)\ . \tag{3.7.2}$$

This skewsymmetry becomes clearer if one instead starts from expression (2.4.6) and define

$$R_{AB,CD} := R_{ABC}{}^{E}\tilde{g}_{ED} = (-1)^{\varepsilon_D(\varepsilon_A+\varepsilon_B+\varepsilon_C)}R_{D,ABC}\ . \tag{3.7.3}$$

Then the skewsymmetry (3.7.2) simply translates into a skewsymmetry between the third and fourth index:

$$R_{AB,CD} = -(-1)^{\varepsilon_C\varepsilon_D}(C\leftrightarrow D)\ . \tag{3.7.4}$$

We note that the torsionfree condition has not been used so far in this Section 3.7. The first Bianchi identity (2.4.7) reads (in the torsionfree case):

$$0 = \sum_{\text{cycl. } A,B,C}(-1)^{\varepsilon_A\varepsilon_C}R_{AB,CD}\ . \tag{3.7.5}$$

The $(A\leftrightarrow B)$ antisymmetry, the $(C\leftrightarrow D)$ antisymmetry (3.7.4) and the first Bianchi identity (3.7.5) imply that Riemann curvature tensor $R_{AB,CD}$ is symmetric with respect to an $(AB\leftrightarrow CD)$ exchange of two pairs of indices:

$$R_{AB,CD} = (-1)^{(\varepsilon_A+\varepsilon_B)(\varepsilon_C+\varepsilon_D)}(AB\leftrightarrow CD)\ . \tag{3.7.6}$$

This, in turn, implies that there is a version of the first Bianchi identity (3.7.5), where one sums cyclically over the three last indices:

$$0 = \sum_{\text{cycl. } B,C,D}(-1)^{\varepsilon_B\varepsilon_D}R_{AB,CD}\ . \tag{3.7.7}$$

It is interesting to compare Riemann tensors in the Riemannian case with the antisymplectic case. In both cases, the $(A\leftrightarrow B)$ antisymmetry and the Bianchi identity (3.7.5) hold, but the $(C\leftrightarrow D)$ antisymmetry (3.7.4) turns in the antisymplectic case into an $(C\leftrightarrow D)$ symmetry (4.6.4), and there is *no* antisymplectic analogue of the $(AB\leftrightarrow CD)$ exchange symmetry (3.7.6), cf. Subsection 4.6.

3.8 Scalar Curvature

The scalar curvature is defined as

$$R := (-1)^{\varepsilon_B}g^{BA}R_{AB} = (-1)^{\varepsilon_A}R_{AB}g^{BA}\ . \tag{3.8.1}$$

Proposition 3.3 *The Levi–Civita scalar curvature R is proportional to the scalar ν_{ρ_g},*

$$R \;=\; -4\nu_{\rho_g}\ . \tag{3.8.2}$$

Sketched proof of Proposition 3.3: Straightforward calculations shows that

$$R \;=\; -4\nu^{(0)}_{\rho_g} - \nu^{(1)} + (-1)^{\varepsilon_A} g^{AB}\ \Gamma_B{}^D{}_C\ \Gamma^C{}_{DA}\ , \tag{3.8.3}$$

where

$$2(-1)^{\varepsilon_A} g^{AB}\ \Gamma_B{}^D{}_C\ \Gamma^C{}_{DA} \;=\; -(-1)^{\varepsilon_A+\varepsilon_B}\Gamma^A{}_{BC}(g^{CB}\frac{\overleftarrow{\partial^r}}{\partial z^A}) \;=\; \nu^{(2)} + \frac{\nu^{(3)}}{2}\ . \tag{3.8.4}$$

□

As a corollary of Proposition 3.3 one gets that the ν_ρ scalar (3.4.1) for arbitrary ρ is given by the formula

$$\nu_\rho \;=\; \nu(\rho;\rho_g,g) + \nu_{\rho_g} \;=\; \sqrt{\frac{\rho_g}{\rho}}(\Delta_{\rho_g}\sqrt{\frac{\rho}{\rho_g}}) - \frac{R}{4}\ . \tag{3.8.5}$$

3.9 The Δ Operator At $\rho = \rho_g$

When one restricts to $\rho = \rho_g$, the Δ operator (3.5.2) reduces to the Laplace–Beltrami operator minus a quarter of the Levi–Civita scalar curvature:

$$\Delta|_{\rho=\rho_g} \;=\; \Delta_{\rho_g} + \nu_{\rho_g} \;=\; \Delta_{\rho_g} - \frac{R}{4}\ . \tag{3.9.1}$$

This is the even operator (1.0.3) already mentioned in the Introduction. But the important question is: Does the zeroth–order term $\nu_{\rho_g} = -R/4$ in the operator (3.9.1) have a property that distinguish it from all the other zeroth–order terms? Yes, in the following sense:

1. Firstly, consider the most general ρ–independent operator of the form

   $$\Delta_{\rho_g} + \nu\ , \tag{3.9.2}$$

   where Δ_{ρ_g} is the Laplace–Beltrami operator and ν is a general zeroth–order term. (Here it is important that we only allow ρ–independent ν's from the very beginning.)

2. Secondly, apply Proposition 3.2 to classify the possible zeroth–order terms ν. In detail, one sees that $\nu = \beta\nu_{\rho_g}$ is proportional to ν_{ρ_g} for some proportionality factor β. Hence the operator (3.9.2) is actually

   $$\Delta_{\rho_g} + \beta\nu_{\rho_g}\ . \tag{3.9.3}$$

3. Thirdly, replace the canonical density $\rho_g \to \rho$ by an arbitrary density ρ. In other words, replace the ρ–independent operator (3.9.3) with the corresponding ρ–dependent operator

   $$\Delta' := \Delta_\rho + \beta\nu_\rho\ . \tag{3.9.4}$$

   More rigorously, one should consider an algebra homomorphism $s : \mathcal{A}_g \to \mathcal{A}_{\rho,g}$ from the algebra $\mathcal{A}_g$ of differential operators, that only depend on the metric g, to the algebra $\mathcal{A}_{\rho,g}$ of differential operators, that depend on both the density ρ and the metric g. The s homomorphism should satisfy $\pi \circ s = \mathrm{Id}_{\mathcal{A}_g}$, where $\pi : \mathcal{A}_{\rho,g} \to \mathcal{A}_g$ denotes the restriction map $|_{\rho=\rho_g}$ and "$\circ$" denotes composition. Clearly such a procedure is in general highly ambiguous, but in the present situation, where we are only interested in the ρ–extension of just two operators, namely the second–order operator Δ_{ρ_g} and the zeroth–order operator ν_{ρ_g}, there is a preferred candidate for the s homomorphism in this sector, *i.e.*, $\Delta_{\rho_g} \stackrel{s}{\mapsto} \Delta_{\rho}$ and $\nu_{\rho_g} \stackrel{s}{\mapsto} \nu_{\rho}$, respectively.

4. Fourthly, apply the $\sqrt{\rho}$–independence argument of Subsection 3.5. It follows that the $\sqrt{\rho}$–conjugated Δ' operator $\sqrt{\rho}\Delta'\frac{1}{\sqrt{\rho}}$ becomes independent of ρ if and only if $\beta\!=\!1$. (In the antisymplectic case Δ' is also nilpotent if and only if $\beta\!=\!1$.) Thus we conclude that the coefficient $\beta\!=\!1$, and hence the even Δ operator (3.5.2) is singled out.

5. Fifthly, restrict to $\rho = \rho_g$. Hence one arrives at the preferred operator (3.9.1).

Needless to say, that the above argument depends crucially on the order of the above five steps. In particular, if step 3 is performed before step 1 and 2, *i.e.*, if one considers the most general ρ–dependent zeroth–order term ν from the very beginning, the β coefficient in front of the zeroth–order term ν_{ρ_g} would remain arbitrary.

3.10 Particle In Curved Space

In this Subsection 3.10 we indicate how the Δ operator (3.5.2) is related to quantization of a particle in a curved Riemannian target space [7, 8, 9, 10, 11, 12, 13, 14, 15, 16] with a measure density ρ not necessarily equal to the canonical density (3.1.10). The classical Hamiltonian action $S_{\rm cl}$ is

$$S_{\rm cl} \;=\; \int dt \left(p_A \dot{z}^A - H_{\rm cl}\right) \;, \qquad H_{\rm cl} \;=\; \frac{1}{2}p_A p_B g^{BA} + V \;, \qquad \{z^A, p_B\}_{PB} \;=\; \delta^A_B \;, \tag{3.10.1}$$

where p_A denote the momenta for the z^A variables. We shall for simplicity not consider reparametrizations of the time variable t. Moreover, we assume that the Riemannian metric $g_{AB}\!=\!g_{AB}(z)$, the density $\rho\!=\!\rho(z)$, the potential $V\!=\!V(z)$ and general coordinate transformations $z^A \to z'^B\!=\!f^B(z)$ do not depend explicitly on time t. The naive Hamiltonian operator $\hat{H}_\rho$ is [7, 9, 10, 11]

$$\begin{aligned}
\hat{H}_\rho - V(\hat{z}) \;&=\; \frac{1}{2}\hat{p}^r_A \; g^{AB}(\hat{z}) \; \hat{p}^\ell_B \;=\; \frac{1}{2\sqrt{\rho(\hat{z})}} \; \hat{p}_A \; \rho(\hat{z}) \; g^{AB}(\hat{z}) \; \hat{p}_B \; \frac{(-1)^{\varepsilon_B}}{\sqrt{\rho(\hat{z})}} && (3.10.2)\\
&=\; \frac{1}{2}[\hat{p}_A + \frac{\hbar}{i}\ln\sqrt{\rho(\hat{z})}\frac{\overleftarrow{\partial^r}}{\partial \hat{z}^A}] \; g^{AB}(\hat{z}) \; [\hat{p}_B(-1)^{\varepsilon_B} - \frac{\hbar}{i}\frac{\overrightarrow{\partial^\ell}}{\partial \hat{z}^B}\ln\sqrt{\rho(\hat{z})}] && (3.10.3)\\
&=\; \frac{1}{2}\hat{p}_A \; g^{AB}(\hat{z}) \; \hat{p}_B(-1)^{\varepsilon_B} + \frac{\hbar^2}{2}\nu^{(0)}_\rho(\hat{z}) && (3.10.4)\\
&=\; \frac{1}{2}\left(p_A p_B g^{BA}(z)\right)^\wedge + \frac{\hbar^2}{2}\left(\nu^{(0)}_\rho(\hat{z}) + \frac{\nu^{(1)}(\hat{z})}{4}\right) \;. && (3.10.5)
\end{aligned}$$

The left, middle, and right momentum operators, denoted by $\hat{p}^\ell_A$, $\hat{p}_A$, and $\hat{p}^r_A$, respectively, are related as

$$\frac{(-1)^{\varepsilon_A}}{\sqrt{\rho(\hat{z})}} \; \hat{p}^\ell_A \; \sqrt{\rho(\hat{z})} \;=\; \hat{p}_A \;=\; \sqrt{\rho(\hat{z})} \; \hat{p}^r_A \; \frac{1}{\sqrt{\rho(\hat{z})}} \;. \tag{3.10.6}$$

The non–zero canonical equal–time commutator relations read

$$-[\hat{p}^\ell_B, \hat{z}^A] \;=\; [\hat{z}^A, \hat{p}_B] \;=\; [\hat{z}^A, \hat{p}^r_B] \;=\; i\hbar\delta^A_B\mathbf{1} \;. \tag{3.10.7}$$

The hat "∧" in eq. (3.10.5) denotes the corresponding Weyl–ordered operator. We mention for completeness a temporal point–splitting operation "T" defined as [12]

$$T\left(\hat{F}_1(t)\cdots\hat{F}_n(t)\right) \;=\; \frac{1}{n!}\sum_{\pi\in\mathcal{S}_n}(-1)^{\varepsilon_{F,\pi}} \lim_{\substack{t_1,\ldots,t_n\to t\\ t_{\pi(1)}>\ldots>t_{\pi(n)}}} \hat{F}_{\pi(1)}(t_{\pi(1)})\cdots\hat{F}_{\pi(n)}(t_{\pi(n)}) \;, \qquad (3.10.8)$$

where $\varepsilon_{F,\pi}$ denotes the Grassmann sign factor arising from permuting

$$\hat{F}_1(t_1)\cdots\hat{F}_n(t_n) \quad\longrightarrow\quad \hat{F}_{\pi(1)}(t_{\pi(1)})\cdots\hat{F}_{\pi(n)}(t_{\pi(n)}) \;. \qquad (3.10.9)$$

For most practical purposes, the temporal point–splitting "T" is the same as Weyl ordering "∧". In particular, Weyl–ordering "∧" and temporal point–splitting "T" yield the same two–loop quantum correction:

$$\left.\begin{matrix}\left(p_A p_B g^{BA}(z)\right)^{\wedge}\\[2ex] T\left(\hat{p}_A\hat{p}_B g^{BA}(\hat{z})\right)\end{matrix}\right\} - \hat{p}_A\; g^{AB}(\hat{z})\;\hat{p}_B(-1)^{\varepsilon_B} \;=\; \frac{1}{4}[\hat{p}_A,[\hat{p}_B,g^{BA}(\hat{z})]] \;=\; -\frac{\hbar^2}{4}\nu^{(1)}(\hat{z}) \;. \qquad (3.10.10)$$

Note that Weyl–ordering "∧" and temporal point–splitting "T" are *not* covariant operations.

Now what should be the quantum Hamiltonian $\hat{H}$ for the operator formalism? Obviously, one must (among other things) demand that

1. *$\hat{H}$ is a scalar invariant.*
2. *$\hat{H}$ is Hermitean.*
3. *$\hat{H}$ has dimension of energy.*
4. *$\hat{H}$ reduces to the classical Hamiltonian $H_{\rm cl}$ in the classical limit $\hbar\to 0$.*

The naive Hamiltonian operator (3.10.2) satisfies all these conditions 1–4. It is a scalar invariant, since the momentum operators transform by definition under coordinate transformations $z^A \to z'^B = f^B(z)$ as

$$\hat{p}'^{\ell}_B \;=\; \left(\frac{\overrightarrow{\partial^{\ell}}}{\partial f^B(\hat{z})}\hat{z}^A\right)\,\hat{p}^{\ell}_A \;, \qquad (3.10.11)$$

$$\hat{p}'^{r}_B \;=\; \hat{p}^{r}_A\,\left(\hat{z}^A\frac{\overleftarrow{\partial^{r}}}{\partial f^B(\hat{z})}\right) \;, \qquad (3.10.12)$$

$$\hat{p}'_B \;=\; \left(p_A\,\left(z^A\frac{\overleftarrow{\partial^{r}}}{\partial f^B(z)}\right)\right)^{\wedge} \;=\; \frac{1}{2}\left\{\hat{p}_A,\hat{z}^A\frac{\overleftarrow{\partial^{r}}}{\partial f^B(\hat{z})}\right\}_+ \;. \qquad (3.10.13)$$

Note however that conditions 1–4 do not specify the quantum corrections to the quantum Hamiltonian $\hat{H}$. For instance, one could add any multiple of $\hbar^2\nu_\rho(\hat{z})$ to $\hat{H}$ without affecting conditions 1–4. Now recall that every choice of $\hat{H}$ in the operator formalism corresponds to a choice of action functional in the path integral. One may fix the ambiguity by additionally demanding the following:

5. *The operator formalism with the Hamiltonian operator $\hat{H}$ should correspond to a Hamiltonian path integral formulation where the path integral action is the pure classical action $S_{\rm cl}$ with no quantum corrections, i.e.,*

$$\langle z_f|\exp\left[-\frac{i}{\hbar}\hat{H}\Delta t\right]|z_i\rangle \;=\; \langle z_f,t_f|z_i,t_i\rangle \;\sim\; \int_{z(t_i)=z_i}^{z(t_f)=z_f}[dz][dp]\;\exp\left[\frac{i}{\hbar}S_{\rm cl}[z,p]\right] \;. \qquad (3.10.14)$$

The reader may wonder why we invoke the path integral formulation. The point is that, on one hand, there is *no* unique way of telling what part of an operator should be considered as quantum corrections, while on the other hand, there is a well–defined quantum part of an action functional, namely all the terms of order $\mathcal{O}(\hbar)$. The phase space path integral in quantum mechanics does not need to be renormalized (unlike configuration space path integrals or quantum field theories), so it is consistent to demand that the bare quantum corrections of the action functional vanish. Condition 5 determines in principle the Hamiltonian operator $\hat{H}$ to all orders in $\hbar$, but we shall in this paper truncate $\hat{H}$ at two–loop order, *i.e.*, ignore possible higher–order quantum corrections of order $\mathcal{O}(\hbar^3)$ for simplicity. According to standard heuristic arguments, it follows from condition 5 that the quantum Hamiltonian

$$\hat{H} \;\sim\; T(H_{\rm cl}) \tag{3.10.15}$$

is equal to the time–ordered classical Hamiltonian $T(H_{\rm cl})$. However, time–ordering "T" is not a geometrically well–defined operation, at least not if one uses the temporal point–splitting (3.10.8). It should only be trusted modulo terms that contains single–derivatives of the metric [12]. In detail, the time–ordered classical Hamiltonian $T(H_{\rm cl})$ is given by the following non–covariant expression

$$T(H_{\rm cl}) \;=\; \hat{H}_{\rho} - \frac{\hbar^2}{2}\left(\nu_{\rho}^{(0)}(\hat{z}) + \frac{\nu^{(1)}(\hat{z})}{4}\right) , \tag{3.10.16}$$

cf. eqs. (3.10.5) and (3.10.10). The combination

$$\nu_{\rho}^{(0)} + \frac{\nu^{(1)}}{4} \;=\; \nu_{\rho} + \frac{\nu^{(2)}}{8} + \frac{\nu^{(3)}}{16} \;=\; \nu_{\rho} + \frac{(-1)^{\varepsilon_A}}{4} g^{AB}\,\Gamma_{B}{}^{D}{}_{C}\,\Gamma^{C}{}_{DA} \tag{3.10.17}$$

is the ν_{ρ} scalar (3.4.1) plus non–covariant terms that contain single–derivatives of the metric, cf. eq. (3.8.3). Equations (3.10.15), (3.10.16) and (3.10.17) therefore strongly suggest that the full quantum Hamiltonian $\hat{H}$ is

$$\hat{H} \;=\; \hat{H}_{\rho} - \frac{\hbar^2}{2}\nu_{\rho}(\hat{z}) \;=\; T(H_{\rm cl}) - \frac{\hbar^2}{16}\left(\nu^{(2)}(\hat{z}) + \frac{\nu^{(3)}(\hat{z})}{2}\right) , \tag{3.10.18}$$

where we are neglecting possible quantum corrections of order $\mathcal{O}(\hbar^3)$. The operator (3.10.18) satisfies condition 1–5. For instance, it is a scalar invariant because of Lemma 3.1. We shall provide further details concerning condition 5 in eq. (3.10.39) below. The preferred operator (3.10.18) also has an extra feature:

6. *The three operators*

$$\hat{H}_{g} \;=\; \sqrt{\rho(\hat{z})}\;\hat{H}\;\frac{1}{\sqrt{\rho(\hat{z})}} , \qquad \hat{H} , \qquad or \qquad \frac{1}{\sqrt{\rho(\hat{z})}}\;\hat{H}\;\sqrt{\rho(\hat{z})} \tag{3.10.19}$$

are independent of ρ, if one declares that the left, middle, or right momentum operators $\hat{p}^{\ell}_{A}$, $\hat{p}_{A}$, or $\hat{p}^{r}_{A}$ are independent of ρ, respectively.

We are now ready to relate the Δ operator (3.5.2) to a particle in a curved space. The main point is that the Hamiltonian (3.10.18) becomes $\Delta\!=\!\Delta_{\rho}\!+\!\nu_{\rho}$ from eq. (3.5.2) if we identify

$$\hat{z}^{A} \;\leftrightarrow\; z^{A} , \qquad \hat{p}^{\ell}_{A} \;\leftrightarrow\; \frac{\hbar}{i}\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}} , \tag{3.10.20}$$

$$\hat{H}_{\rho} - \hat{V} \;\leftrightarrow\; -\frac{\hbar^2}{2}\Delta_{\rho} , \qquad \hat{H} - \hat{V} \;\leftrightarrow\; -\frac{\hbar^2}{2}\Delta , \qquad \hat{H}_{g} - \hat{V} \;\leftrightarrow\; -\frac{\hbar^2}{2}\Delta_{g} . \tag{3.10.21}$$

In detail, let $|z,t\rangle_\rho := |z,t\rangle/\sqrt{\rho(z)}$ denote the instantaneous eigenstate $\hat{z}^A(t)|z,t\rangle_\rho = z^A|z,t\rangle_\rho$, and let the eigenstate $|z,t\rangle$ be the corresponding semidensity state with normalization $\int d^N z\ |z,t\rangle\langle z,t| = \mathbf{1}$ and Grassmann–parity $\varepsilon\left(|z,t\rangle\right)\!=\!0$. As a check, note that the formula (3.10.14) is covariant since it is implicitly understood that the path integral contains one more momentum integration $\prod_{i=1}^{n}\int dp(t_{2i-1})$ than coordinate integration $\prod_{i=1}^{n-1}\int dz(t_{2i})$ for any temporal discretization

$$t_i \equiv t_0 < t_1 < \ldots < t_{2n-1} < t_{2n} \equiv t_f \ . \tag{3.10.22}$$

The momentum operators $\hat{p}^\ell_A$, $\hat{p}_A$, or $\hat{p}^r_A$ act on the eigenstates as follows:

$$\begin{aligned}
{}_\rho\langle z,t|\hat{p}^\ell_A(t) &= \frac{\hbar}{i}\frac{\overrightarrow{\partial^\ell}}{\partial z^A}\ {}_\rho\langle z,t| \ , & \langle z,t|\hat{p}_A(t) &= \frac{\hbar}{i}\langle z,t|\frac{\overleftarrow{\partial^r}}{\partial z^A} \ , && (3.10.23)\\
\hat{p}^r_A(t)|z,t\rangle_\rho &= i\hbar|z,t\rangle_\rho\frac{\overleftarrow{\partial^r}}{\partial z^A} \ , & \hat{p}_A(t)|z,t\rangle &= i\hbar|z,t\rangle\frac{\overleftarrow{\partial^r}}{\partial z^A} \ . && (3.10.24)
\end{aligned}$$

Therefore, the Hamiltonians $\hat{H}_\rho$, $\hat{H}$, and $\hat{H}_g$ translate into the Laplace operators Δ_ρ, Δ, and Δ_g:

$$\begin{aligned}
{}_\rho\langle z,t|\left(\hat{H}_\rho(t)-\hat{V}(t)\right) &= -\frac{\hbar^2}{2}\Delta_\rho\ {}_\rho\langle z,t| \ , && (3.10.25)\\
{}_\rho\langle z,t|\left(\hat{H}(t)-\hat{V}(t)\right) &= -\frac{\hbar^2}{2}\Delta\ {}_\rho\langle z,t| \ , && (3.10.26)\\
\langle z,t|\left(\hat{H}_g(t)-\hat{V}(t)\right) &= -\frac{\hbar^2}{2}\Delta_g\ \langle z,t| \ , && (3.10.27)
\end{aligned}$$

cf. eqs. (3.2.1), (3.5.2) and (3.5.3), respectively. The time–evolution of states and operators are governed by

$$\begin{aligned}
{}_\rho\langle z,t|\hat{H}(t) &= i\hbar\frac{d}{dt}\ {}_\rho\langle z,t| \ , & \langle z,t|\hat{H}(t) &= i\hbar\frac{d}{dt}\langle z,t| \ , && (3.10.28)\\
\hat{H}(t)|z,t\rangle_\rho &= \frac{\hbar}{i}\frac{d}{dt}|z,t\rangle_\rho \ , & \hat{H}(t)|z,t\rangle &= \frac{\hbar}{i}\frac{d}{dt}|z,t\rangle \ , && (3.10.29)
\end{aligned}$$

$$i\hbar\left(\frac{d}{dt}-\frac{\partial}{\partial t}\right)\hat{F}(t) = [\hat{F}(t),\hat{H}(t)] \ , \tag{3.10.30}$$

where $\hat{F}(t)\!=\!F(\hat{z}(t),\hat{p}(t),t)$ is an arbitrary operator that may depend explicitly on time t. We should mention that semidensity states appear in geometric quantization [28].

Let us now calculate the left–hand side of eq. (3.10.14), *i.e.*, the transition element $\langle z_f|e^{-\beta\hat{H}}|z_i\rangle$ in the operator formalism using expression (3.10.18) as the Hamiltonian [13]. Here we define

$$\Delta t := t_f - t_i \ , \qquad \beta := \frac{i}{\hbar}\Delta t \ , \qquad \gamma := \frac{i}{\hbar\Delta t} \ . \tag{3.10.31}$$

It is better to change coordinates $(z_i^A;z_f^A) \to (z_m^A;\Delta z^A)$ from the start–point z_i^A and the endpoint z_f^A to the midpoint z_m^A and the displacement Δz^A, where

$$z_m^A := \frac{z_f^A + z_i^A}{2} \ , \qquad \Delta z^A := z_f^A - z_i^A \ . \tag{3.10.32}$$

In fact, we will suppress the subscript "m" since it is implicitly understood from now on that all quantities are to be evaluated at the midpoint. We are going to rewrite all operators in terms of symbols [29]. The Weyl symbol H_W for the quantum Hamiltonian (3.10.18) reads

$$H_W := (\hat{H})_W = H_{\rm cl} + \frac{\hbar^2}{16}\left(\nu^{(2)} + \frac{\nu^{(3)}}{2}\right) \ , \tag{3.10.33}$$

cf. eq. (3.10.18). Two Weyl symbols F and G are multiplied together via the Groenewold/Moyal $*$ product. It can be graphically represented as:

$$
\begin{aligned}
F*G &= F\exp[\mapsto]G \;=\; FG+(F\mapsto G)+\frac{1}{2}(F \stackrel{\mapsto}{\mapsto} G)+\mathcal{O}(\mapsto^{3})\ , && (3.10.34)\\
\frac{2}{i\hbar}(F\mapsto G) &= (F\leftarrow G)-(F\rightarrow G) \;=\; \{F,G\}_{PB}\ , && (3.10.35)\\
\leftarrow &:= \frac{\overleftarrow{\partial^{r}}}{\partial z^{A}}\frac{\overrightarrow{\partial^{\ell}}}{\partial p_{A}}\ , \qquad \rightarrow \;:=\; \frac{\overleftarrow{\partial^{r}}}{\partial p_{A}}(-1)^{\varepsilon_{A}}\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}\ . && (3.10.36)
\end{aligned}
$$

We will also need the zp–ordered and the pz–ordered symbols. They can be expressed in terms of the Weyl symbol $(\cdot)_{W}$ as

$$
\begin{aligned}
F_{z_{f}p} &= \frac{\langle z_{f}|\hat{F}|p\rangle}{\langle z_{f}|p\rangle} \;=\; \exp\left[-\frac{i\hbar}{2}\frac{\overrightarrow{\partial^{\ell}}}{\partial p_{A}}\frac{\overrightarrow{\partial^{\ell}}}{\partial z_{f}^{A}}\right]F_{W_{f}}\\
&= \exp\left[(\frac{\Delta z^{A}}{2}-\frac{i\hbar}{2}\frac{\overrightarrow{\partial^{\ell}}}{\partial p_{A}})\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}\right]F_{W}\ , && (3.10.37)\\
F_{pz_{i}} &= \frac{\langle p|\hat{F}|z_{i}\rangle}{\langle p|z_{i}\rangle} \;=\; \exp\left[\frac{i\hbar}{2}\frac{\overrightarrow{\partial^{\ell}}}{\partial p_{A}}\frac{\overrightarrow{\partial^{\ell}}}{\partial z_{i}^{A}}\right]F_{W_{i}}\\
&= \exp\left[(\frac{i\hbar}{2}\frac{\overrightarrow{\partial^{\ell}}}{\partial p_{A}}-\frac{\Delta z^{A}}{2})\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}\right]F_{W}\ . && (3.10.38)
\end{aligned}
$$

The transition element (or propagator) in the operator formalism now becomes

$$
\begin{aligned}
\langle z_{f}|e^{-\beta\hat{H}}|z_{i}\rangle &= \int d^{N}p\ \langle z_{f}|e^{-\frac{1}{2}\beta\hat{H}}|p\rangle\ \langle p|e^{-\frac{1}{2}\beta\hat{H}}|z_{i}\rangle \;=\; \int d^{N}p\ \langle z_{f}|p\rangle\ \langle p|z_{i}\rangle\ (e^{-\frac{1}{2}\beta\hat{H}})_{z_{f}p}\ (e^{-\frac{1}{2}\beta\hat{H}})_{pz_{i}}\\
&= \int\frac{d^{N}p}{(2\pi\hbar)^{N}}\ e^{\frac{i}{\hbar}p_{A}\Delta z^{A}}(e^{-\frac{1}{2}\beta\hat{H}})_{W}*(e^{-\frac{1}{2}\beta\hat{H}})_{W} \;=\; \int\frac{d^{N}p}{(2\pi\hbar)^{N}}\ e^{\frac{i}{\hbar}p_{A}\Delta z^{A}}(e^{-\beta\hat{H}})_{W}\\
&= \int\frac{d^{N}p}{(2\pi\hbar)^{N}}\ e^{\frac{i}{\hbar}p_{A}\Delta z^{A}}e^{-\beta H_{W}}\\
&\quad\times\left(1+\frac{\beta^{2}}{4}(H_{W}\stackrel{\mapsto}{\mapsto}H_{W})+\frac{\beta^{3}}{6}(H_{W}\mapsto H_{W}\mapsto H_{W})+\mathcal{O}(\mapsto^{4})\right)\\
&= (2\pi i\hbar\Delta t)^{-\frac{N}{2}}\rho_{g}\ e^{\frac{1}{2}\gamma\Delta z^{A}g_{AB}\Delta z^{B}}e^{-\beta V}\left(1-\frac{\hbar^{2}\beta}{16}\left(\nu^{(2)}+\frac{\nu^{(3)}}{2}\right)\right.\\
&\quad+\frac{\hbar^{2}\beta^{2}}{8}\left((H_{\rm cl}\stackrel{\rightarrow}{\leftarrow}H_{\rm cl})-(H_{\rm cl}\stackrel{\rightarrow}{\rightarrow}H_{\rm cl})\right)\\
&\quad+\frac{\hbar^{2}\beta^{3}}{24}\left((H_{\rm cl}\rightarrow H_{\rm cl}\leftarrow H_{\rm cl})+(H_{\rm cl}\leftarrow H_{\rm cl}\rightarrow H_{\rm cl})-2(H_{\rm cl}\rightarrow H_{\rm cl}\rightarrow H_{\rm cl})\right)\\
&\quad\left.\left.+\ \mathcal{O}(\rightarrow^{4},\hbar^{3})\right)\right|_{p=\frac{\hbar}{i}\frac{\overleftarrow{\partial^{r}}}{\partial\Delta z}}\\
&= (2\pi i\hbar\Delta t)^{-\frac{N}{2}}\rho_{g}\left(e^{-\beta V}\left(1-\frac{\hbar^{2}\beta}{24}R\right)+\frac{\hbar^{2}\beta}{6}e^{-\frac{1}{2}\beta V}(\Delta_{\rho_{g}}e^{-\frac{1}{2}\beta V})+\mathcal{O}((\Delta z)^{2},\hbar^{3})\right)\\
&= (2\pi i\hbar\Delta t)^{-\frac{N}{2}}\rho\ e^{-\frac{1}{2}\beta V}\left(1+\frac{\hbar^{2}\beta}{6}\overrightarrow{\Delta}+\mathcal{O}((\Delta z)^{2},\hbar^{3})\right)e^{-\frac{1}{2}\beta V}\Big|_{\rho=\rho_{g}}\ . && (3.10.39)
\end{aligned}
$$

In the first equality of eq. (3.10.39) we summed over a complete set of momentum states $|p\rangle$, so that it becomes possible to replace operators by symbols. The zp–ordered and the pz–ordered symbols (3.10.37) and (3.10.38) were used in the second and third equality. We performed integration by part of p_A in the third equality. In the sixth equality, we replaced all non–Gaussian appearances of the momenta p_A by derivatives with respect to the displacement Δz^A, and performed the p_A integration. After the integration over p_A, the terms downstairs in the seventh expression (that are either quadratic or cubic in $H_{\rm cl}$) read

$$(H_{\rm cl} \rightleftarrows H_{\rm cl}) \sim \frac{1}{\beta}\nu^{(2)} + \mathcal{O}((\Delta z)^2) , \qquad (3.10.40)$$

$$(H_{\rm cl} \rightrightarrows H_{\rm cl}) \sim g^{AB}\frac{\overrightarrow{\partial^{\ell}}}{\partial z^B}\frac{\overrightarrow{\partial^{\ell}}}{\partial z^A}\left(V - \frac{1}{2\beta}\ln g\right) - \frac{1}{2\beta}\nu^{(3)} + \mathcal{O}((\Delta z)^2) , \qquad (3.10.41)$$

$$(H_{\rm cl} \to H_{\rm cl} \to H_{\rm cl}) \sim \frac{1}{\beta^2}\nu^{(2)} + \frac{(-1)^{\varepsilon_A}}{\beta}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^A}g^{AB})\frac{\overrightarrow{\partial^{\ell}}}{\partial z^B}\left(V - \frac{1}{2\beta}\ln g\right) + \mathcal{O}((\Delta z)^2) , \qquad (3.10.42)$$

$$(H_{\rm cl} \to H_{\rm cl} \leftarrow H_{\rm cl}) \sim \frac{1}{\beta^2}\left(\nu^{(1)} - \frac{\nu^{(3)}}{2}\right) + \frac{1}{\beta}g^{AB}\frac{\overrightarrow{\partial^{\ell}}}{\partial z^B}\frac{\overrightarrow{\partial^{\ell}}}{\partial z^A}\left(V - \frac{1}{2\beta}\ln g\right) + \mathcal{O}((\Delta z)^2) , \qquad (3.10.43)$$

$$(H_{\rm cl} \leftarrow H_{\rm cl} \to H_{\rm cl}) \sim \left(V - \frac{1}{2\beta}\ln g,\ V - \frac{1}{2\beta}\ln g\right) - \frac{1}{2\beta^2}\nu^{(3)} + \mathcal{O}((\Delta z)^2) . \qquad (3.10.44)$$

All the individual contributions of eqs. (3.10.40)–(3.10.44) have been collected in the eighth and ninth expression of eq. (3.10.39). The eighth expression is the well-known covariant formula for the path integral, *i.e.*, the right–hand side of eq. (3.10.14). In the phase space path integral, the $R/24$ term arises from the integration over quantum fluctuations [12]. In the ninth (and last) expression of eq. (3.10.39), the $R/24$ term conspires with the Beltrami–Laplace operator to produce yet another appearance of the Δ operator (3.5.2).

3.11 First–Order S^{AB} Matrices

After considering quantization of a particle on a curved space in Subsection 3.10, we shall continue with the investigation of Riemannian manifolds. We will assume for the remainder of the Riemannian Sections 3 and 6 that the density $\rho = \rho_g$ is equal to the canonical density (3.1.10).

Because of the presence of the metric tensor g^{AB}, the symmetry of the general linear (= gl) Lie–algebra (2.7.6) reduces to an orthogonal Lie–subalgebra. Its generators $S^{AB}_{\mp}$ read

$$S^{AB}_{\mp} := C^A\, g^{BC}\frac{\overrightarrow{\partial^{\ell}}}{\partial C^C} + Y^A\, g^{BC}\frac{\overrightarrow{\partial^{\ell}}}{\partial Y^C} \mp (-1)^{\varepsilon_A\varepsilon_B}(A \leftrightarrow B) , \qquad (3.11.1)$$

$$\varepsilon(S^{AB}_{\mp}) = \varepsilon_A + \varepsilon_B , \qquad p(S^{AB}_{\mp}) = 0 , \qquad (3.11.2)$$

$$S^{A}_{\mp C} := S^{AB}_{\mp}\, g_{BC}(-1)^{\varepsilon_C} . \qquad (3.11.3)$$

The $S^{AB}_{\mp}$ matrices are called first–order matrices, because they are first–order differential operators in the C^A and Y^A variables. The S^{AB}_{-} matrices satisfy an orthogonal Lie–algebra:

$$[S^{AB}_{\mp}, S^{CD}_{\mp}] = (-1)^{\varepsilon_A(\varepsilon_B+\varepsilon_C)}\left(g^{BC}\, S^{AD}_{-} + S^{BC}_{-}\, g^{AD}\right) \mp (-1)^{\varepsilon_A\varepsilon_B}(A \leftrightarrow B) , \qquad (3.11.4)$$

$$[S^{AB}_{\mp}, S^{CD}_{\pm}] = (-1)^{\varepsilon_A(\varepsilon_B+\varepsilon_C)}\left(g^{BC}\, S^{AD}_{+} - S^{BC}_{+}\, g^{AD}\right) \mp (-1)^{\varepsilon_A\varepsilon_B}(A \leftrightarrow B) . \qquad (3.11.5)$$

Note that the eqs. (3.11.4) and (3.11.5) remain invariant under a c–number shift

$$S_{+}^{AB} \;\to\; S_{+}^{\prime AB} \;:=\; S_{+}^{AB} + \alpha g^{AB}\mathbf{1} \;, \tag{3.11.6}$$

where α is a parameter.

3.12 Γ^{A} Matrices

The standard Dirac operator is only defined on a spin manifold, it depends on the vielbein, and we shall describe it in Subsections 6.4–6.6. But first we shall introduce a poor man's version of Γ^{A} matrices and the so–called Hodge–Dirac operator in the next Subsections 3.12–3.15. This construction will work for a general Riemannian manifold, which is not necessarily a spin manifold.

The Γ^{A} matrices can be defined via a Berezin–Fradkin operator representation [30, 31]

$$\Gamma^{A}_{\lambda} \;\equiv\; \Gamma^{A} \;:=\; C^{A} + \lambda P^{A} \;, \qquad\qquad P^{A} \;:=\; g^{AB}\frac{\overrightarrow{\partial^{\ell}}}{\partial C^{B}} \;, \tag{3.12.1}$$

$$\varepsilon(\Gamma^{A}) \;=\; \varepsilon_{A} \;, \qquad\qquad p(\Gamma^{A}) \;=\; 1 \pmod 2 \;. \tag{3.12.2}$$

where λ is a Bosonic parameter with $\varepsilon(\lambda)\!=\!0\!=\!p(\lambda)$, which is introduced to bring our presentation of the Riemannian case in closer analogy with the antisymplectic case, see Subsection 4.9. One may interpret λ as a Planck constant. The Γ^{A} matrices satisfy a Clifford algebra

$$[\Gamma^{A}, \Gamma^{B}] \;=\; 2\lambda g^{AB}\mathbf{1} \;. \tag{3.12.3}$$

The Γ^{A} matrices form a fundamental representation of the an orthogonal Lie–algebra (3.11.4):

$$[S_{\mp}^{AB}, \Gamma^{C}] \;=\; \Gamma^{A}_{\pm\lambda}\, g^{BC} \mp (-1)^{\varepsilon_{A}\varepsilon_{B}}(A \leftrightarrow B) \;. \tag{3.12.4}$$

If one commutes a metric connection $\nabla^{(T)}_{A}$ in the $T^{A}{}_{B}$ representation (2.7.4) with a Γ^{B} matrix, one gets

$$[\nabla^{(T)}_{A}, \Gamma^{B}] \;=\; -\Gamma_{A}{}^{B}{}_{C}\, \Gamma^{C} \;. \tag{3.12.5}$$

The minus sign on the right–hand side of eq. (3.12.5) can be explained as follows: The contravariant flat Γ^{A} matrices are passive bookkeeping devices that ultimately should be contracted with an active covariant tensor field η_{A}. It is this implicitly written η_{A} that we are really differentiating. Thus there should be a minus sign.

The $\nabla^{(T)}_{A}$ realization (2.7.4) can be identically rewritten into the following $S_{\pm}$ matrix realization

$$\nabla^{(S)}_{A} \;:=\; \frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}} - \frac{1}{2}\sum_{\pm} \Gamma^{\pm}_{A,BC}\, S_{\pm}^{CB}(-1)^{\varepsilon_{B}} \;, \tag{3.12.6}$$

i.e., $\nabla^{(T)}_{A} = \nabla^{(S)}_{A}$, where

$$\Gamma^{\pm}_{A,BC}(-1)^{\varepsilon_{C}} \;:=\; \frac{1}{2}(-1)^{\varepsilon_{A}\varepsilon_{B}}\Gamma_{BAC} \pm (-1)^{\varepsilon_{B}\varepsilon_{C}}(B \leftrightarrow C) \;. \tag{3.12.7}$$

The Levi–Civita $\Gamma^{\pm}_{A,BC}$ connection reads:

$$\begin{aligned} \Gamma^{+}_{A,BC} &= \frac{1}{2}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}} g_{BC}) \;, \\ \Gamma^{-}_{A,BC} &= \frac{1}{2}(\tilde{g}_{AB}\frac{\overleftarrow{\partial^{r}}}{\partial z^{C}}) + (-1)^{(\varepsilon_{B}+1)(\varepsilon_{C}+1)}(B \leftrightarrow C) \;. \end{aligned} \tag{3.12.8}$$

Note that both the S_{-}^{AB} and the S_{+}^{AB} matrices are needed in the matrix realization (3.12.6).

3.13 C Versus Y

The S^{AB} matrices (3.11.1) treat the C^A and the Y^A variables on complete equal footing, whereas the Γ^A matrices (6.4.1) contain only the C's. Just from demanding that the Γ^A matrices carry definite Grassmann– and form–parity, such $C \leftrightarrow Y$ symmetry breaking seems unavoidable. Further analysis of the Riemannian case reveals that it is only possible to write a Berezin–Fradkin operator representation (6.4.1) of the Clifford algebra (6.4.3) using the C^A variables. (The C^A variables are also preferred in the antisymplectic case as well, see Subsection 4.B below.) One may ponder if there are situations where the Y variables are useful instead? Yes. The democracy between C and Y gets restored in a bigger framework that allows for both even and odd, Riemannian and symplectic manifolds, cf. Table 1. For instance, the Y^A variables are the only ones suitable for writing down a Berezin–Fradkin–like representation

$$\tilde{\Gamma}^A \;:=\; Y^A + \lambda\omega^{AB}\frac{\overrightarrow{\partial^\ell}}{\partial Y^B}\;, \qquad \varepsilon(\tilde{\Gamma}^A) \;=\; \varepsilon_A\;, \qquad p(\tilde{\Gamma}^A) \;=\; 0\;, \tag{3.13.1}$$

of the Heisenberg algebra

$$[\tilde{\Gamma}^A, \tilde{\Gamma}^B] \;=\; 2\lambda\omega^{AB}\mathbf{1} \;=\; -(-1)^{\varepsilon_A\varepsilon_B}(A\leftrightarrow B) \tag{3.13.2}$$

in even symplectic geometry [32, 33, 34]. (The Y^A variables are also preferred in the odd Riemannian case [25, 35, 36].)

Returning to the even Riemannian case, we will for simplicity only consider the C^A variables from now on, *i.e.*, we shall from now on put the Y^A variables to zero $Y^A \to 0$ everywhere, in particular inside the $T^A{}_B$ matrices (2.7.5) and the S^{AB} matrices (3.11.1).

3.14 Hodge $*$ Operation

One may formally define a Hodge $*$ operation on exterior forms $\eta = \eta(z;C) \in \Omega_{\bullet 0}(M)$ as a fiberwise Fourier transformation

$$(*\eta)(z;C) \;:=\; \int\frac{d^N C'}{\rho}\; e^{\frac{i}{\hbar}C'\wedge C}\eta(z;C')\;, \tag{3.14.1}$$

where we have introduced the shorthand notation

$$C'\wedge C \;:=\; C'^A\, g_{AB}\wedge C^B\;. \tag{3.14.2}$$

The Hodge $*$ operation is an involution $*^2 \sim \mathrm{Id}$. Note that the Hodge dual $*\eta$ in general is a distribution.

In detail, the Hodge $*$ operation is built out of two operations: Firstly, a fiberwise Fourier transform

$$\Gamma\left(\bigwedge{}^\bullet(T^*M)\right) \equiv \Omega_{\bullet 0}(M) \;\ni\; \eta \;\overset{\mathcal{F}}{\mapsto}\; \pi = \mathcal{F}\eta \;\in\; \Gamma\left(\bigwedge{}^\bullet(TM)\right)\;, \tag{3.14.3}$$

that takes exterior forms $\eta{=}\eta(z;C)$ to multivectors

$$\pi \;=\; \pi(z;B) \;=\; \frac{1}{m!}\pi^{A_1\cdots A_m}(z)\; B^\ell_{A_m}\wedge\cdots\wedge B^\ell_{A_1}\;, \tag{3.14.4}$$

where $B^\ell_A{\equiv}(-1)^{\varepsilon_A}B^r_A$ and

$$B^\ell_A\wedge B^\ell_C \;=\; -(-1)^{\varepsilon_A\varepsilon_C}B^\ell_C\wedge B^\ell_A\;, \qquad \varepsilon(B^\ell_A) \;=\; \varepsilon_A\;, \qquad p(B^\ell_A) \;=\; 1\;. \tag{3.14.5}$$

The Fourier transform $\mathcal{F}$ itself only depends on the density ρ:

$$(\mathcal{F}\eta)(z;B) \ := \ \int \frac{d^N C}{\rho} \ e^{\frac{i}{\hbar}C^A \wedge B^\ell_A}\eta(z;C) \ . \qquad (3.14.6)$$

Secondly, a flat map

$$\Gamma(TM) \ \ni \ X \ \stackrel{\flat}{\mapsto} \ \eta = X^\flat \ \in \ \Gamma(T^*M) \ , \qquad (3.14.7)$$

that takes vectors $X = X^A B^\ell_A$ to co–vectors $\eta = \eta_A C^A$. The Riemannian flat map $\flat$ is $X^\flat_A = X^B g_{BA}$, or equivalently, in terms of basis elements,

$$B^\ell_A \ = \ g_{AB} C^B \ . \qquad (3.14.8)$$

Altogether, the Hodge $*$ operation can be written as

$$(*\eta)(z;C) \ = \ (\mathcal{F}\eta)(z;B)\Big|_{B^\ell_A = g_{AB}C^B} \ . \qquad (3.14.9)$$

In contrast to the Riemannian case, there is no good way to construct an antisymplectic Hodge $*$ operation. This is because the antisymplectic flat map $B^\ell_A = E_{AB}C^B$ carries the opposite Grassmann–parity $\varepsilon(B^\ell_A) = \varepsilon_A + 1$, cf. Subsection 4.1.

Proposition 3.4 *The Hodge adjoint de Rham operator, also known as the Hodge codifferential, is:*

$$\begin{aligned} *d* \ \sim \ \delta \ := \ & (-1)^{\varepsilon_A} \left(\frac{1}{\rho} \frac{\overrightarrow{\partial^\ell}}{\partial z^A} \rho - (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} g_{BC}) C^C P^B (-1)^{\varepsilon_B} \right) P^A \\ = \ & (-1)^{\varepsilon_A} \left(\frac{1}{\rho} \frac{\overrightarrow{\partial^\ell}}{\partial z^A} \rho - \frac{1}{2} (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} g_{BC}) S^{CB}_+ (-1)^{\varepsilon_B} \right) P^A \ . \end{aligned} \qquad (3.14.10)$$

Proof of Proposition 3.4:

$$\begin{aligned} (*d*\eta)(z,C) \ = \ & \int \frac{d^N C'}{\rho} \ e^{\frac{i}{\hbar}C' \wedge C} C'^A \frac{\overrightarrow{\partial^\ell}}{\partial z^A} \int \frac{d^N C''}{\rho} \ e^{\frac{i}{\hbar}C'' \wedge C'} \eta(z,C'') \\ = \ & (-1)^{\varepsilon_A} \int \frac{d^N C'}{\rho} \left(\frac{\overrightarrow{\partial^\ell}}{\partial z^A} + \frac{i}{\hbar} (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} C \wedge C') \right) \int \frac{d^N C''}{\rho} \ C'^A e^{\frac{i}{\hbar}(C''-C) \wedge C'} \eta(z,C'') \\ = \ & -(-1)^{\varepsilon_A} \frac{i}{\hbar} \int \frac{d^N C''}{\rho} \left(\frac{\overrightarrow{\partial^\ell}}{\partial z^A} - (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} C \wedge P) \right) \int \frac{d^N C'}{\rho} \ P^A e^{\frac{i}{\hbar}(C''-C) \wedge C'} \eta(z,C'') \\ \sim \ & \frac{(-1)^{\varepsilon_A}}{\rho} \left(\frac{\overrightarrow{\partial^\ell}}{\partial z^A} - (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} C \wedge P) \right) \rho P^A \eta(z,C) \ . \end{aligned} \qquad (3.14.11)$$

□

3.15 Hodge–Dirac Operator $D^{(T)} = d + \lambda\delta$

We shall for the remainder of Section 3 assume that the connection is the Levi–Civita connection.

Central for our discussion are the $T^A{}_B$ generators (2.7.5). They act on exterior forms $\eta \in \Omega_{\bullet 0}(M)$, *i.e.*, functions $\eta{=}\eta(z;C)$ of z and C. (Recall that the Y^A variables are put to zero $Y^A \to 0$.)

The Dirac operator $D^{(T)}$ in the $T^A{}_B$ representation (2.7.4) is a Γ^A matrix (3.12.1) times the covariant derivative (2.7.4)

$$D^{(T)} \ := \ \Gamma^A \nabla^{(T)}_A \ = \ C^A \nabla^{(T)}_A + \lambda P^A \nabla^{(T)}_A \ = \ d + \lambda\delta \ , \qquad (3.15.1)$$

$$\varepsilon(D^{(T)}) \ = \ 0 \ , \qquad p(D^{(T)}) \ = \ 1 \ (\text{mod } 2) \ . \qquad (3.15.2)$$

The component of the Dirac operator to zeroth order in λ,

$$D^{(T)}\Big|_{\lambda=0} \ = \ C^A \nabla^{(T)}_A \ = \ C^A \left(\frac{\overrightarrow{\partial^\ell}}{\partial z^A} - \Gamma_A{}^B{}_C \ C^C \frac{\overrightarrow{\partial^\ell}}{\partial C^B} \right) \ = \ C^A \frac{\overrightarrow{\partial^\ell}}{\partial z^A} \ = \ d \ , \qquad (3.15.3)$$

is just the exterior de Rham derivative d, because the connection is torsionfree. The component of the Dirac operator to first order in λ,

$$\begin{aligned} [\frac{\overrightarrow{\partial^\ell}}{\partial \lambda}, D^{(T)}] \ &= \ P^A \nabla^{(T)}_A \ = \ [P^A, \nabla^{(T)}_A] + (-1)^{\varepsilon_A} \nabla^{(T)}_A P^A \\ &= \ \Gamma^A{}_{AC} P^C + (-1)^{\varepsilon_A} \left(\frac{\overrightarrow{\partial^\ell}}{\partial z^A} - (-1)^{(\varepsilon_A+1)\varepsilon_B+\varepsilon_C} \Gamma_{BAC} \ C^C P^B \right) P^A \\ &= \ (-1)^{\varepsilon_A} \left(\frac{1}{\rho_g} \frac{\overrightarrow{\partial^\ell}}{\partial z^A} \rho_g - (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} g_{BC}) C^C P^B (-1)^{\varepsilon_B} \right) P^A \ \stackrel{(3.14.10)}{=}: \ \delta \ , \end{aligned} \qquad (3.15.4)$$

is the Hodge adjoint de Rham operator. Equations (3.15.3) and (3.15.4) prove the last equality in eq. (3.15.1).

The Laplacian $\Delta^{(T)}_{\rho_g}$ in the $T^A{}_B$ representation (2.7.4) is

$$\begin{aligned} \Delta^{(T)}_{\rho_g} \ &:= \ (-1)^{\varepsilon_A} \nabla_A g^{AB} \nabla^{(T)}_B \ = \ (-1)^{\varepsilon_A} \nabla^{(T)}_A g^{AB} \nabla^{(T)}_B + \Gamma^A{}_{AC} \ g^{CB} \ \nabla^{(T)}_B \\ &= \ \frac{(-1)^{\varepsilon_A}}{\rho_g} \nabla^{(T)}_A \rho_g g^{AB} \nabla^{(T)}_B \ . \end{aligned} \qquad (3.15.5)$$

Theorem 3.5 (Weitzenböck's formula for exterior forms) *The difference between the square of the Dirac operator $D^{(T)}$ and the Laplacian $\Delta^{(T)}_{\rho_g}$ in the $T^A{}_B$ representation (2.7.4) is*

$$D^{(T)} D^{(T)} \ - \ \lambda \Delta^{(T)}_{\rho_g} \ = \ -\frac{\lambda}{4} S^{BA}_- \ R_{AB,CD} \ S^{DC}_- (-1)^{\varepsilon_C+\varepsilon_D} \qquad (3.15.6)$$

$$= \ -\lambda C^A \ R_{AB} \ P^B + \frac{\lambda}{2} C^B C^A \ R_{AB,CD} \ P^D P^C (-1)^{\varepsilon_C+\varepsilon_D} \ . \qquad (3.15.7)$$

Remarks: The square $D^{(T)} D^{(T)} = \lambda(d\delta + \delta d)$ is known as the form Laplacian. The Laplacian $\Delta^{(T)}_{\rho_g}$ is equal to the Bochner Laplacian.

PROOF OF THEOREM 3.5: The square is a sum of three terms

$$D^{(T)}D^{(T)} \;=\; \frac{1}{2}[D^{(T)},D^{(T)}] \;=\; I+II+III\;. \tag{3.15.8}$$

The first term is

$$I \;:=\; \frac{1}{2}[\Gamma^{B},\Gamma^{A}]\nabla^{(T)}_{A}\nabla^{(T)}_{B} \;=\; \lambda g^{BA}\;\nabla^{(T)}_{A}\nabla^{(T)}_{B}\;. \tag{3.15.9}$$

The second term is

$$\begin{aligned} II &:= \Gamma^{A}[\nabla^{(T)}_{A},\Gamma^{B}]\nabla^{(T)}_{B} \stackrel{(3.12.5)}{=} -\Gamma^{A}\;\Gamma_{A}{}^{B}{}_{C}\;\Gamma^{C}\;\nabla^{(T)}_{B} \;=\; -(-1)^{\varepsilon_{C}}\Gamma^{B}{}_{CA}\;\Gamma^{A}\;\Gamma^{C}\;\nabla^{(T)}_{B}\\ &= -(-1)^{\varepsilon_{C}}\lambda\Gamma^{B}{}_{CA}\;g^{AC}\;\nabla^{(T)}_{B} \;=\; \lambda\frac{(-1)^{\varepsilon_{A}}}{\rho_{g}}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}\rho_{g}g^{AB})\nabla^{(T)}_{B}\;. \end{aligned} \tag{3.15.10}$$

Together, the first two terms $I+II$ form the Laplace operator (3.15.5):

$$I+II \;=\; \lambda\Delta^{(T)}_{\rho_{g}}\;. \tag{3.15.11}$$

The third term yields the curvature terms:

$$\begin{aligned} III &:= -\frac{1}{2}\Gamma^{B}\Gamma^{A}[\nabla^{(T)}_{A},\nabla^{(T)}_{B}] \;=\; \frac{1}{2}\Gamma^{B}\Gamma^{A}\;R_{AB}{}^{D}{}_{C}\;T^{C}{}_{D} \;=\; -\frac{1}{4}\Gamma^{B}\Gamma^{A}\;R_{AB,CD}\;S^{DC}_{-}(-1)^{\varepsilon_{C}+\varepsilon_{D}}\\ &= -\frac{1}{4}\left(C^{B}C^{A}+\lambda(S^{BA}_{-}+g^{BA})+\lambda^{2}P^{B}P^{A}\right)R_{AB,CD}\;S^{DC}_{-}(-1)^{\varepsilon_{C}+\varepsilon_{D}}\\ &= -\frac{1}{2}C^{B}C^{A}\;R_{AB,CD}\;C^{C}P^{D}(-1)^{(\varepsilon_{C}+1)(\varepsilon_{D}+1)}-\frac{\lambda}{4}S^{BA}_{-}\;R_{AB,CD}\;S^{DC}_{-}(-1)^{\varepsilon_{C}+\varepsilon_{D}}\\ &\quad -\frac{\lambda^{2}}{2}P^{B}P^{A}\;R_{AB,CD}\;P^{C}C^{D}(-1)^{(\varepsilon_{C}+1)(\varepsilon_{D}+1)}\\ &= -\frac{\lambda}{4}S^{BA}_{-}\;R_{AB,CD}\;S^{DC}_{-}(-1)^{\varepsilon_{C}+\varepsilon_{D}} \;=\; -\lambda C^{B}P^{A}\;R_{AB,CD}\;C^{D}P^{C}(-1)^{\varepsilon_{C}+\varepsilon_{D}}\\ &= -\lambda C^{B}\;R_{BA,CD}g^{DA}\;P^{C}(-1)^{(\varepsilon_{A}+1)(\varepsilon_{C}+1)+\varepsilon_{D}}+\lambda C^{B}\;R_{BA,DC}\;C^{D}P^{C}P^{A}(-1)^{\varepsilon_{A}+(\varepsilon_{C}+1)(\varepsilon_{D}+1)}\\ &= -\lambda C^{B}\;R_{BAC}{}^{A}\;P^{C}(-1)^{(\varepsilon_{A}+1)(\varepsilon_{C}+1)}+\lambda C^{D}C^{B}\;R_{BA,DC}\;P^{C}P^{A}(-1)^{\varepsilon_{A}(\varepsilon_{D}+1)+\varepsilon_{C}}\\ &= -\lambda C^{B}\;R_{BC}\;P^{C}+\frac{\lambda}{2}C^{B}C^{D}\;R_{DB,AC}\;P^{C}P^{A}(-1)^{\varepsilon_{A}+\varepsilon_{C}}\;. \end{aligned} \tag{3.15.12}$$

Here the first Bianchi identity (3.7.5) was used to cancel terms proportional to zeroth and second order in λ.

□

3.A Appendix: Is There A Second–Order Formalism?

For the standard Dirac operator, which will be discussed in Subsections 6.4–6.6, it is natural to replace the first–order s^{ab}_{-} matrices (6.3.1) with the second–order σ^{ab}_{-} matrices (6.6.1). Therefore, it is natural to speculate if it is possible to replace the first–order $S^{AB}_{\pm}$ matrices (3.11.1) with the following second–order matrices:

$$\Sigma^{AB}_{\mp} \;:=\; \frac{1}{4\lambda}\Gamma^{A}\Gamma^{B}\mp(-1)^{\varepsilon_{A}\varepsilon_{B}}(A\leftrightarrow B)\;, \tag{3.A.1}$$

$$\varepsilon(\Sigma^{AB}_{\mp}) \;=\; \varepsilon_{A}+\varepsilon_{B}\;,\qquad p(\Sigma^{AB}_{\mp}) \;=\; 0\;. \tag{3.A.2}$$

(The names first– and second–order refer to the number of C^{A}–derivatives.) On one hand, the matrices

$$\Sigma^{AB}_{-} \;=\; \frac{1}{4\lambda}\{\Gamma^{A},\Gamma^{B}\}_{+} \;=\; \frac{1}{2\lambda}C^{A}C^{B}+\frac{1}{2}S^{AB}_{-}+\frac{\lambda}{2}P^{A}P^{B}\;. \tag{3.A.3}$$

yield precisely the same non–Abelian Lie–algebra (3.11.4) and fundamental representation (3.12.4) as the S_{-}^{AB} matrices. Moreover, the S_{-}^{AB} matrices rotate the Σ_{-}^{AB} matrices

$$[\Sigma_{-}^{AB}, S_{-}^{CD}] \;=\; (-1)^{\varepsilon_{A}(\varepsilon_{B}+\varepsilon_{C})}\left(g^{BC}\,\Sigma_{-}^{AD} + \Sigma_{-}^{BC}\,g^{AD}\right) - (-1)^{\varepsilon_{A}\varepsilon_{B}}(A \leftrightarrow B)\;. \tag{3.A.4}$$

However, the commutator of Σ_{-}^{AB} and S_{+}^{CD} does not close,

$$[\Sigma_{-}^{AB}, S_{+}^{CD}] \;=\; (-1)^{\varepsilon_{A}(\varepsilon_{B}+\varepsilon_{C})}\left(g^{BC}\,\tilde{\Sigma}^{AD} - \tilde{\Sigma}^{BC}\,g^{AD}\right) - (-1)^{\varepsilon_{A}\varepsilon_{B}}(A \leftrightarrow B)\;, \tag{3.A.5}$$

where the tilde generators

$$\tilde{\Sigma}^{AB} \;:=\; -\frac{1}{2\lambda}C^{A}C^{B} + \frac{1}{2}S_{+}^{AB} + \frac{\lambda}{2}P^{A}P^{B} \tag{3.A.6}$$

have no $(A \leftrightarrow B)$ symmetry or antisymmetry. On the other hand, the matrices

$$\Sigma_{+}^{AB} \;:=\; \frac{1}{4\lambda}[\Gamma^{A}, \Gamma^{B}] \;\stackrel{(3.12.3)}{=}\; \frac{1}{2}g^{AB}\mathbf{1} \tag{3.A.7}$$

are proportional to the identity operator, and thus behave very differently from the non–Abelian S_{+}^{AB} matrices.

The problem with a substitution $S_{\mp}^{AB} \to \Sigma_{\mp}^{AB}$ is that the S_{+}^{AB} matrices appear in the matrix realization (3.12.6). On one hand, the Σ_{-}^{AB} representation (3.A.1) is not suitable, because it couples pathologically to the non–vanishing S_{+}^{AB} sector, and, on the other hand, the Σ_{+}^{AB} matrices are Abelian, and therefore pathological by themselves. Hence, it is doubtful if the substitution $S_{\mp}^{AB} \to \Sigma_{\mp}^{AB}$ makes any sense at all. In any case, we shall dismiss the second–order $\Sigma_{\mp}^{AB}$ matrices (3.A.1) from now on.

4 Antisymplectic Geometry

4.1 Metric

Let there be given an antisymplectic metric, *i.e.*, a closed two–form

$$E \;=\; \frac{1}{2}C^{A}\;E_{AB} \wedge C^{B} \;=\; -\frac{1}{2}E_{AB}\;C^{B} \wedge C^{A} \;\in\; \Omega_{20}(M)\;, \tag{4.1.1}$$

of Grassmann–parity

$$\varepsilon(E_{AB}) \;=\; \varepsilon_{A} + \varepsilon_{B} + 1\;, \qquad \varepsilon(E) \;=\; 1\;, \qquad p(E_{AB}) \;=\; 0\;, \tag{4.1.2}$$

and with antisymmetry

$$E_{BA} \;=\; -(-1)^{\varepsilon_{A}\varepsilon_{B}}E_{AB}\;. \tag{4.1.3}$$

The closeness condition

$$dE \;=\; 0 \tag{4.1.4}$$

reads in components

$$\sum_{\text{cycl. } A,B,C}(-1)^{\varepsilon_{A}\varepsilon_{C}}\Big(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}E_{BC}\Big) \;=\; 0\;. \tag{4.1.5}$$

The antisymplectic metric E_{AB} is assumed to be non–degenerate, *i.e.*, there exists an inverse contravariant $(2,0)$ tensor field E^{AB} such that

$$E_{AB}\;E^{BC} \;=\; \delta_{A}^{C}\;. \tag{4.1.6}$$

The inverse E^{AB} has Grassmann–parity

$$\varepsilon(E^{AB}) \;=\; \varepsilon_A + \varepsilon_B + 1 \;, \tag{4.1.7}$$

and symmetry

$$E^{BA} \;=\; -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} E^{AB} \;. \tag{4.1.8}$$

The closeness condition (4.1.4) has no Riemannian analogue. It is the integrability condition for the local existence of Darboux coordinates.

4.2 Odd Laplacian Δ_ρ

The odd Laplacian Δ_ρ, which takes scalar functions in scalar functions, is defined as

$$2\Delta_\rho \;:=\; \frac{(-1)^{\varepsilon_A}}{\rho} \frac{\overrightarrow{\partial^\ell}}{\partial z^A} \rho E^{AB} \frac{\overrightarrow{\partial^\ell}}{\partial z^B} \;, \qquad \varepsilon(\Delta_\rho) \;=\; 1 \;, \qquad p(\Delta_\rho) \;=\; 0 \;. \tag{4.2.1}$$

Note the factor of 2 in the odd Laplacian (4.2.1) as compared with the Riemannian case (3.2.1). It is similar in nature to the factor of 2 in difference between eqs. (3.1.1) and (4.1.1). Both are introduced to avoid factors of 2 in Darboux coordinates.

The antibracket (f,g) of two functions $f=f(z)$ and $g=g(z)$ can be defined via a double commutator with the odd Laplacian, acting on the constant unit function 1,

$$\begin{aligned} (f,g) \;&:=\; (-1)^{\varepsilon_f} [[\overrightarrow{\Delta_\rho}, f], g]1 \;\equiv\; (-1)^{\varepsilon_f} \Delta_\rho(fg) - (-1)^{\varepsilon_f} (\Delta_\rho f) g - f(\Delta_\rho g) + (-1)^{\varepsilon_g} fg(\Delta_\rho 1) \\ &=\; (f\frac{\overleftarrow{\partial^r}}{\partial z^A}) E^{AB} (\frac{\overrightarrow{\partial^\ell}}{\partial z^B} g) \;=\; -(-1)^{(\varepsilon_f+1)(\varepsilon_g+1)} (g,f) \;. \end{aligned} \tag{4.2.2}$$

The antibracket (4.2.2) satisfies a Jacobi identity,

$$\sum_{\text{cycl. } f,g,h} (-1)^{(\varepsilon_f+1)(\varepsilon_h+1)} (f,(g,h)) \;=\; 0 \;, \tag{4.2.3}$$

because of the closeness condition (4.1.4).

4.3 Odd Scalar ν_ρ

A Grassmann–odd function ν_ρ can be constructed from the antisymplectic metric E and an arbitrary density ρ as

$$\nu_\rho \;:=\; \nu_\rho^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24} \;, \tag{4.3.1}$$

where

$$\nu_\rho^{(0)} \;:=\; \frac{1}{\sqrt{\rho}} (\Delta_1 \sqrt{\rho}) \;, \tag{4.3.2}$$

$$\nu^{(1)} \;:=\; (-1)^{\varepsilon_A} (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} E^{AB} \frac{\overleftarrow{\partial^r}}{\partial z^B}) (-1)^{\varepsilon_B} \;, \tag{4.3.3}$$

$$\begin{aligned} \nu^{(2)} \;&:=\; -(-1)^{\varepsilon_B} (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} E_{BC}) (z^C, (z^B, z^A)) \\ &=\; (-1)^{\varepsilon_A \varepsilon_D} (\frac{\overrightarrow{\partial^\ell}}{\partial z^D} E^{AB}) E_{BC} (E^{CD} \frac{\overleftarrow{\partial^r}}{\partial z^A}) \;. \end{aligned} \tag{4.3.4}$$

Here Δ_1 is the odd Laplacian (4.2.1) with $\rho = 1$, and $(\cdot,\cdot)$ is the antibracket (4.2.2).

Lemma 4.1 *The odd quantity ν_ρ is a scalar, i.e., it does not depend on the coordinate system.*

The proof of Lemma 4.1 is given in Ref. [22]. Below follows an antisymplectic version of Proposition 3.2.

Proposition 4.2 (Classification of 2–order differential invariants) *If a function $\nu = \nu(z)$ has the following properties:*

1. *The function ν is a scalar.*
2. *$\nu(z)$ is a polynomial of the metric $E_{AB}(z)$, the density $\rho(z)$, their inverses, and z–derivatives thereof in the point z.*
3. *ν is invariant under constant rescaling of the density $\rho \to \lambda\rho$, where λ is a z–independent parameter.*
4. *ν scales as $\nu \to \lambda\nu$ under constant Weyl scaling $E^{AB} \to \lambda E^{AB}$, where λ is a z–independent parameter.*
5. *Each term in ν contains precisely two z–derivatives.*

Then ν is proportional to the odd scalar ν_ρ

$$\nu \;=\; \alpha\, \nu_\rho \;, \tag{4.3.5}$$

where α is z–independent proportionality constant.

The proof of Proposition 4.2 is similar to the proof of Proposition 3.2.

4.4 Δ And Δ_E

Khudaverdian's Δ_E operator [17, 18, 19, 20, 21, 22, 23], which takes semidensities to semidensities, is defined using arbitrary coordinates as

$$\Delta_E \;:=\; \Delta_1 + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24} \;. \tag{4.4.1}$$

It is obviously manifestly independent of ρ. That it takes semidensities to semidensities will become clear because of eq. (4.4.3) below. The Jacobi identity (4.2.3) precisely encodes the nilpotency of Δ_E. The Grassmann–odd nilpotent Δ operator, which takes scalar functions to scalar functions, can be defined as defined as

$$\Delta \;:=\; \Delta_\rho + \nu_\rho \;. \tag{4.4.2}$$

In fact, every Grassmann–odd, nilpotent, second–order operator is of the form (4.4.2) up to a Grassmann–odd constant [4]. We shall dismiss Grassmann–odd constants since they do not satisfy all the five assumptions of Proposition 4.2. The Δ_E operator and the Δ operator are related via $\sqrt{\rho}$–conjugation [22, 4]

$$\Delta_E \;=\; \sqrt{\rho}\Delta\frac{1}{\sqrt{\rho}} \;. \tag{4.4.3}$$

The proof is almost identical to the corresponding Riemannian calculation (3.5.4).

Recall how the zeroth–order term is determined in the Riemannian case, where no nilpotency principle was available, cf. Subsections 3.5 and 3.9. There we applied a ρ independence test. Could one do a similar analysis in the antisymplectic case? Yes. In detail, consider an operator

$$\Delta' \;:=\; \Delta+\nu = \Delta_{\rho}+\nu_{\rho}+\nu \ , \tag{4.4.4}$$

where ν is a most general zeroth–order term. It is easy to see from eqs. (4.4.3) and (4.4.4) that the corresponding $\sqrt{\rho}$–conjugated operator $\sqrt{\rho}\Delta'\frac{1}{\sqrt{\rho}}$ is independent of ρ if and only if the shift term ν is ρ–independent. From Proposition 4.2, one then concludes that $\nu=0$ has to be zero, *i.e.*, the form of the Δ operator (4.4.2) can be uniquely reproduced from a ρ–independence test and knowledge about possible scalar structures.

4.5 Antisymplectic Connection

A connection $\nabla^{(\Gamma)}$ is called *antisymplectic*, if it preserves the antisymplectic metric E,

$$0 \;=\; (\nabla^{(\Gamma)}_{A}E)_{BC} \;=\; (\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}E_{BC}) - ((-1)^{\varepsilon_{A}\varepsilon_{B}}\Gamma_{BAC} - (-1)^{\varepsilon_{B}\varepsilon_{C}}(B\leftrightarrow C)) \ . \tag{4.5.1}$$

Here we have lowered the Christoffel symbol with the metric

$$\Gamma_{ABC} \;:=\; E_{AD}\Gamma^{D}{}_{BC}(-1)^{\varepsilon_{B}} \ . \tag{4.5.2}$$

We should stress that there is not a unique choice of an antisymplectic, torsionfree, and ρ–compatible connection $\nabla^{(\Gamma)}$, *i.e.*, a connection that satisfies eqs. (4.5.1), (2.2.3) and (2.3.3). On the other hand, it can be demonstrated that such connections $\nabla^{(\Gamma)}$ exist locally for $N>2$, where $N=\dim(M)$ denotes the dimension of the manifold M. (There are counterexamples for $N=2$ where $\nabla^{(\Gamma)}$ need not exist.) The mere existence of an antisymplectic and torsionfree connection $\nabla^{(\Gamma)}$ implies that the two–form E is closed (4.1.4), if we hadn't already assumed it in the first place. (Curiously, while it is impossible to impose closeness relations in Riemannian geometry, the closeness relations are almost impossible to avoid in geometric structures defined by two–forms.) The antisymplectic condition (4.5.1) reads in terms of the contravariant (inverse) metric

$$0 \;=\; (\nabla^{(\Gamma)}_{A}E)^{BC} \;\equiv\; (\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}E^{BC}) + \Big(\Gamma_{A}{}^{B}{}_{D}E^{DC} - (-1)^{(\varepsilon_{B}+1)(\varepsilon_{C}+1)}(B\leftrightarrow C)\Big) \ . \tag{4.5.3}$$

4.6 The Riemann Curvature

For an antisymplectic connection $\nabla^{(\Gamma)}$, we prefer to work with a $(0,4)$ Riemann tensor (as opposed to a $(1,3)$ tensor) by lowering the upper index with the metric (4.1.1). In terms of Christoffel symbols it is easiest to work with expression (2.4.2):

$$\begin{aligned} R_{D,ABC} \;&:=\; E_{DF}R^{F}{}_{ABC} \\ &=\; (-1)^{\varepsilon_{A}(\varepsilon_{D}+1)}\left((-1)^{\varepsilon_{B}}\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}\Gamma_{DBC} + (-1)^{\varepsilon_{F}(\varepsilon_{A}+\varepsilon_{D})}\Gamma_{FAD}\Gamma^{F}{}_{BC}\right. \\ &\quad \left. -(-1)^{\varepsilon_{A}\varepsilon_{B}}(A\leftrightarrow B)\right) . \end{aligned} \tag{4.6.1}$$

In the second equality of eq. (4.6.1) is used the antisymplectic condition (4.5.1). If the antisymplectic condition (4.5.1) is used one more time on the first term in eq. (4.6.1), one derives the following symmetry

$$R_{D,ABC} \;=\; (-1)^{(\varepsilon_{A}+\varepsilon_{B})(\varepsilon_{C}+\varepsilon_{D})+\varepsilon_{C}\varepsilon_{D}}(C\leftrightarrow D) \ . \tag{4.6.2}$$

This symmetry becomes clearer if one instead starts from expression (2.4.6) and defines

$$R_{AB,CD} \;:=\; R_{ABC}{}^{F} E_{FD} \;=\; -(-1)^{\varepsilon_A+\varepsilon_B+(\varepsilon_A+\varepsilon_B+\varepsilon_C)\varepsilon_D} R_{D,ABC} \; . \tag{4.6.3}$$

Then the symmetry (4.6.2) simply translates into a symmetry between the third and fourth index:

$$R_{AB,CD} \;=\; (-1)^{\varepsilon_C\varepsilon_D}(C \leftrightarrow D) \; . \tag{4.6.4}$$

The Ricci 2–form is then

$$\mathcal{R}_{AB} \;=:\; R_{AB}{}^{C}{}_{C}(-1)^{\varepsilon_C} \;=\; R_{AB,CD}E^{DC}(-1)^{\varepsilon_C} \; . \tag{4.6.5}$$

We note that the torsionfree condition has not been used so far in this Section 4.6. The first Bianchi identity (2.4.7) reads (in the torsionfree case):

$$0 \;=\; \sum_{\text{cycl. } A,B,C} (-1)^{\varepsilon_A\varepsilon_C} R_{AB,CD} \; . \tag{4.6.6}$$

4.7 Odd Scalar Curvature

The odd scalar curvature is defined as

$$R \;:=\; E^{BA}\mathcal{R}_{AB} \;=\; \mathcal{R}_{AB}E^{BA} \; . \tag{4.7.1}$$

Proposition 4.3 *For an arbitrary, antisymplectic, torsionfree, and ρ–compatible connections ∇^{Γ}, the scalar curvature R does only depend on E and ρ through the odd ν_{ρ} scalar [4]*

$$R \;=\; -8\nu_{\rho} \; . \tag{4.7.2}$$

The proof of Proposition 4.3 is given in Ref. [4]. It is extended to degenerate anti–Poisson structures in Ref. [23, 37]. In particular, one concludes that the odd scalar curvature R does not depend on the connection used, and the odd Δ operator (4.4.2) reduces to the odd Δ operator (1.0.1) in the Introduction.

Altogether, we have now established a link between the zeroth–order terms in the even and odd Δ operators (1.0.3) and (1.0.1):

$$\begin{array}{ccc} \text{Riemannian zeroth order term} & & \text{Antisymplectic zeroth order term} \\ -\dfrac{R}{4} \;=\; \nu_{\rho_g} & \longleftrightarrow & 2\nu_{\rho} \;=\; -\dfrac{R}{4} \; . \end{array} \tag{4.7.3}$$

The left (resp. right) equality is due to Proposition 3.3 (resp. 4.3). Both zeroth–order terms are characterized by the same ρ–independence test described in Subsections 3.9 and 4.4 (up to a subtlety on how to switch back and forth between ρ–dependent and ρ–independent formalism in the Riemannian case). It is no coincidence that the same coefficient minus–a–quarter appears on both sides of the correspondence (after the odd Δ operator has been multiplied with an appropriate factor 2). At the mathematical level, this is basically because the zeroth–order terms are determined by the $\nu_{\rho}^{(0)}$ building blocks alone, where the inverse metrics g^{AB} and E^{AB} enter in a similar manner, and only linearly. For expressions that do not depend on the metric tensors g_{AB} and E_{AB}, and only have an linear dependence of the inverse metrics g^{AB} and E^{AB}, respectively, one does not see the effects that distinguish Riemannian and antisymplectic geometry, such as *e.g.*, opposite Grassmann–parity, closeness relations and the Jacobi identities.

4.8 First–Order S^{AB} Matrices

Because of the presence of the antisymplectic tensor E^{AB}, the symmetry of the general linear (= gl) Lie–algebra (2.7.6) reduces to an antisymplectic Lie–subalgebra. Its generators $S^{AB}_{\pm}$ read

$$S^{AB}_{\pm} \ := \ C^{A}(-1)^{\varepsilon_{B}}P^{B} \ \mp \ (-1)^{(\varepsilon_{A}+1)(\varepsilon_{B}+1)}(A \leftrightarrow B) \ , \qquad P^{A} \ := \ E^{AB}\frac{\overrightarrow{\partial^{\ell}}}{\partial C^{B}} \ , \qquad (4.8.1)$$

$$\varepsilon(S^{AB}_{\pm}) \ = \ \varepsilon_{A}+\varepsilon_{B}+1 \ , \qquad p(S^{AB}_{\pm}) \ = \ 0 \ , \qquad (4.8.2)$$

$$S^{A}_{\pm C} \ := \ S^{AB}_{\pm} \ E_{BC}(-1)^{\varepsilon_{C}} \ . \qquad (4.8.3)$$

The S^{AB}_{+} matrices satisfy an antisymplectic Lie–algebra:

$$\begin{aligned}[S^{AB}_{\pm}, S^{CD}_{\pm}] \ &= \ (-1)^{\varepsilon_{A}(\varepsilon_{B}+\varepsilon_{C}+1)+\varepsilon_{B}}\left(E^{BC} \ S^{AD}_{+} - S^{BC}_{+} \ E^{AD}\right) \\ &\quad \mp(-1)^{(\varepsilon_{A}+1)(\varepsilon_{B}+1)}(A \leftrightarrow B) \ , \qquad (4.8.4) \\ [S^{AB}_{\pm}, S^{CD}_{\mp}] \ &= \ (-1)^{\varepsilon_{A}(\varepsilon_{B}+\varepsilon_{C}+1)+\varepsilon_{B}}\left(E^{BC} \ S^{AD}_{-} + S^{BC}_{-} \ E^{AD}\right) \\ &\quad \mp(-1)^{(\varepsilon_{A}+1)(\varepsilon_{B}+1)}(A \leftrightarrow B) \ . \qquad (4.8.5)\end{aligned}$$

Note that the eqs. (4.8.4) and (4.8.5) remain invariant under a c–number shift

$$S^{AB}_{+} \ \to \ S'^{AB}_{+} \ := \ S^{AB}_{+} + \alpha E^{AB}\mathbf{1} \ , \qquad (4.8.6)$$

where α is a parameter.

4.9 Γ^{A} Matrices

Guided by the analysis of Appendix 4.B, we now define antisymplectic Γ^{A} matrices via the following Berezin–Fradkin operator representation [30, 31]

$$\Gamma^{A}_{\theta} \ \equiv \ \Gamma^{A} \ := \ C^{A} + (-1)^{\varepsilon_{A}}\theta P^{A} \ = \ C^{A} - P^{A}\theta \ , \qquad \varepsilon(\Gamma^{A}) \ = \ \varepsilon_{A} \ , \qquad p(\Gamma^{A}) \ = \ 1 \ (\text{mod } 2) \ , \qquad (4.9.1)$$

where θ is a nilpotent Fermionic parameter with $\varepsilon(\theta)=1$ and $p(\theta)=0$. The Γ^{A} matrices satisfy a Clifford–like algebra

$$[\Gamma^{A}, \Gamma^{B}] \ = \ 2(-1)^{\varepsilon_{A}}\theta E^{AB}\mathbf{1} \ . \qquad (4.9.2)$$

The Γ^{A} matrices form a fundamental representation of the antisymplectic Lie–algebra (4.8.4):

$$[S^{AB}_{\pm}, \Gamma^{C}] \ = \ \Gamma^{A}_{\pm\theta}(-1)^{\varepsilon_{B}}E^{BC} \mp (-1)^{(\varepsilon_{A}+1)(\varepsilon_{B}+1)}(A \leftrightarrow B) \ . \qquad (4.9.3)$$

If one commutes an antisymplectic connection $\nabla^{(T)}_{A}$ in the $T^{A}{}_{B}$ representation (2.7.4) with a Γ^{B} matrix, one gets

$$[\nabla^{(T)}_{A}, \Gamma^{B}] \ = \ -\Gamma_{A}{}^{B}{}_{C} \ \Gamma^{C} \ . \qquad (4.9.4)$$

4.10 Dirac Operator $D^{(T)} = d + \theta\delta$

We shall for the remainder of Section 4 assume that the connection is antisymplectic, torsionfree and ρ–compatible.

The Dirac operator $D^{(T)}$ in the $T^{A}{}_{B}$ representation (2.7.4) is a Γ^{A} matrix (4.9.1) times the covariant derivative (2.7.4)

$$D^{(T)} ~:=~ \Gamma^{A}\nabla^{(T)}_{A} ~=~ d+\theta\delta ~, \qquad \varepsilon(D^{(T)}) ~=~ 0 ~, \qquad p(D^{(T)}) ~=~ 1 ~(\mathrm{mod}~2) ~. \tag{4.10.1}$$

Unlike the Riemannian case of Subsection 3.15, the component δ of the Dirac operator to first order in θ does not have an interpretation as a Hodge codifferential, since there is no antisymplectic Hodge $*$ operation. Even worse, it depends explicitly on the Christoffel symbols:

$$\begin{aligned}\delta ~&:=~ (-1)^{\varepsilon_{A}}P^{A}\nabla^{(T)}_{A} ~=~ (-1)^{\varepsilon_{A}}[P^{A},\nabla^{(T)}_{A}] + (-1)^{\varepsilon_{A}}\nabla^{(T)}_{A}P^{A} \\ &=~ \Gamma^{A}{}_{AC}P^{C} + (-1)^{\varepsilon_{A}}\left(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}} + (-1)^{\varepsilon_{A}\varepsilon_{B}}\Gamma_{BAC}~C^{C}P^{B}\right)P^{A} \\ &=~ (-1)^{\varepsilon_{A}}\left(\frac{1}{\rho}\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}\rho + \Gamma_{ABC}~C^{C}P^{B}\right)P^{A} ~.\end{aligned} \tag{4.10.2}$$

Nevertheless, there exists a close antisymplectic analogue of Weitzenböck's formula (3.15.7), cf. eq. (4.10.5) below. The odd Laplacian $\Delta^{(T)}_{\rho}$ in the $T^{A}{}_{B}$ representation (2.7.4) is

$$2\Delta^{(T)}_{\rho} ~:=~ (-1)^{\varepsilon_{A}}\nabla_{A}E^{AB}\nabla^{(T)}_{B} ~=~ \frac{(-1)^{\varepsilon_{A}}}{\rho}\nabla^{(T)}_{A}\rho E^{AB}\nabla^{(T)}_{B} ~. \tag{4.10.3}$$

Theorem 4.4 (Antisymplectic Weitzenböck type formula for exterior forms) *The difference between the square of the Dirac operator $D^{(T)}$ and twice the odd Laplacian $\Delta^{(T)}_{\rho}$ in the $T^{A}{}_{B}$ representation is*

$$\begin{aligned}D^{(T)}D^{(T)} ~-~ 2\theta\Delta^{(T)}_{\rho} ~&=~ \frac{\theta}{4}(-1)^{\varepsilon_{B}+\varepsilon_{C}}S^{BA}_{-}~R_{AB,CD}~S^{DC}_{+} & (4.10.4)\\ &=~ -\theta C^{A}~R_{AB}~P^{B} + \frac{\theta}{2}C^{B}C^{A}~R_{AB,CD}~P^{D}P^{C}(-1)^{\varepsilon_{C}} ~. & (4.10.5)\end{aligned}$$

Proof of Theorem 4.4: The square is a sum of three terms

$$D^{(T)}D^{(T)} ~=~ \frac{1}{2}[D^{(T)},D^{(T)}] ~=~ I+II+III ~. \tag{4.10.6}$$

The first term is

$$I ~:=~ \frac{1}{2}[\Gamma^{B},\Gamma^{A}]\nabla^{(T)}_{A}\nabla^{(T)}_{B} ~=~ (-1)^{\varepsilon_{B}}\theta E^{BA}~\nabla^{(T)}_{A}\nabla^{(T)}_{B} ~. \tag{4.10.7}$$

The second term is

$$\begin{aligned}II ~&:=~ \Gamma^{A}[\nabla^{(T)}_{A},\Gamma^{B}]\nabla^{(T)}_{B} ~\stackrel{(4.9.4)}{=}~ -\Gamma^{A}~\Gamma_{A}{}^{B}{}_{C}~\Gamma^{C}~\nabla^{(T)}_{B} ~=~ -(-1)^{\varepsilon_{C}}\Gamma^{B}{}_{CA}~\Gamma^{A}~\Gamma^{C}~\nabla^{(T)}_{B} \\ &=~ -(-1)^{\varepsilon_{B}}\theta\Gamma^{B}{}_{CA}~E^{AC}~\nabla^{(T)}_{B} ~=~ \theta\frac{(-1)^{\varepsilon_{A}}}{\rho}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}\rho E^{AB})\nabla^{(T)}_{B} ~.\end{aligned} \tag{4.10.8}$$

Together, the first two terms $I+II$ form the odd Laplacian (4.10.3):

$$I+II \;=\; 2\theta\Delta_{\rho}^{(T)}\ . \tag{4.10.9}$$

The third term yields the curvature terms:

$$\begin{aligned}
III &:= -\frac{1}{2}\Gamma^{B}\Gamma^{A}[\nabla^{(T)}_{A},\nabla^{(T)}_{B}] \;=\; \frac{1}{2}\Gamma^{B}\Gamma^{A}\ R_{AB}{}^{D}{}_{C}\ T^{C}D \;=\; \frac{1}{4}\Gamma^{B}\Gamma^{A}\ R_{AB,CD}\ S_{+}^{DC}(-1)^{\varepsilon_{C}} \\
&= \frac{1}{4}\left(C^{B}C^{A}+(-1)^{\varepsilon_{B}}\theta(S_{-}^{BA}+E^{BA})\right)R_{AB,CD}\ S_{+}^{DC}(-1)^{\varepsilon_{C}} \\
&= \frac{1}{2}C^{B}C^{A}\ R_{AB,CD}\ C^{C}P^{D}(-1)^{\varepsilon_{C}\varepsilon_{D}}+\frac{\theta}{4}(-1)^{\varepsilon_{B}+\varepsilon_{C}}S_{-}^{BA}\ R_{AB,CD}\ S_{+}^{DC} \\
&= \frac{\theta}{4}(-1)^{\varepsilon_{B}+\varepsilon_{C}}S_{-}^{BA}\ R_{AB,CD}\ S_{+}^{DC} \;=\; (-1)^{\varepsilon_{A}+\varepsilon_{B}}\theta C^{B}P^{A}\ R_{AB,CD}\ C^{D}P^{C} \\
&= -(-1)^{(\varepsilon_{A}+1)(\varepsilon_{C}+1)}\theta C^{B}\ R_{BA,CD}E^{DA}\ P^{C}-\theta C^{B}\ R_{BA,DC}\ C^{D}P^{C}P^{A}(-1)^{\varepsilon_{A}+\varepsilon_{C}\varepsilon_{D}} \\
&= -(-1)^{(\varepsilon_{A}+1)(\varepsilon_{C}+1)}\theta C^{B}\ R_{BAC}{}^{A}\ P^{C}+\theta C^{D}C^{B}\ R_{BA,DC}\ P^{C}P^{A}(-1)^{\varepsilon_{A}(\varepsilon_{D}+1)} \\
&= -\theta C^{B}\ R_{BC}\ P^{C}+\frac{\theta}{2}C^{B}C^{D}\ R_{DB,AC}\ P^{C}P^{A}(-1)^{\varepsilon_{A}}\ .
\end{aligned} \tag{4.10.10}$$

Here the first Bianchi identity (4.6.6) was used one time in the θ–independent sector.

□

4.A Appendix: Is There A Second–Order Formalism?

There are no deformations of the first–order S_{-}^{AB} matrices (4.8.1). The general second–order deformation of the S_{+}^{AB} matrices (4.8.1) reads

$$\Sigma_{+}^{AB} \;:=\; S_{+}^{AB}+\alpha E^{AB}\mathbf{1}+\beta P^{A}P^{B}\theta\ , \tag{4.A.1}$$

where α and β are two parameters. The second–order Σ_{+}^{AB} matrices satisfy precisely the same antisymplectic Lie–algebra (4.8.4) as the S_{+}^{AB} matrices. Moreover, the S_{+}^{AB} matrices rotate the Σ_{+}^{AB} matrices,

$$[\Sigma_{+}^{AB},S_{+}^{CD}] \;=\; (-1)^{\varepsilon_{A}(\varepsilon_{B}+\varepsilon_{C}+1)+\varepsilon_{B}}\left(E^{BC}\ \Sigma_{+}^{AD}-\Sigma_{+}^{BC}\ E^{AD}\right)-(-1)^{(\varepsilon_{A}+1)(\varepsilon_{B}+1)}(A\leftrightarrow B)\ . \tag{4.A.2}$$

The Σ_{+}^{AB} matrices interact with the Γ^{C} and the S_{-}^{CD} matrices as follows

$$[\Sigma_{+}^{AB},\Gamma^{C}] \;=\; \Gamma^{A}_{(1+\beta)\theta}(-1)^{\varepsilon_{B}}E^{BC}-(-1)^{(\varepsilon_{A}+1)(\varepsilon_{B}+1)}(A\leftrightarrow B)\ , \tag{4.A.3}$$

$$\begin{aligned}
[\Sigma_{+}^{AB},S_{-}^{CD}] \;=\;& (-1)^{\varepsilon_{A}(\varepsilon_{B}+\varepsilon_{C}+1)+\varepsilon_{B}}\left(E^{BC}\ \tilde{\Sigma}^{AD}+\tilde{\Sigma}^{BC}\ E^{AD}\right) \\
&-(-1)^{(\varepsilon_{A}+1)(\varepsilon_{B}+1)}(A\leftrightarrow B)\ ,
\end{aligned} \tag{4.A.4}$$

where the generators

$$\tilde{\Sigma}^{AB} \;:=\; S_{-}^{AB}+\beta P^{A}P^{B}\theta \tag{4.A.5}$$

have no $(A\leftrightarrow B)$ symmetry or antisymmetry. According to eq. (4.A.3), one must choose the parameter $\beta=0$ to be zero, in order to ensure that the Σ_{+}^{AB} matrices rotates the Γ^{A} matrices in the correct way. One concludes that a consistent antisymplectic second–order formulation does not exist, regardless of whether the pathological S_{-}^{AB} sector decouples or not, and we shall abandon the subject. See also comment in the Conclusions.

4.B Appendix: What Is An Antisymplectic Clifford Algebra?

In this Appendix 4.B, we shall motivate the definition (4.9.2) of an antisymplectic Clifford algebra given in Subsection 4.9. Intuitively, one would probably assume that an antisymplectic Clifford algebra should be

$$\Gamma^{A}\star\Gamma^{B}-(-1)^{(\varepsilon_{A}+1)(\varepsilon_{B}+1)}(A\leftrightarrow B)\ \stackrel{?}{=}\ 2E^{AB}\mathbf{1}\ , \tag{4.B.1}$$

where the "$\star$" denotes a Fermionic multiplication, $\varepsilon(\star)=1$, cf. question 3 in the Introduction. We will now expose some of the weaknesses of the proposal (4.B.1). (A question mark "?" on top of an equality sign "=" indicates that a formula may be ultimately wrong.) It follows from eq. (1.0.6) that the form degree of the $\star$ multiplication must vanish, $p(\star)=0$. Let us assume that the $\star$ multiplication is invertible and commute with the Γ^{A} matrices,

$$\Gamma^{A}\star-(-1)^{\varepsilon(\Gamma_{A})}\star\Gamma^{A}\ \equiv\ [\Gamma^{A},\star]\ \stackrel{?}{=}\ 0\ . \tag{4.B.2}$$

Then one can bring the Clifford algebra (4.B.1) on a Riemannian form,

$$\Gamma^{A}\Gamma^{B}+(-1)^{\varepsilon_{A}\varepsilon_{B}}(A\leftrightarrow B)\ =\ 2g^{AB}\mathbf{1}\ , \tag{4.B.3}$$

where the Riemannian metric g^{AB} is a product of $\star^{-1}$ and the antisymplectic metric E^{AB},

$$g^{AB}\ :=\ (-1)^{\varepsilon(\Gamma_{A})}\star^{-1}E^{AB}\ =\ (-1)^{\varepsilon_{A}\varepsilon_{B}}(A\leftrightarrow B)\ ,\qquad \varepsilon(g^{AB})\ =\ \varepsilon_{A}+\varepsilon_{B}\ . \tag{4.B.4}$$

The Riemannian structure (4.B.4) is non–commutative,

$$[g^{AB},g^{CD}]\ =\ -2(-1)^{\varepsilon_{B}+\varepsilon_{C}}\star^{-2}E^{AB}E^{CD}\ \neq\ 0\ , \tag{4.B.5}$$

since $[\star^{-1},\star^{-1}]=2\star^{-2}\neq 0$, and hence the metric (4.B.4) is not a classical Riemannian metric. We would like to interpret the left–hand side of eq. (4.B.3) as a commutator $[\Gamma^{A},\Gamma^{B}]$, cf. definition (1.1.3). This implies that the Grassmann– and form–parity of the Γ^{A} matrices are

$$\varepsilon(\Gamma^{A})\ =\ \varepsilon_{A}\ ,\qquad p(\Gamma^{A})\ =\ 1\ (\mathrm{mod}\ 2)\ . \tag{4.B.6}$$

The only natural candidate for a Berezin–Fradkin operator representation [30, 31] is

$$\Gamma^{A}\ =\ C^{A}+g^{AB}\frac{\overrightarrow{\partial^{\ell}}}{\partial C^{B}}\ \equiv\ C^{A}-P^{A}\star^{-1}\ ,\qquad \varepsilon(C^{A})\ =\ \varepsilon_{A}\ ,\qquad p(C^{A})\ =\ 1\ , \tag{4.B.7}$$

where the C^{A} variables commute with the $\star$ multiplication, $[C^{A},\star]=0$, and they carry the same Grassmann– and form–parities as the Γ^{A} matrices. The P^{A} derivatives are defined in eq. (4.8.1). However, the Berezin–Fradkin operator representation (4.B.7) does not satisfy the Clifford algebra (4.B.3) due to the non–commutative metric (4.B.5). The representation does also violate the commutation relation (4.B.2). There appear extra terms on the respective right–hand sides,

$$[\Gamma^{A},\star^{-1}]\ =\ -2\star^{-2}P^{A}\ , \tag{4.B.8}$$

$$[\Gamma^{A},\Gamma^{B}]\ =\ 2g^{AB}\mathbf{1}-2\star^{-2}P^{A}P^{B}(-1)^{\varepsilon_{B}}\ . \tag{4.B.9}$$

The original antisymplectic Clifford algebra (4.B.1) looks even more complicated:

$$\frac{1}{2}\Gamma^{A}\star\Gamma^{B}-(-1)^{(\varepsilon_{A}+1)(\varepsilon_{B}+1)}(A\leftrightarrow B)\ =\ S_{+}^{AB}-E^{AB}\mathbf{1}+P^{A}P^{B}\star^{-1}\ . \tag{4.B.10}$$

One idea would be to try to correct the Clifford algebra (4.B.9) by adding higher–order terms $\mathcal{O}(\star^{-2})$ to the Berezin–Fradkin operator representation (4.B.7), but unfortunately there is no obvious way to do that. Another idea is to take the limit $\star^{-1} \to 0$ in some appropriate way at the end of the calculations. The approach that we shall pursuit in this paper is to take $\theta \equiv \star^{-1}$ as a fundamental object, *i.e.*, forgetting that it originally was an inverse of $\star$, and then assume that it is nilpotent $\theta^2 = \star^{-2} = 0$. Then the Γ^A matrices and the θ variable commute $[\Gamma^A, \theta] = 0$, the Riemannian metric (4.B.4) becomes an ordinary commutative structure, and the Clifford algebra (4.B.3) is restored. The price is that the Fermionic $\star$ multiplication (4.B.1), which ironically was our initial clue, does not exist.

5 General Spin Theory

5.1 Spin Manifold

Let W be a vector space of the same dimension as the manifold M. Let the vectors (=points) in W have coordinates w^a of Grassmann–parity $\varepsilon(w^a) = \varepsilon_a$ (and form–degree $p(w^a) = 0$). It is assumed that the *flat* index "a" (denoted with a small roman letter) of the vector space W runs over the same index–set as the *curved* index "A" (denoted with a capital roman letter) of the manifold M. In a slight misuse of notation, let $TW := M \times W$ (resp. $T^*W := M \times W^*$) denote the trivial vector bundle over M with the vector space W (resp. dual vector space W^*) as fiber. Let ∂^r_a and $\overrightarrow{dw}^a$ denote dual bases in W and W^*, respectively, of Grassmann–parity $\varepsilon(\overrightarrow{dw}^a) = \varepsilon_a = \varepsilon(\partial^r_a)$. The form–parities $p(\overrightarrow{dw}^a) = p(\partial^r_a)$ are either all 0 or all 1, depending on applications, whereas a 1–form dw^a with no arrow "$\to$" always carries odd form–parity $p(dw^a) = 1$ (and Grassmann–parity $\varepsilon(dw^a) = \varepsilon_a$).

Let us assume that M is a spin manifold, *i.e.*, that there exists a bijective bundle map

$$e \;=\; \partial^r_a \; e^a{}_A \; \overrightarrow{dz}^A \;:\; \Gamma(TM) \to \Gamma(TW) \; , \tag{5.1.1}$$

$$e^{-1} \;=\; \partial^r_A \; e^A{}_a \; \overrightarrow{dw}^a \;:\; \Gamma(TW) \to \Gamma(TM) \; . \tag{5.1.2}$$

The intertwining tensor field $e^a{}_A$ is known as a vielbein. (There are topological obstructions for the existence of a global vielbein. However, it would be out of scope to describe global notions for supermanifolds here, such as, orientability and Stiefel–Whitney classes. The interesting topic of index theorems for Dirac operators will for similar reasons be omitted in this paper.)

Note that the superdeterminant $\mathrm{sdet}(e^a{}_A) \neq 0$ of the vielbein transforms as a density under general coordinate transformations. In general, the vielbein $e^a{}_A$ is called compatible with the measure density ρ, if

$$\rho \;\sim\; \mathrm{sdet}(e^a{}_A) \tag{5.1.3}$$

is proportional to the vielbein superdeterminant $\mathrm{sdet}(e^a{}_A)$ with a z–independent proportionality factor. In this case, the notion of volume is unique (up to an overall rescaling).

5.2 Spin Connection $\nabla^{(\omega)} = d + \omega$

A connection $\nabla^{(\omega)} = d + \omega \;:\; \Gamma(TM) \times \Gamma(TW) \to \Gamma(TW)$ in the bundle TW is known as a spin connection, where

$$\nabla^{(\omega)}_A \;=\; \frac{\overrightarrow{\partial^\ell}}{\partial z^A} + \partial^r_b \; \omega^b{}_{Ac} \; \overrightarrow{dw}^c \quad . \tag{5.2.1}$$

The total connection $\nabla = d + \Gamma + \omega$ contains both a Christoffel symbol $\Gamma^B{}_{AC}$, which acts on curved indices, and a spin connection $\omega^b{}_{Ac}$, which acts on flat indices. We will always demand that the total connection ∇ preserves the vielbein

$$0 \;=\; (\nabla_A e^b{}_C) \;=\; (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} e^b{}_C) - (-1)^{\varepsilon_A\varepsilon_b} e^b{}_B\ \Gamma^B{}_{AC} + \omega_A{}^b{}_c\ e^c{}_C\ . \tag{5.2.2}$$

This condition (5.2.2) fixes uniquely the spin connection as

$$\begin{aligned}
\omega^b{}_{Ac} &:= \Gamma^b{}_{Ac} - f^b{}_{Ac}\ , &(5.2.3)\\
\omega_A{}^b{}_c &:= \Gamma_A{}^b{}_c - f_A{}^b{}_c \;=\; (-1)^{\varepsilon_A\varepsilon_b}\omega^b{}_{Ac}\ , &(5.2.4)\\
\omega_a{}^b{}_c &:= \Gamma_a{}^b{}_c - f_a{}^b{}_c \;=\; (e^T)_a{}^A\ \omega_A{}^b{}_c\ , &(5.2.5)\\
\omega^b{}_{ac} &:= \Gamma^b{}_{ac} - f^b{}_{ac} \;=\; (-1)^{\varepsilon_a\varepsilon_b}\omega_a{}^b{}_c\ , &(5.2.6)
\end{aligned}$$

where

$$\begin{aligned}
\Gamma^b{}_{Ac} &:= e^b{}_B\ \Gamma^B{}_{AC}\ e^C{}_c\ , &(5.2.7)\\
\Gamma_A{}^b{}_c &:= (-1)^{\varepsilon_A\varepsilon_b}\Gamma^b{}_{Ac}\ , &(5.2.8)\\
\Gamma_a{}^b{}_c &:= (e^T)_a{}^A\ \Gamma_A{}^b{}_c\ , &(5.2.9)\\
\Gamma^b{}_{ac} &:= (-1)^{\varepsilon_a\varepsilon_b}\Gamma_a{}^b{}_c\ , &(5.2.10)\\
f_A{}^b{}_c &:= (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} e^b{}_D) e^D{}_c\ , &(5.2.11)\\
f^b{}_{Ac} &:= (-1)^{\varepsilon_A\varepsilon_b} f_A{}^b{}_c\ , &(5.2.12)\\
f_a{}^b{}_c &:= (e^T)_a{}^A\ f_A{}^b{}_c\ , &(5.2.13)\\
f^b{}_{ac} &:= (-1)^{\varepsilon_a\varepsilon_b} f_a{}^b{}_c\ . &(5.2.14)
\end{aligned}$$

Here the transposed vielbein is

$$(e^T)_A{}^a \;:=\; (-1)^{(\varepsilon_a+1)\varepsilon_A} e^a{}_A\ . \tag{5.2.15}$$

The condition (5.2.2) implies in many cases that one can transfer concepts/objects back and forth between TM and TW by simply multiplying with appropriate factors of the vielbein. Firstly, the spin connection $\nabla^{(\omega)}_A : \Gamma(TW) \to \Gamma(TW)$ can in a certain sense be thought of as the connection $\nabla^{(\Gamma)}_A : \Gamma(TM) \to \Gamma(TM)$ conjugated with the vielbein $e : \Gamma(TM) \to \Gamma(TW)$, *i.e.*, roughly speaking a product of three matrices,

$$\begin{aligned}
e\nabla^{(\Gamma)}_A e^{-1} &= \partial^r_b\ e^b{}_B\ \overrightarrow{dz^B}\ (\frac{\overrightarrow{\partial^\ell}}{\partial z^A} + \partial^r_D\ \Gamma^D{}_{AE}\ \overrightarrow{dz^E})\ \partial^r_C\ e^C{}_c\ \overrightarrow{dw^c}\\
&= \frac{\overrightarrow{\partial^\ell}}{\partial z^A} + (-1)^{\varepsilon_A\varepsilon_D}\partial^r_b\ e^b{}_D(\frac{\overrightarrow{\partial^\ell}}{\partial z^A} e^D{}_c)\ \overrightarrow{dw^c} + \partial^r_b\ \Gamma^b{}_{Ac}\ \overrightarrow{dw^c} \overset{(5.2.2)}{=} \nabla^{(\omega)}_A\ .
\end{aligned} \tag{5.2.16}$$

Secondly, the torsion tensors $T^{(\omega)b}{}_{AC}$ for the $\nabla^{(\omega)}$ connection is equal to the torsion tensor $T^{(\Gamma)B}{}_{AC}$ for the $\nabla^{(\Gamma)}$ connection up to a vielbein factor:

$$T^{(\omega)a}{}_{BC} \;=\; e^a{}_A\ T^{(\Gamma)A}{}_{BC}\ . \tag{5.2.17}$$

This follows from

$$\begin{aligned}
T^{(\omega)} &\equiv \frac{1}{2}dz^A \wedge \partial^r_b\ T^{(\omega)b}{}_{AC}\ dz^C \;:=\; [\nabla^{(\omega)} \overset{\wedge}{,} e] \;=\; [dz^A \frac{\overrightarrow{\partial^\ell}}{\partial z^A} + dz^A\ \partial^r_b\ \omega^b{}_{Ad}\ \overrightarrow{dw^d} \overset{\wedge}{,} \partial^r_c\ e^c{}_C\ dz^C]\\
&= dz^A \wedge \partial^r_b \left((-1)^{\varepsilon_A\varepsilon_b} \frac{\overrightarrow{\partial^\ell}}{\partial z^A} e^b{}_C + \omega^b{}_{Ac}\ e^c{}_C \right) dz^C \overset{(5.2.2)}{=} dz^A \wedge \partial^r_b\ e^b{}_B\ \Gamma^B{}_{AC}\ dz^C
\end{aligned}$$

$$= \frac{1}{2} dz^A \wedge \partial^r_b \; e^b{}_B \; T^{(\Gamma)B}{}_{AC} \; dz^C \; . \tag{5.2.18}$$

In particular, the two connections $\nabla^{(\Gamma)}$ and $\nabla^{(\omega)}$ are torsionfree at the same time.

Thirdly, if the $\nabla^{(\Gamma)}_A$ connection and the vielbein $e^a{}_A$ are both compatible with the density ρ, cf. eqs. (2.3.3) and (5.1.3), then the spin connection $\nabla^{(\omega)}_A$ becomes traceless,

$$\omega_A{}^b{}_b(-1)^{\varepsilon_b} \stackrel{(5.2.2)}{=} 0 \; . \tag{5.2.19}$$

Fourthly, the two Riemann curvature tensor $R^{(\Gamma)}$ and $R^{(\omega)}$ are related, see next Subsection 5.3. Fifthly, the two connections $\nabla^{(\Gamma)}$ and $\nabla^{(\omega)}$ respect an additional structure, such as a Riemannian (resp. an antisymplectic) structure at the same time, cf. Subsection 6.1 (resp. Subsection 7.1).

5.3 Spin Curvature

The spin curvature $R^{(\omega)}$ is defined as (half) the commutator of the $\nabla^{(\omega)}$ connection (5.2.1),

$$\begin{aligned} R^{(\omega)} &= \frac{1}{2}[\nabla^{(\omega)} \stackrel{\wedge}{,} \nabla^{(\omega)}] \;=\; -\frac{1}{2} dz^B \wedge dz^A \;\otimes\; [\nabla^{(\omega)}_A, \nabla^{(\omega)}_B] \\ &= -\frac{1}{2} dz^B \wedge dz^A \;\otimes\; \partial^r_d \; R^{(\omega)d}{}_{ABc} \; \overrightarrow{dw^c} \; , \end{aligned} \tag{5.3.1}$$

$$\begin{aligned} R^{(\omega)d}{}_{ABc} &= \overrightarrow{dw^d} \left([\nabla^{(\omega)}_A, \nabla^{(\omega)}_B]\partial^r_c\right) \\ &= (-1)^{\varepsilon_d\varepsilon_A}\left(\frac{\overrightarrow{\partial^\ell}}{\partial z^A}\omega^d{}_{Bc}\right) + \omega^d{}_{Ae}\;\omega^e{}_{Bc} - (-1)^{\varepsilon_A\varepsilon_B}(A \leftrightarrow B) \; . \end{aligned} \tag{5.3.2}$$

The two types of Riemann curvature tensors $R^{(\Gamma)}$ and $R^{(\omega)}$ are equal up to conjugation with vielbein factors

$$R^{(\omega)d}{}_{ABc} \;=\; e^d{}_D \; R^{(\Gamma)D}{}_{ABC} \; e^C{}_c \; , \tag{5.3.3}$$

basically because curvature is a commutator of connections,

$$\begin{aligned} e\;\partial^r_D \; R^{(\Gamma)D}{}_{ABC} \; \overrightarrow{dz^C} \;\; e^{-1} &= e[\nabla^{(\Gamma)}_A, \nabla^{(\Gamma)}_B]e^{-1} \stackrel{(5.2.16)}{=} [\nabla^{(\omega)}_A, \nabla^{(\omega)}_B] \\ &= \partial^r_d \; R^{(\omega)d}{}_{ABc} \; \overrightarrow{dw^c} \; . \end{aligned} \tag{5.3.4}$$

5.4 Covariant Tensors with Flat Indices

Let

$$\Omega_{mn}(W) \;:=\; \Gamma\left(\bigwedge{}^m(T^*W) \otimes \bigvee{}^n(T^*W)\right) \tag{5.4.1}$$

be the vector space of $(0, m{+}n)$–tensors $\eta_{a_1\cdots a_m b_1\cdots b_n}(z)$ that are antisymmetric with respect to the first m indices $a_1 \ldots a_m$, and symmetric with respect to the last n indices $b_1 \ldots b_n$. As usual, it is practical to introduce a coordinate–free notation

$$\eta(z;c;y) \;=\; \frac{1}{m!n!} c^{a_m} \wedge \cdots \wedge c^{a_1} \;\; \eta_{a_1\cdots a_m b_1\cdots b_n}(z) \otimes y^{b_n} \vee \cdots \vee y^{b_1} \; . \tag{5.4.2}$$

Here the variables y^a are symmetric counterparts to the one–form basis $c^a \equiv dw^a$.

$$\begin{aligned} c^a \wedge c^b &= -(-1)^{\varepsilon_a\varepsilon_b} c^b \wedge c^a \; , \qquad & \varepsilon(c^a) &= \varepsilon_a \; , \qquad & p(c^a) &= 1 \; , \\ y^a \vee y^b &= (-1)^{\varepsilon_a\varepsilon_b} y^b \vee y^a \; , \qquad & \varepsilon(y^a) &= \varepsilon_a \; , \qquad & p(y^a) &= 0 \; . \end{aligned} \tag{5.4.3}$$

The covariant derivative can be realized on covariant tensors $\eta \in \Omega_{mn}(W)$ by a linear differential operator

$$\nabla^{(t)}_{A} \;:=\; \frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}} - \omega_{A}{}^{b}{}_{c}\; t^{c}{}_{b}\;, \tag{5.4.4}$$

where

$$t^{a}{}_{b} \;:=\; c^{a}\frac{\overrightarrow{\partial^{\ell}}}{\partial c^{b}} + y^{a}\frac{\overrightarrow{\partial^{\ell}}}{\partial y^{b}} \tag{5.4.5}$$

are generators of the Lie–algebra $gl(W)$, which reflects infinitesimal change of frame/basis in W, cf. eq. (2.7.6). The relation with the $\nabla^{(T)}_{A}$ realization (2.7.4) is

$$\nabla^{(T)}_{A}\eta(z;e^{b}{}_{B}C^{B};e^{c}{}_{C}Y^{C}) \;=\; (\nabla^{(t)}_{A}\eta)(z;c;y)\Bigg|_{\substack{c^{b}\,=\,e^{b}{}_{B}C^{B}\\ y^{c}\,=\,e^{c}{}_{C}Y^{C}}}\;, \tag{5.4.6}$$

because of condition (5.2.2), where $\eta\!=\!\eta(z;c;y)\!\in\!\Omega_{\bullet\bullet}(W)$ is a flat covariant tensor. The relationship (5.4.6) between the $\nabla^{(T)}$ and the $\nabla^{(t)}$ realizations, where one puts $c^{b}=e^{b}{}_{B}C^{B}$ and $y^{c}=e^{c}{}_{C}Y^{C}$, is of course just a particular case of the more general correspondence (5.2.16) between the $\nabla^{(\Gamma)}$ and the $\nabla^{(\omega)}$ connections.

5.5 Local Gauge Transformations

Consider for simplicity a flat one–form $\eta=\eta_{a}(z)c^{a}\in\Omega_{10}(W)$. The covariant derivative reads

$$(\nabla_{A}\eta)_{c} \;=\; (\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}\eta_{c}) - \eta_{b}\;\omega^{b}{}_{Ac}\;. \tag{5.5.1}$$

Under a local gauge transformation

$$\eta_{a} \;=\; \eta'_{b}\;\Lambda^{b}{}_{a}\;, \qquad\qquad c'^{a} \;=\; c^{a}\;, \tag{5.5.2}$$

where the group element $\Lambda^{a}{}_{b}\!=\!\Lambda^{a}{}_{b}(z)$ is z–dependent, the spin connection $\omega^{b}{}_{Ac}$ obeys the well–known affine transformation law for gauge potentials,

$$\Lambda^{b}{}_{a}\;\omega^{a}{}_{Ac} \;=\; (-1)^{\varepsilon_{A}\varepsilon_{b}}(\frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}}\Lambda^{b}{}_{c}) + \omega'^{b}{}_{Ad}\;\Lambda^{d}{}_{c}\;, \tag{5.5.3}$$

so that the covariant derivative transforms covariantly,

$$(\nabla_{A}\eta)_{a} \;=\; (\nabla_{A}\eta')_{b}\;\Lambda^{b}{}_{a}\;. \tag{5.5.4}$$

6 Riemannian Spin Geometry

6.1 Spin Geometry

Assume that the vector space W is endowed with a constant Riemannian metric

$$g^{(0)} \;=\; y^{a}\;g^{(0)}_{ab}\vee y^{b} \;\in\; \Omega_{02}(W)\;, \tag{6.1.1}$$

called the *flat* metric. It has Grassmann–parity

$$\varepsilon(g^{(0)}_{ab}) \;=\; \varepsilon_{a}+\varepsilon_{b}\;, \qquad \varepsilon(g^{(0)}) \;=\; 0\;, \qquad p(g^{(0)}_{AB}) \;=\; 0\;, \tag{6.1.2}$$

and symmetry

$$g^{(0)}_{ba} \;=\; -(-1)^{(\varepsilon_a+1)(\varepsilon_b+1)} g^{(0)}_{ab} \;. \tag{6.1.3}$$

Furthermore, assume that the vielbein $e^a{}_A$ intertwines between the curved g_{AB} metric and the flat $g^{(0)}_{ab}$ metric:

$$g_{AB} \;=\; (e^T)_A{}^a \; g^{(0)}_{ab} \; e^b{}_B \;. \tag{6.1.4}$$

As a consequence, the canonical Riemannian density (3.1.10) is compatible with the vielbein, *i.e.*, it is proportional to the vielbein superdeterminant,

$$\rho_g \;:=\; \sqrt{\mathrm{sdet}(g_{AB})} \;=\; \sqrt{\mathrm{sdet}(g^{(0)}_{ab})} \;\mathrm{sdet}(e^a{}_A) \;\;\sim\;\; \mathrm{sdet}(e^a{}_A) \;, \tag{6.1.5}$$

cf. eq. (5.1.3). A spin connection $\nabla^{(\omega)}$ is called *metric*, if it preserves the flat metric,

$$0 \;=\; -\nabla^{(\omega)}_A g^{(0)}_{bc} \;=\; \omega_{A,bc} - (-1)^{(\varepsilon_b+1)(\varepsilon_c+1)}\omega_{A,cb} \;, \tag{6.1.6}$$

i.e., the lowered $\omega_{A,bc}$ symbol should be skewsymmetric in the flat indices. Here we have lowered the $\omega_{A,bc}$ symbol with the flat metric

$$\omega_{A,bc} \;:=\; (-1)^{\varepsilon_A\varepsilon_b}\omega_{bAc}(-1)^{\varepsilon_c} \;, \qquad \omega_{bAc}(-1)^{\varepsilon_c} \;:=\; g^{(0)}_{bd}\,\omega^d{}_{Ac} \;. \tag{6.1.7}$$

In particular, the two connections $\nabla^{(\Gamma)}$ and $\nabla^{(\omega)}$ are metric at the same time, as a consequence of the correspondence (5.2.2) and (6.1.4). Note that we shall from now on put the y^a variables to zero $y^a \to 0$ everywhere, in analogy with the Y^a variables of Subsection 3.13.

6.2 Levi–Civita Spin Connection

The Levi–Civita spin connection $\nabla^{(\omega)}$ is by definition the unique spin connection that corresponds to the Levi–Civita connection $\nabla^{(\Gamma)}$ via the identifications (5.2.2) and (6.1.4). It is both torsionfree $T^{(\omega)}\!=\!0$ and preserves the metric (6.1.6). The Levi–Civita formula for the spin connection in terms of the vielbein reads

$$-2\omega_{bac} \;=\; (-1)^{\varepsilon_a\varepsilon_b} f_{a[bc]} + (-1)^{(\varepsilon_a+\varepsilon_b)\varepsilon_c} f_{c[ba]} + f_{b[ac]} \;, \tag{6.2.1}$$

where

$$f_{bac} \;:=\; g^{(0)}_{bd} f^d{}_{ac}(-1)^{\varepsilon_c} \;, \qquad \omega_{bac} \;:=\; g^{(0)}_{bd}\,\omega^d{}_{ac}(-1)^{\varepsilon_c} \;, \tag{6.2.2}$$

and where $f_{a[bc]} := f_{abc} - (-1)^{\varepsilon_b\varepsilon_c} f_{acb}$, cf. eqs. (5.2.11)–(5.2.14).

6.3 First–Order s^{ab} Matrices

Because of the presence of the flat metric $g^{ab}_{(0)}$, the symmetry of the general linear Lie–algebra $gl(W)$ reduces to an orthogonal Lie–subalgebra $o(W)$. Its generators $s^{ab}_{\mp}$ read

$$s^{ab}_{\mp} \;:=\; c^a p^b \;\mp\; (-1)^{\varepsilon_a\varepsilon_b}(a \leftrightarrow b) \;, \qquad p^a \;:=\; g^{ab}_{(0)} \frac{\overrightarrow{\partial^\ell}}{\partial c^b} \;, \tag{6.3.1}$$

$$\varepsilon(s^{ab}_{\mp}) \;=\; \varepsilon_a + \varepsilon_b \;, \qquad p(s^{ab}_{\mp}) \;=\; 0 \;, \tag{6.3.2}$$

$$s^a_{\mp c} \;:=\; s^{ab}_{\mp}\, g^{(0)}_{bc}(-1)^{\varepsilon_c} \;. \tag{6.3.3}$$

The transposed operator of a differential operator that depend on the flat c^a–variables is now defined to imitate integration by part. (This becomes important in Lemma 6.4 below.) Explicitly, the transposed fundamental operators are

$$\mathbf{1}^T \;=\; \mathbf{1} \;, \qquad (c^a)^T \;=\; c^a \;, \qquad (p^a)^T \;=\; -p^a \;. \tag{6.3.4}$$

Therefore the transposed $s^{ab}_{\mp}$ matrices read

$$(s^{ab}_{-})^T \;=\; -s^{ab}_{-} \;, \qquad (s^{ab}_{+})^T \;=\; 2g^{ab}_{(0)}\mathbf{1} - s^{ab}_{+} \;. \tag{6.3.5}$$

The $\nabla^{(t)}_A$ realization (5.4.4) can be identically rewritten into the following s^{ab} matrix realization

$$\nabla^{(s)}_A \;:=\; \frac{\overrightarrow{\partial^\ell}}{\partial z^A} - \frac{1}{2}\omega_{A,bc}\; s^{cb}_{-}(-1)^{\varepsilon_b} \;=\; \frac{\overrightarrow{\partial^\ell}}{\partial z^A} - \frac{1}{2}\omega_A{}^b{}_c\; s^c_{-b} \;, \tag{6.3.6}$$

i.e., $\nabla^{(t)}_A = \nabla^{(s)}_A$ for a metric spin connection. One gets a projection onto the s^{ab}_{-} matrices (rather than the s^{ab}_{+} matrices), because a metric spin connection $\omega_{A,bc}$ is antisymmetric, cf. eq. (6.1.6). Note that in the s^{ab} representation — not only the connection (6.3.6) — but also the curvature — carries a minus–a–half normalization:

$$[\nabla^{(s)}_A, \nabla^{(s)}_B] \;=\; -\frac{1}{2}R_{AB}{}^d{}_c\; s^c_{-d} \;. \tag{6.3.7}$$

This can be explained as follows: The minus sign is caused by that the s^{ab} representation acts on covariant tensors (as opposed to contravariant tensors), and the factor $\frac{1}{2}$ because the $t^a{}_b$ generator (5.4.5) becomes $\frac{1}{2}s^a_{-b}$ after the metric symmetrization.

The s^{ab}_{-} matrices satisfy an $o(W)$ Lie–algebra:

$$[s^{ab}_{\mp}, s^{cd}_{\mp}] \;=\; (-1)^{\varepsilon_a(\varepsilon_b+\varepsilon_c)}\left(g^{bc}_{(0)}\; s^{ad}_{-} + s^{bc}_{-}\; g^{ad}_{(0)}\right) \mp (-1)^{\varepsilon_a\varepsilon_b}(a \leftrightarrow b) \;. \tag{6.3.8}$$

6.4 γ^a Matrices And Clifford Algebras

The flat γ^a matrices can be defined via a Berezin–Fradkin operator representation [30, 31]

$$\gamma^a_\lambda \;\equiv\; \gamma^a \;:=\; c^a + \lambda p^a \;, \qquad \varepsilon(\gamma^a) \;=\; \varepsilon_a \;, \qquad p(\gamma^a) \;=\; 1 \text{ (mod 2)} \;. \tag{6.4.1}$$

The transposed γ^a matrices correspond to a change in the parameter $\lambda \leftrightarrow -\lambda$:

$$(\gamma^a)^T \;:=\; c^a - \lambda p^a \;=\; \gamma^a_{-\lambda} \;. \tag{6.4.2}$$

The γ^a matrices satisfy a Clifford algebra

$$[\gamma^a, \gamma^b] \;=\; 2\lambda g^{ab}_{(0)}\mathbf{1} \;. \tag{6.4.3}$$

The γ^a matrices commute with the transposed $(\gamma^b)^T$ matrices

$$[\gamma^a, (\gamma^b)^T] \;=\; 0 \;. \tag{6.4.4}$$

Let V be the vector space

$$V \;:=\; \text{span } c^a \;\oplus\; \text{span } p^a \;=\; \text{span } \gamma^a \;\oplus\; \text{span } (\gamma^a)^T \;, \tag{6.4.5}$$

and let

$$T(V) \;:=\; \bigoplus_{m=0}^{\infty} V^{\otimes m} \;=\; (\text{span } \mathbf{1}) \;\oplus\; V \;\oplus\; V{\otimes}V \;\oplus\; V{\otimes}V{\otimes}V \;\oplus \ldots \tag{6.4.6}$$

be the corresponding tensor algebra. Let $I(V)$ be the two–sided ideal generated by

$$[c^a \overset{\otimes}{,} c^b] \ , \qquad [p^a \overset{\otimes}{,} c^b] - g^{ab}\mathbf{1} \ , \qquad [p^a \overset{\otimes}{,} p^b] \ , \tag{6.4.7}$$

or equivalently, the two–sided ideal generated by

$$[\gamma^a \overset{\otimes}{,} \gamma^b] - 2g^{ab}\mathbf{1} \ , \qquad [\gamma^a \overset{\otimes}{,} (\gamma^b)^T] \ , \qquad [(\gamma^a)^T \overset{\otimes}{,} (\gamma^b)^T] + 2g^{ab}\mathbf{1} \ . \tag{6.4.8}$$

Then the Heisenberg algebra, or equivalently, the Clifford algebra $\mathrm{Cl}(V)$ is isomorphic to the quotient

$$\mathrm{Cl}(V) \ \cong \ T(V)/I(V) \ . \tag{6.4.9}$$

Each element of $\mathrm{Cl}(V)$ is a differential operator in the c^a–variables, and may be Wick/normal–ordered in a unique way, so that all the c–derivatives (the p's) stands to the right of all the c's. This is also known as *cp*–ordering.

There is another important description of the Clifford algebra $\mathrm{Cl}(V)$ as a tensor product of two (mutually commutative) Clifford algebras

$$\mathrm{Cl}(V) \ \cong \ \mathrm{Cl}(\gamma) \otimes \mathrm{Cl}(\gamma^T) \ , \tag{6.4.10}$$

where

$$\mathrm{Cl}(\gamma) \ = \ \bigoplus_{m=0}^{\infty} \mathrm{span}\ \gamma^{a_1}\gamma^{a_2}\cdots\gamma^{a_m} \ \cong \ T(\gamma)/I(\gamma) \ , \tag{6.4.11}$$

$$\mathrm{Cl}(\gamma^T) \ = \ \bigoplus_{m=0}^{\infty} \mathrm{span}\ (\gamma^{a_1})^T(\gamma^{a_2})^T\cdots(\gamma^{a_m})^T \ \cong \ T(\gamma^T)/I(\gamma^T) \ . \tag{6.4.12}$$

Since the γ matrices commute with the transposed γ^T matrices, it is possible to unshuffle an arbitrary element in $\mathrm{Cl}(V)$ into a $\gamma\gamma^T$–ordered form, *i.e.*, so that all the γ matrices stand to the left of all the γ^T matrices. For instance, the $\gamma\gamma^T$–ordered form of the γ^a and the $(\gamma^a)^T$ matrices are

$$\begin{aligned} \gamma^a \ &= \ \gamma^a \otimes \mathbf{1} \ , \\ (\gamma^a)^T \ &= \ \mathbf{1} \otimes (\gamma^a)^T \ , \end{aligned} \tag{6.4.13}$$

respectively. For more complicated expressions, the $\gamma\gamma^T$–ordered form will in general not be unique, since *e.g.*, the γ matrices do not commute among themselves. Nevertheless, the $\gamma\gamma^T$–ordering bears some resemblance with, *e.g.*, the method of holomorphic and antiholomorphic blocks in conformal field theory.

The γ^a matrices form a fundamental representation of the $o(W)$ Lie–algebra (6.3.8):

$$[s^{ab}_{\mp}, \gamma^c] \ = \ \gamma^a_{\pm\lambda}\ g^{bc}_{(0)} \mp (-1)^{\varepsilon_a\varepsilon_b}(a \leftrightarrow b) \ . \tag{6.4.14}$$

As a consequence, if one commutes a metric spin connection (6.3.6) with a flat γ^a matrix, one gets

$$[\nabla^{(s)}_A, \gamma^b] \ = \ -\omega_A{}^b{}_c\ \gamma^c \ . \tag{6.4.15}$$

A curved γ^A matrix is now defined as a flat γ^a matrix dressed with the inverse vielbein in the obvious way:

$$\gamma^A \ := \ e^A{}_a\ \gamma^a \ = \ \gamma^a\ (e^T)_a{}^A \ , \qquad \varepsilon(\gamma^A) \ = \ \varepsilon_A \ , \qquad p(\gamma^A) \ = \ 1 \ (\mathrm{mod}\ 2) \ . \tag{6.4.16}$$

(Similar straightforward rules applies to other objects when switching between flat and curved indices.)

If one commutes a metric spin connection (6.3.6) with a curved γ^A matrix, one gets

$$[\nabla^{(s)}_A, \gamma^B] \ = \ -\Gamma_A{}^B{}_C\ \gamma^C \ , \tag{6.4.17}$$

cf. eqs. (5.2.4) and (6.4.15). The result (6.4.17) can be summarized as saying that the total connection $\nabla = d + \Gamma + \omega$ commutes with the γ^A matrices: $[\nabla_A, \gamma^B] = 0$.

6.5 Dirac Operator $D^{(s)}$

For a general discussion of Dirac operators, see *e.g.*, Ref. [38]. We shall for the remainder of the Section 6 assume that the connection is the Levi–Civita connection.

Central for our discussion are the s^{ab} matrices (6.3.1). They act on flat exterior forms $\eta \in \Omega_{\bullet 0}(W)$, *i.e.*, functions $\eta{=}\eta(z;c)$ of the z^A and c^a variables.

The Dirac operator $D^{(s)}$ in the s^{ab} representation (6.3.6) is a γ^A matrix (6.4.16) times a covariant derivative (6.3.6)

$$D^{(s)} \ := \ \gamma^A \nabla^{(s)}_A \ , \qquad \varepsilon(D^{(s)}) \ = \ 0 \ , \qquad p(D^{(s)}) \ = \ 1 \ (\mathrm{mod}\ 2) \ . \tag{6.5.1}$$

The Laplace operator $\Delta^{(s)}_{\rho_g}$ in the s^{ab} representation (6.3.6) is

$$\begin{aligned}\Delta^{(s)}_{\rho_g} \ &:= \ (-1)^{\varepsilon_A}\nabla_A g^{AB}\nabla^{(s)}_B \ = \ (-1)^{\varepsilon_A}\nabla^{(s)}_A g^{AB}\nabla^{(s)}_B + \Gamma^A{}_{AC}\ g^{CB}\ \nabla^{(s)}_B \\ &= \ \frac{(-1)^{\varepsilon_A}}{\rho_g}\nabla^{(s)}_A \rho_g g^{AB}\nabla^{(s)}_B \ .\end{aligned} \tag{6.5.2}$$

Theorem 6.1 (*cp*–ordered Weitzenböck formula for flat exterior forms) *The difference between the square of the Dirac operator $D^{(s)}$ and the Laplace operator $\Delta^{(s)}_{\rho_g}$ in the s^{ab} representation (6.3.6) is*

$$D^{(s)}D^{(s)} \ - \ \lambda\Delta^{(s)}_{\rho_g} \ = \ -\frac{\lambda}{4}s^{BA}_{-}\ R_{AB,CD}\ s^{DC}_{-}(-1)^{\varepsilon_C+\varepsilon_D} \tag{6.5.3}$$

$$= \ -\lambda c^A\ R_{AB}\ p^B + \frac{\lambda}{2}c^B c^A\ R_{AB,CD}\ p^D p^C(-1)^{\varepsilon_C+\varepsilon_D} \ . \tag{6.5.4}$$

Proof of Theorem 6.1: Almost identical to the proof of Theorem 3.5 because of eq. (5.3.3).

□

6.6 Second–Order σ^{ab} Matrices

We now replace the first–order $s^{ab}_{\mp}$ matrices (6.3.1) with second–order matrices:

$$\sigma^{ab}_{\mp}(\lambda) \ \equiv \ \sigma^{ab}_{\mp} \ := \ \frac{1}{4\lambda}\gamma^a\gamma^b \mp (-1)^{\varepsilon_a\varepsilon_b}(a\leftrightarrow b) \ = \ \sigma^{ab}_{\mp}\otimes \mathbf{1} \ , \tag{6.6.1}$$

$$\varepsilon(\sigma^{ab}_{\mp}) \ = \ \varepsilon_a+\varepsilon_b \ , \qquad p(\sigma^{ab}_{\mp}) \ = \ 0 \ , \tag{6.6.2}$$

$$\sigma^{a}_{\mp c} \ := \ \sigma^{ab}_{\mp}\ g^{(0)}_{bc}(-1)^{\varepsilon_c} \ . \tag{6.6.3}$$

(The names first– and second–order refer to the number of c^a–derivatives.) The transposed $\sigma^{ab}_{\mp}$ matrices read

$$(\sigma^{ab}_{\mp})^T \ = \ \pm\frac{1}{4\lambda}(\gamma^a)^T(\gamma^b)^T \mp (-1)^{\varepsilon_a\varepsilon_b}(a\leftrightarrow b) \ = \ \mp\sigma^{ab}_{\mp}(-\lambda) \ = \ \mathbf{1}\otimes(\sigma^{ab}_{\mp})^T \ . \tag{6.6.4}$$

In the last expression of eqs. (6.6.1) and (6.6.4) we wrote the $\sigma^{ab}_{\mp}$ and the $(\sigma^{ab}_{\mp})^T$ matrices on a $\gamma\gamma^T$–ordered form. In particular, the $\sigma^{ab}_{\mp}$ matrices decouple completely from the $(\sigma^{ab}_{\mp})^T$ matrices,

$$[\sigma^{ab}_{\mp},(\sigma^{cd}_{\mp})^T] \ = \ 0 \ , \qquad [\sigma^{ab}_{\mp},(\sigma^{cd}_{\pm})^T] \ = \ 0 \ . \tag{6.6.5}$$

On one hand, the matrices

$$\sigma_{-}^{ab} \;=\; \frac{1}{4\lambda}\{\gamma^{a},\gamma^{b}\}_{+} \;=\; \frac{1}{2\lambda}c^{a}c^{b}+\frac{1}{2}s_{-}^{ab}+\frac{\lambda}{2}p^{a}p^{b} \qquad (6.6.6)$$

satisfy precisely the same non–Abelian $o(W)$ Lie–algebra (6.3.8) and fundamental representation (6.4.14) as the s_{-}^{ab} matrices. On the other hand, the matrices

$$\sigma_{+}^{ab} \;:=\; \frac{1}{4\lambda}[\gamma^{a},\gamma^{b}] \;\stackrel{(6.4.3)}{=}\; \frac{1}{2}g_{(0)}^{ab}\mathbf{1} \qquad (6.6.7)$$

are proportional to the identity operator, and thus Abelian.

The s_{-}^{ab} matrices can be expressed in terms of the σ_{-}^{ab} matrices and their transposed,

$$s_{-}^{ab} \;=\; \sigma_{-}^{ab}+\sigma_{-}^{ab}(-\lambda) \;=\; \sigma_{-}^{ab}\otimes\mathbf{1}-\mathbf{1}\otimes(\sigma_{-}^{ab})^{T}\ , \qquad (6.6.8)$$

as a consequence of eq. (6.6.6). In contrast, the s_{+}^{ab} matrices can *not* be expressed in terms of the $\sigma_{\mp}^{ab}$ matrices and their transposed.

The first–order $\nabla_{A}^{(s)}$ realization (6.3.6) can be identically rewritten into the following second–order $\sigma\sigma^{T}$ matrix realization

$$\nabla_{A}^{(\sigma\sigma^{T})} \;:=\; \frac{\stackrel{\rightarrow}{\partial^{\ell}}}{\partial z^{A}}-\frac{1}{2}\omega_{A,bc}\left(\sigma_{-}^{cb}\otimes\mathbf{1}-\mathbf{1}\otimes(\sigma_{-}^{cb})^{T}\right)(-1)^{\varepsilon_{b}} \;=\; \frac{\stackrel{\rightarrow}{\partial^{\ell}}}{\partial z^{A}}-\frac{1}{2}\omega_{A}{}^{b}{}_{c}\left(\sigma_{-}^{c}{}_{b}\otimes\mathbf{1}-\mathbf{1}\otimes(\sigma_{-}^{c}{}_{b})^{T}\right)\ , \qquad (6.6.9)$$

i.e., $\nabla_{A}^{(t)}=\nabla_{A}^{(s)}=\nabla_{A}^{(\sigma\sigma^{T})}$ for a metric spin connection. In contrast, the first–order $\nabla_{A}^{(S)}$ realization (3.12.6) does in general not have a second–order formulation for a metric connection, even if the manifold is a spin manifold, cf. Appendix 3.A. This is despite the fact that the first–order realizations $\nabla_{A}^{(S)}$ and $\nabla_{A}^{(s)}$ are closely related via condition (5.2.2),

$$\nabla_{A}^{(S)}\eta(z;e^{b}{}_{B}C^{B}) \;=\; (\nabla_{A}^{(s)}\eta)(z;c)\Big|_{c^{b}=e^{b}{}_{B}C^{B}}\ , \qquad (6.6.10)$$

where $\eta=\eta(z;c;y)\in\Omega_{\bullet 0}(W)$ is a flat exterior form. Here the $S_{\mp}^{AB}$ and $s_{\mp}^{ab}$ matrices act by adjoint action on the C^{C} and c^{c} variables as

$$[S_{\mp}^{AB},C^{C}] \;=\; C^{A}g^{BC}\mp(-1)^{\varepsilon_{A}\varepsilon_{B}}(A\leftrightarrow B)\ , \qquad [s_{\mp}^{ab},c^{c}] \;=\; c^{a}g_{(0)}^{bc}\mp(-1)^{\varepsilon_{a}\varepsilon_{b}}(a\leftrightarrow b)\ , \qquad (6.6.11)$$

cf. eqs. (3.11.1) and (6.3.1), respectively. The crucial difference is that the $\nabla_{A}^{(S)}$ realization (3.12.6) contains a non–trivial S_{+} sector, while the $\nabla_{A}^{(s)}$ realization (6.3.6) has *no* s_{+} sector. This has its root in the fact that the flat metric condition (6.1.6) is an algebraic condition, while the curved metric condition (3.6.1) is a differential condition. (Curiously, it is just opposite for the torsionfree conditions: the curved torsionfree condition is an algebraic condition, while the flat torsionfree condition is a differential condition, cf. eqs. (2.2.2) and (5.2.18).)

6.7 Lichnerowicz' Formula

It is convenient to define a totally symmetrized combination of three γ^{a} matrices as

$$\gamma^{a_{1}a_{2}a_{3}} \;:=\; \frac{1}{3!}\sum_{\pi\in S_{3}}(-1)^{\varepsilon_{\pi,a}}\gamma^{a_{\pi(1)}}\,\gamma^{a_{\pi(2)}}\,\gamma^{a_{\pi(3)}}\ , \qquad (6.7.1)$$

where $(-1)^{\varepsilon_{\pi,a}}$ is the sign factor that arises when one does a π–permutation of three supercommuting objects with the same Grassmann– and form–parity as the γ^a matrices, say, the c's

$$c^{a_1}\wedge c^{a_2}\wedge c^{a_3} \;=\; (-1)^{\varepsilon_{\pi,a}} c^{a_{\pi(1)}}\wedge c^{a_{\pi(2)}}\wedge c^{a_{\pi(3)}} \;, \tag{6.7.2}$$

cf. (5.4.3). The symmetrized γ^{abc} matrix can be reduced with the help of the Clifford relation (6.4.3) as

$$\gamma^{abc} \;=\; \gamma^a\gamma^b\gamma^c - \lambda g^{ab}_{(0)}\;\gamma^c + (-1)^{\varepsilon_b\varepsilon_c}\lambda g^{ac}_{(0)}\;\gamma^b - \gamma^a\;\lambda g^{bc}_{(0)}\;. \tag{6.7.3}$$

Theorem 6.2 ($\gamma\gamma^T$–ordered Lichnerowicz' formula [6]) *The square of the Dirac operator $D^{(\sigma\sigma^T)}$ in the $\sigma\sigma^T$ representation (6.6.1) is*

$$D^{(\sigma\sigma^T)}D^{(\sigma\sigma^T)} \;=\; \lambda\Delta^{(\sigma\sigma^T)}_{\rho_g} - \frac{\lambda}{4}R + \frac{\lambda}{2}\sigma_-^{BA}\;R_{AB,CD}\otimes(\sigma_-^{DC})^T(-1)^{\varepsilon_C+\varepsilon_D}\;. \tag{6.7.4}$$

PROOF OF THEOREM 6.2: One derives that the square of the Dirac operator $D^{(\sigma\sigma^T)}$ is the Laplacian $\Delta^{(\sigma\sigma^T)}_{\rho_g}$ plus a curvature term, by proceeding along the lines of the proof of Theorem 3.5:

$$D^{(\sigma\sigma^T)}D^{(\sigma\sigma^T)} \;=\; \frac{1}{2}[D^{(\sigma\sigma^T)},D^{(\sigma\sigma^T)}] \;=\; \lambda\Delta^{(\sigma\sigma^T)}_{\rho_g} - \frac{1}{2}\gamma^B\gamma^A[\nabla^{(\sigma\sigma^T)}_A,\nabla^{(\sigma\sigma^T)}_B]\;. \tag{6.7.5}$$

When one $\gamma\gamma^T$–decomposes the curvature term, it splits in two parts:

$$-\frac{1}{2}\gamma^B\gamma^A[\nabla^{(\sigma\sigma^T)}_A,\nabla^{(\sigma\sigma^T)}_B] \;=\; \frac{1}{4}\gamma^B\gamma^A\;R_{AB}{}^d{}_c\left(\sigma^c_{-d}\otimes\mathbf{1}-\mathbf{1}\otimes(\sigma^c_{-d})^T\right) \;=\; III+III^T\;, \tag{6.7.6}$$

where

$$III^T \;:=\; \frac{\lambda}{2}\sigma_-^{BA}\;R_{AB,CD}\otimes(\sigma_-^{DC})^T(-1)^{\varepsilon_C+\varepsilon_D}\;, \tag{6.7.7}$$

and

$$\begin{aligned}
III \;&:=\; -\frac{1}{4}\gamma^B\gamma^A\;R_{AB,CD}\;\sigma_-^{DC}(-1)^{\varepsilon_C+\varepsilon_D} \;=\; -\frac{1}{8\lambda}\gamma^B\gamma^A\;R_{AB,CD}\;\gamma^D\gamma^C(-1)^{\varepsilon_C+\varepsilon_D}\\
&=\; \frac{1}{8\lambda}(-1)^{(\varepsilon_A+\varepsilon_B)\varepsilon_C}\gamma^B\gamma^A\gamma^C\;R_{AB,CD}\;\gamma^D(-1)^{\varepsilon_D}\\
&\overset{(6.7.3)}{=}\; \frac{1}{8\lambda}\left(\gamma^{CBA}+\gamma^C\;\lambda g^{BA}-\lambda g^{CB}\;\gamma^A+(-1)^{\varepsilon_A\varepsilon_B}\lambda g^{CA}\;\gamma^B\right)R_{AB,CD}\;\gamma^D(-1)^{\varepsilon_D}\\
&=\; -\frac{1}{4}g^{CB}\;\gamma^A\;R_{AB,CD}\;\gamma^D(-1)^{\varepsilon_D} \;=\; \frac{1}{4}(-1)^{(\varepsilon_A+\varepsilon_B)(\varepsilon_D+1)}R_{ABD}{}^B\;\gamma^A\gamma^D(-1)^{\varepsilon_D}\\
&=\; -\frac{1}{4}R_{DA}\;\gamma^A\gamma^D(-1)^{\varepsilon_D} \;=\; -\frac{\lambda}{4}R_{DA}\;g^{AD}(-1)^{\varepsilon_D} \;=\; -\frac{\lambda}{4}R\;.
\end{aligned} \tag{6.7.8}$$

Here the first Bianchi identity (3.7.5) was used one time.

□

6.8 Clifford Representations

The spinor representations $\mathcal{S}$ and $\mathcal{S}^T$ can be defined as Fock spaces

$$\mathcal{S} \;:=\; \mathrm{Cl}(\gamma)|0\rangle \;=\; \bigoplus_{m=0}^{\infty}\mathrm{span}\;c^{a_1}c^{a_2}\cdots c^{a_m}|0\rangle\;, \qquad p^a|0\rangle \;=\; 0\;, \tag{6.8.1}$$

$$\mathcal{S}^T \;:=\; \mathrm{Cl}(\gamma^T)|0^T\rangle \;=\; \bigoplus_{m=0}^{\infty}\mathrm{span}\;p^{a_1}p^{a_2}\cdots p^{a_m}|0^T\rangle\;, \qquad c^a|0^T\rangle \;=\; 0\;. \tag{6.8.2}$$

The constraints $p^a|0\rangle = 0$ (resp. $c^a|0^T\rangle = 0$) are consistent, because the p^a's (resp. the c^a's) commute. The representation (6.8.1) and (6.8.2) are of course just two possibilities out of infinitely many equivalent choices of Fock space representations. A different class of vacua $|1\rangle$ and $|1^T\rangle$ are defined via

$$\sigma_-^{ab}|1\rangle \ = \ 0 \ , \qquad\qquad (\sigma_-^{ab})^T|1^T\rangle \ = \ 0 \ . \tag{6.8.3}$$

They both represent the singlet/trivial representation of the orthogonal Lie–group $O(W)$. Again, the constraints (6.8.3) for the vacua are consistent, since the σ_-^{ab} (resp. the $(\sigma_-^{ab})^T$) matrices form Lie–algebras. All the above constraints are examples of first–class constraints. More generally, assume that $|\Omega\rangle$ and $|\Omega^T\rangle$ are two arbitrary consistent vacua (that are not necessarily related). Let $\mathcal{V}$ and $\mathcal{V}^T$ denote the corresponding vector spaces

$$\mathcal{V} \ := \ \mathrm{Cl}(\gamma)|\Omega\rangle \ , \qquad\qquad \mathcal{V}^T \ := \ \mathrm{Cl}(\gamma^T)|\Omega^T\rangle \ . \tag{6.8.4}$$

The Clifford algebra $\mathrm{Cl}(V) \cong \mathrm{Cl}(\gamma) \otimes \mathrm{Cl}(\gamma^T)$ is defined to act on the tensor product $\mathcal{V} \otimes \mathcal{V}^T$ via a $\gamma\gamma^T$–ordered form, *i.e.*, the γ^a matrices act on the first factor $\mathcal{V}$ and the transposed $(\gamma^a)^T$ matrices act on the second factor $\mathcal{V}^T$. In detail, if $|v\rangle \in \mathcal{V}$ and $|v^T\rangle \in \mathcal{V}^T$ are two (not necessarily related) states, then

$$\gamma^a.(|v\rangle \otimes |v^T\rangle) \ := \ (\gamma^a|v\rangle) \otimes |v^T\rangle \ , \tag{6.8.5}$$

$$(\gamma^a)^T.(|v\rangle \otimes |v^T\rangle) \ := \ (-1)^{\vec{\varepsilon}(\gamma^a)\cdot\vec{\varepsilon}(v)}|v\rangle \otimes (\gamma^a)^T|v^T\rangle \ . \tag{6.8.6}$$

By definition, $\mathcal{V}$ is a Clifford bundle, while $\mathcal{V}^T$ is a dual/contragredient Clifford bundle.

A Lie–algebra element $x \in so(W)$ is of the form

$$x \ = \ \frac{1}{2}(-1)^{\varepsilon_a} x_{ab} s_-^{ba} \ = \ \frac{1}{2} x^a{}_b s^b{}_{-a} \ = \ \frac{1}{2} x^a{}_b \left(\sigma^b{}_{-a} \otimes \mathbf{1} - \mathbf{1} \otimes (\sigma^b{}_{-a})^T\right) \ , \tag{6.8.7}$$

where

$$x_{ab} \ = \ (-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}(a \leftrightarrow b) \ , \qquad x^a{}_c \ := g_{(0)}^{ab} x_{bc} \ . \tag{6.8.8}$$

A $\gamma\gamma^T$–ordered form of a generic special orthogonal Lie–group element $g = e^x \in SO(W)$ is

$$\exp\left[\frac{1}{2}x^a{}_b s^b{}_{-a}\right] \ = \ \exp\left[\frac{1}{2}x^a{}_b \sigma^b{}_{-a}\right] \ \otimes \ \exp\left[-\frac{1}{2}x^c{}_d (\sigma^d{}_{-c})^T\right] \ . \tag{6.8.9}$$

In this way the vector space $\mathcal{V}^T$ becomes a dual/contragredient representation of the special orthogonal Lie–group $SO(W)$, hence the name.

6.9 Intertwining Operator

Consider the intertwining operator

$$s \ := \ \int d^N\theta \ e^{\theta_a \gamma^a} \otimes e^{\theta_b (\gamma^b)^T} \ , \tag{6.9.1}$$

where θ_a are integration variables with Grassmann–parity $\varepsilon(\theta_a) = \varepsilon_a$ and form–parity $p(\theta_a) = 1$ (mod 2).

Lemma 6.3 *The intertwining operator s is invariant under the adjoint action $e^x s e^{-x} = s$ of the special orthogonal Lie–group $SO(W)$. Equivalently, the intertwining operator s commute with the $so(W)$ Lie–algebra generators $[s_-^{ab}, s] = 0$.*

PROOF OF LEMMA 6.3: The adjoint action rotates the γ^a matrices,

$$\begin{aligned}
\exp\left[\frac{1}{2}x^c{}_d\sigma^d_{-c}\right]\gamma^a\exp\left[-\frac{1}{2}x^e{}_f\sigma^f_{-e}\right] &= (e^x)^a{}_b\gamma^b \ , \\
\exp\left[-\frac{1}{2}x^c{}_d(\sigma^d_{-c})^T\right](\gamma^a)^T\exp\left[\frac{1}{2}x^e{}_f(\sigma^f_{-e})^T\right] &= (e^x)^a{}_b(\gamma^b)^T \ ,
\end{aligned} \tag{6.9.2}$$

where

$$(e^x)^a{}_b \ := \ \delta^a_b + x^a{}_b + \frac{1}{2!}x^a{}_cx^c{}_b + \frac{1}{3!}x^a{}_cx^c{}_dx^d{}_b + \frac{1}{4!}x^a{}_cx^c{}_dx^d{}_ex^e{}_b + \ldots \ . \tag{6.9.3}$$

Hence one may change integration variables $\theta_a \to \theta'_b = \theta_a(e^x)^a{}_b$ in the integral (6.9.1). The Jacobian vanishes for special orthogonal transformations

$$\ln\mathrm{sdet}(e^x)^a{}_b \ = \ (-1)^{\varepsilon_a}x^a{}_a \ = \ (-1)^{\varepsilon_a}g^{ab}_{(0)}x_{ba} \ = \ 0 \ . \tag{6.9.4}$$

□

Lemma 6.4 *The corresponding intertwining state*

$$||s\rangle\rangle \ := \ s.(|\Omega\rangle\otimes|\Omega^T\rangle) \ = \ \int d^N\theta \ e^{\theta_a\gamma^a}|\Omega\rangle\otimes e^{\theta_b(\gamma^b)^T}|\Omega^T\rangle \tag{6.9.5}$$

is invariant under the action of the special orthogonal Lie–group $SO(W)$. Equivalently, the $so(W)$ Lie–algebra generators s^{ab}_- annihilate the intertwining state $s^{ab}_-||s\rangle\rangle = 0$.

PROOF OF LEMMA 6.4:

$$\begin{aligned}
e^x||s\rangle\rangle &= \int d^N\theta \ e^{\theta_a\gamma^a}\exp\left[\frac{1}{4\lambda}(-1)^{\varepsilon_c}x_{cd}\gamma^d\gamma^c\right]|\Omega\rangle\otimes e^{\theta_b(\gamma^b)^T}\exp\left[-\frac{1}{4\lambda}(-1)^{\varepsilon_e}x_{ef}(\gamma^f)^T(\gamma^e)^T\right]|\Omega^T\rangle \\
&= \int d^N\theta \ \exp\left[\frac{1}{4\lambda}(-1)^{\varepsilon_c}x_{cd}\tilde{\gamma}^d\tilde{\gamma}^c\right]e^{\theta_a\gamma^a}|\Omega\rangle\otimes\exp\left[-\frac{1}{4\lambda}(-1)^{\varepsilon_e}x_{ef}(\tilde{\gamma}^f)^T(\tilde{\gamma}^e)^T\right]e^{\theta_b(\gamma^b)^T}|\Omega^T\rangle \\
&= ||s\rangle\rangle \ ,
\end{aligned} \tag{6.9.6}$$

where we have introduced (a kind of) Fourier transformed γ matrices

$$\tilde{\gamma}^a \ := \ \frac{\overrightarrow{\partial^\ell}}{\partial\theta_a} + g^{ab}_{(0)}\theta_b \ , \qquad (\tilde{\gamma}^a)^T \ := \ -\frac{\overrightarrow{\partial^\ell}}{\partial\theta_a} + g^{ab}_{(0)}\theta_b \ , \tag{6.9.7}$$

which satisfy

$$\tilde{\gamma}^a\exp\left[\theta_b\gamma^b\right] \ = \ \exp\left[\theta_b\gamma^b\right]\gamma^a \ , \qquad -(\tilde{\gamma}^a)^T\exp\left[\theta_b(\gamma^b)^T\right] \ = \ \exp\left[\theta_b(\gamma^b)^T\right](\gamma^a)^T \ . \tag{6.9.8}$$

In the last equality of eq. (6.9.6), we performed integration by part, which turns $\tilde{\gamma}^a$ into $(\tilde{\gamma}^a)^T$, and vice–versa.

□

The algebra bundle (6.4.9) of differential operators in the c^a–variables, or equivalently polynomials in γ and γ^T, is isomorphic to the bispinor bundle $\mathcal{S}\otimes\mathcal{S}^T$. The bundle isomorphism is

$$\mathrm{Cl}(V) \ \cong \ \mathrm{Cl}(\gamma)\otimes\mathrm{Cl}(\gamma^T) \ \ni \ F \ \stackrel{\cong}{\mapsto} \ F||s\rangle\rangle \ \in \ \mathcal{S}\otimes\mathcal{S}^T \ \cong \ \mathrm{End}(\mathcal{S}) \ . \tag{6.9.9}$$

The bispinor bundle $\mathcal{S}\otimes\mathcal{S}^T\cong\mathrm{End}(\mathcal{S})$ is, in turn, isomorphic (as vector bundles) to the bundle of endomorphisms: $\mathcal{S}\to\mathcal{S}$. Let us also mention that the Weyl symbol $\stackrel{\cong}{\mapsto}$ operator isomorphism $\bigwedge^\bullet(V)\stackrel{\cong}{\to}\mathrm{Cl}(V)$ from the exterior algebra $(\bigwedge^\bullet(V);*)$, equipped with the Groenewold/Moyal $*$ product, to the Heisenberg algebra $(\mathrm{Cl}(V);\circ)$, is known as the Chevalley isomorphism in the context of Clifford algebras.

6.10 Schrödinger–Lichnerowicz' Formula

We will be interested in how the Dirac operator acts on a Clifford bundle $\mathcal{V}\otimes|1^T\rangle\cong\mathcal{V}$ and a tensor Clifford bundle $\mathcal{V}\otimes\mathcal{V}^T$.

Theorem 6.5 (Schrödinger–Lichnerowicz' formula [5, 6]) *On a Clifford bundle $\mathcal{V}\otimes|1^T\rangle\cong\mathcal{V}$, the square of the Dirac operator $D^{(\sigma)}$ is equal to the Laplacian $\Delta^{(\sigma)}_{\rho_g}$ minus a quarter of the scalar curvature R,*

$$D^{(\sigma)}D^{(\sigma)} \;=\; \lambda\Delta^{(\sigma)}_{\rho_g}-\frac{\lambda}{4}R\;. \tag{6.10.1}$$

Proof of Theorem 6.5: This is a Corollary to Lichnerowicz' formula (6.7.4).

□

The Schrödinger–Lichnerowicz' formula (6.10.1) corresponds to naively substituting the first–order matrices $s^{ab}_{-}\to\sigma^{ab}_{-}$ in the $\nabla^{(s)}$ realization (6.3.6) with the second–order matrices σ^{ab}_{-}. The analysis in Subsections 6.6 and 6.8 shows in detail why this replacement is geometrically sound and in fact very natural.

Theorem 6.6 *The square of the Dirac operator $D^{(\sigma\sigma^T)}$ on a tensor Clifford bundle $\mathcal{V}\otimes\mathcal{V}^T$ becomes equal to the Laplace–Beltrami operator Δ_{ρ_g} when it is projected on the singlet representation $||s\rangle\rangle$,*

$$D^{(\sigma\sigma^T)}D^{(\sigma\sigma^T)}f||s\rangle\rangle \;=\; \lambda(\Delta_{\rho_g}f)||s\rangle\rangle\;, \tag{6.10.2}$$

where $f\!=\!f(z)$ is an arbitrary scalar function.

Proof of Theorem 6.6: This is a Corollary to the Weitzenböck formula (6.5.3).

□

7 Antisymplectic Spin Geometry

7.1 Spin Geometry

Assume that the vector space W is endowed with a constant antisymplectic metric

$$E^{(0)} \;=\; \frac{1}{2}c^a\,E^{(0)}_{ab}\wedge c^b \;=\; -\frac{1}{2}E^{(0)}_{ab}\;c^b\wedge c^a\;\in\;\Omega_{20}(W)\;, \tag{7.1.1}$$

called the *flat* metric. It has Grassmann–parity

$$\varepsilon(E^{(0)}_{ab}) \;=\; \varepsilon_a+\varepsilon_b+1\;,\qquad \varepsilon(E^{(0)}) \;=\; 1\;,\qquad p(E^{(0)}_{AB}) \;=\; 0\;, \tag{7.1.2}$$

and symmetry

$$E^{(0)}_{ba} \;=\; -(-1)^{\varepsilon_a\varepsilon_b}E^{(0)}_{ab}\;. \tag{7.1.3}$$

Furthermore, assume that the vielbein $e^a{}_A$ intertwines between the curved E_{AB} metric and the flat $E^{(0)}_{ab}$ metric:

$$E_{AB} \;=\; (e^T)_A{}^a\;E^{(0)}_{ab}\;e^b{}_B\;. \tag{7.1.4}$$

A spin connection $\nabla^{(\omega)}$ is called *antisymplectic*, if it preserves the flat metric,

$$0 \;=\; -\nabla^{(\omega)}_{A} E^{(0)}_{bc} \;=\; \omega_{A,bc} - (-1)^{\varepsilon_b\varepsilon_c}\omega_{A,cb} \;, \tag{7.1.5}$$

i.e., the lowered $\omega_{A,bc}$ symbol should be symmetric in the flat indices. Here we have lowered the $\omega_{A,bc}$ symbol with the flat metric

$$\omega_{A,bc} \;:=\; (-1)^{\varepsilon_A\varepsilon_b}\omega_{bAc} \;, \qquad \omega_{bAc} \;:=\; E^{(0)}_{bd}\,\omega^{d}{}_{Ac}(-1)^{\varepsilon_A} \;. \tag{7.1.6}$$

In particular, the two connections $\nabla^{(\Gamma)}$ and $\nabla^{(\omega)}$ are antisymplectic at the same time, as a consequence of the correspondence (5.2.2) and (7.1.4).

7.2 First–Order s^{ab} Matrices

Because of the presence of the flat metric $E^{ab}_{(0)}$, the symmetry of the general linear Lie–algebra $gl(W)$ reduces to an antisymplectic Lie–subalgebra. Its generators $s^{ab}_{\pm}$ read

$$s^{ab}_{\pm} \;:=\; c^{a}(-1)^{\varepsilon_b}p^{b} \;\mp\; (-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}(a \leftrightarrow b) \;, \qquad p^{a} \;:=\; E^{ab}_{(0)}\frac{\overrightarrow{\partial^{\ell}}}{\partial c^{b}} \;, \tag{7.2.1}$$

$$\varepsilon(s^{ab}_{\pm}) \;=\; \varepsilon_a + \varepsilon_b + 1 \;, \qquad p(s^{ab}_{\pm}) \;=\; 0 \;, \tag{7.2.2}$$

$$s^{a}_{\pm c} \;:=\; s^{ab}_{\pm}\, E^{(0)}_{bc}(-1)^{\varepsilon_c} \;. \tag{7.2.3}$$

The $\nabla^{(t)}_{A}$ realization (5.4.4) can be identically rewritten into the following s^{ab} matrix realization

$$\nabla^{(s)}_{A} \;:=\; \frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}} + \frac{1}{2}\omega_{A,bc}\, s^{cb}_{+}(-1)^{\varepsilon_b} \;=\; \frac{\overrightarrow{\partial^{\ell}}}{\partial z^{A}} - \frac{1}{2}\omega_{A}{}^{b}{}_{c}\, s^{c}_{+b} \;, \tag{7.2.4}$$

i.e., $\nabla^{(t)}_{A} = \nabla^{(s)}_{A}$ for an antisymplectic spin connection. One gets a projection onto the s^{ab}_{+} matrices (rather than the s^{ab}_{-} matrices), because an antisymplectic spin connection $\omega_{A,bc}$ is symmetric, cf. eq. (7.1.5).

The s^{ab}_{+} matrices satisfy an antisymplectic Lie–algebra:

$$[s^{ab}_{\pm}, s^{cd}_{\pm}] \;=\; (-1)^{\varepsilon_a(\varepsilon_b+\varepsilon_c+1)+\varepsilon_b}\left(E^{bc}_{(0)}\; s^{ad}_{+} - s^{bc}_{+}\; E^{ad}_{(0)}\right) \mp (-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}(a \leftrightarrow b) \;. \tag{7.2.5}$$

7.3 γ^{a} Matrices

The flat γ^{a} matrices can be defined via a Berezin–Fradkin operator representation [30, 31]

$$\gamma^{a}_{\theta} \;\equiv\; \gamma^{a} \;:=\; c^{a} + (-1)^{\varepsilon_a}\theta p^{a} \;=\; c^{a} - p^{a}\theta \;, \qquad \varepsilon(\gamma^{a}) \;=\; \varepsilon_a \;, \qquad p(\gamma^{a}) \;=\; 1 \pmod 2 \;. \tag{7.3.1}$$

The γ^{a} matrices satisfy a Clifford–like algebra

$$[\gamma^{a}, \gamma^{b}] \;=\; 2(-1)^{\varepsilon_a}\theta E^{ab}_{(0)}\mathbf{1} \;. \tag{7.3.2}$$

The γ^{a} matrices form a fundamental representation of the antisymplectic Lie–algebra (7.2.5):

$$[s^{ab}_{\pm}, \gamma^{c}] \;=\; \gamma^{a}_{\pm\theta}(-1)^{\varepsilon_b}E^{bc}_{(0)} \mp (-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}(a \leftrightarrow b) \;. \tag{7.3.3}$$

As a consequence, if one commutes an antisymplectic spin connection (7.2.4) with a flat γ^a matrix, one gets

$$[\nabla^{(s)}_A, \gamma^b] \;=\; -\omega_A{}^b{}_c\, \gamma^c \; . \tag{7.3.4}$$

Similarly, if one commutes an antisymplectic spin connection (7.2.4) with a curved γ^A matrices, one gets

$$[\nabla^{(s)}_A, \gamma^B] \;=\; -\Gamma_A{}^B{}_C\, \gamma^C \; , \tag{7.3.5}$$

cf. eqs. (5.2.4) and (7.3.4).

7.4 Dirac Operator $D^{(s)}$

We shall for the remainder of Section 7 assume that the connection is antisymplectic, torsionfree and ρ–compatible.

The Dirac operator $D^{(s)}$ in the s^{ab} representation (7.2.4) is a γ^A matrix (7.3.1) times a covariant derivative (7.2.4)

$$D^{(s)} \;:=\; \gamma^A \nabla^{(s)}_A \; , \qquad \varepsilon(D^{(s)}) \;=\; 0 \; , \qquad p(D^{(s)}) \;=\; 1 \text{ (mod 2)} \; . \tag{7.4.1}$$

The odd Laplacian $\Delta^{(s)}_\rho$ in the s representation (7.2.4) is

$$2\Delta^{(s)}_\rho \;:=\; (-1)^{\varepsilon_A} \nabla_A E^{AB} \nabla^{(s)}_B \;=\; \frac{(-1)^{\varepsilon_A}}{\rho} \nabla^{(s)}_A \rho E^{AB} \nabla^{(s)}_B \; . \tag{7.4.2}$$

Theorem 7.1 (Antisymplectic Weitzenböck type formula for flat exterior forms) *The difference between the square of the Dirac operator $D^{(s)}$ and twice the odd Laplacian $\Delta^{(s)}_\rho$ in the s^{ab} representation (7.2.4) is*

$$D^{(s)} D^{(s)} \;-\; 2\theta \Delta^{(s)}_\rho \;=\; \frac{\theta}{4} (-1)^{\varepsilon_B + \varepsilon_C} s_-^{BA}\; R_{AB,CD}\; s_+^{DC} \tag{7.4.3}$$

$$\;=\; -\theta c^A\; R_{AB}\; p^B + \frac{\theta}{2} c^B c^A\; R_{AB,CD}\; p^D p^C (-1)^{\varepsilon_C} \; . \tag{7.4.4}$$

Proof of Theorem 7.1: Almost identical to the proof of Theorem 4.4 because of eq. (5.3.3).

□

7.A Appendix: Shifted s'^{ab}_+ Matrices

We have already seen in Appendix 4.A that there are no consistent antisymplectic second–order deformations of the s^{ab}_+ matrices. The only remaining deformation is a c–number shift,

$$s'^{ab}_+ \;:=\; s^{ab}_+ + \alpha E^{ab}_{(0)} \mathbf{1} \; , \tag{7.A.1}$$

$$s'^{a}_{+b} \;:=\; s^{a}_{+b} + \alpha (-1)^{\varepsilon_a} \delta^a_b \mathbf{1} \; , \tag{7.A.2}$$

with a parameter α, cf. eq. (4.8.6). These shifted s'^{ab}_+ matrices satisfy the same Lie–algebra (7.2.5) and fundamental representation (7.3.3) as the s^{ab}_+ matrices. Moreover, the shift does not affect the s^{ab} representation (7.2.4) of the spin connection, because of tracefree condition (5.2.19). Similarly, the curvature

$$[\nabla^{(s)}_A, \nabla^{(s)}_B] \;=\; -\frac{1}{2} R_{AB}{}^d{}_c\; s^c_{+d} \; . \tag{7.A.3}$$

is unaffected, since the shift–term is proportional to the Ricci two–form $\mathcal{R}_{AB}=0$, which is zero. Thus we conclude that the c–number shift $s^{ab}_+ \to s'^{ab}_+$ has no effects at all on the construction.

8 Conclusions

The main objective of the paper is to gain knowledge about the deepest and most profound geometric levels of the field–antifield formalism [1, 2, 3]. It is imperative to better understand the geometric meaning of the odd scalar curvature R, which sits as a zeroth–order term in the odd Δ operator (1.0.1), and which descends to the quantum master equation $\Delta \exp[\frac{i}{\hbar}W] = 0$ as a two–loop contribution:

$$(W, W) \;=\; 2i\hbar\Delta_{\rho}W - \hbar^{2}\frac{R}{4} \; . \qquad (8.0.1)$$

We have in this paper investigated the hypothesis that the zeroth–order term $-R/4$ of (twice) the odd Δ operator (1.0.1) is related to the zeroth–order term $-R/4$ in the Schrödinger–Lichnerowicz formula (6.10.1). We have so far been unable to give a closed argument that such relationship exists. In fact, Theorem 6.6 indicates that there is no relation, as explained in the Introduction. Some of the main results of the paper are the following.

- We have classified scalar invariants of suitable dimensions that depend on the density ρ and the metric, cf. Proposition 3.2 and Proposition 4.2.
- We have identified (via a ρ–independence argument) a Riemannian counterpart (3.9.1) of the antisymplectic Δ operator (1.0.1), that takes scalars to scalars, and we have, in terms of formulas, traced the minus–a–quarter coefficient in front of R from the Riemannian to the antisymplectic side, cf. Subsection 4.7.
- We have tied the Riemannian Δ operator (3.5.2) to the quantum Hamiltonian $\hat{H}$ for a particle moving in a curved Riemannian space, cf. Subsection 3.10.
- We have derived the Laplace-Beltrami operator Δ_{ρ_g} by projecting the square of the bispinor Dirac operator $D^{(\sigma\sigma^T)}$ to a singlet state $||s\rangle\rangle$, cf. Theorem 6.6.
- We have found a first–order formalism for antisymplectic spinors and proved two Weitzenböck–type identities (Theorem 4.4 and Theorem 7.1) that are in exact one–to–one correspondence with their Riemannian siblings (Theorem 3.5 and Theorem 6.1).

However, there is in our approach *no* antisymplectic analogue of the Riemannian second–order formalism and the Schrödinger–Lichnerowicz formula (6.10.1). A bit oversimplified, this is because the canonical choice for antisymplectic second–order $\Sigma^{AB}_{\pm}$ matrices is

$$\Sigma^{AB}_{\pm} \;\stackrel{?}{=}\; \frac{1}{4}\Gamma^{A}\star\Gamma^{B}\mp(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}(A\leftrightarrow B)\;, \qquad \varepsilon(\Sigma^{AB}_{\pm}) \;=\; \varepsilon_A+\varepsilon_B+1\;, \qquad p(\Sigma^{AB}_{\pm}) \;=\; 0\;, \; (8.0.2)$$

where "$\star$" is a Fermionic multiplication, $\varepsilon(\star) = 1$. This choice (8.0.2) meet all the requirements of Grassmann–parity and symmetry, and is a direct analogue of the Riemannian second–order $\Sigma^{AB}_{\pm}$ matrices (3.A.1). Unfortunately, such $\star$ multiplication does not admit a Berezin–Fradkin representation of the Γ^A matrices, cf. Appendix 4.B. We instead introduced a Fermionic nilpotent parameter θ, which may formally be identified with the inverse $\star^{-1}$, and which serves as a Fermionic analogue of the "Planck constant" λ from the Riemannian case. Then the $\star$ multiplication itself should be identified with the inverse θ^{-1}, which is an ill–defined quantity, and hence the above formula (8.0.2) for the $\Sigma^{AB}_{\pm}$ matrices does not make sense. Note however that the nilpotent θ parameter breaks the non–degeneracy of the Clifford algebra (4.9.2).

Acknowledgement: We would like to thank Poul Henrik Damgaard, the Niels Bohr Institute and the Niels Bohr International Academy for warm hospitality. I.A.B. would also like to thank Michal Lenc and the Masaryk University for warm hospitality. The work of I.A.B. and K.B. is supported by the Ministry of Education of the Czech Republic under the project MSM 0021622409. The work of I.A.B. is supported by grants RFBR 08–01–00737, RFBR 08–02–01118 and LSS–1615.2008.2.

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